



Modeling of Mechanisms and Robotic Manipulators Taking into Account Distributed Inertia of Motion

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Abstract

Strength and stiffness assessment of rod mechanisms and robotic manipulators is commonly performed using finite element methods (FEM), while motion-induced inertia is often represented by resultant forces and moments applied at link centers of mass. This simplification is inherently approximate because inertial and gravitational forces are distributed along the link and depend on instantaneous kinematics. The main contribution of this paper is a systematic FEM-ready algorithm that extends a previously planar distributed-inertia concept to a formulation applicable to both planar and spatial rod models. The proposed approach computes distributed inertial and weight load intensities in the local coordinate system of each moving rod from kinematic quantities (link orientation, angular velocity/acceleration, and pole accelerations), transforms them into consistent equivalent nodal forces and moments, and assembles them into the global FEM equilibrium system for strength and stiffness calculations. Unlike flexible multibody dynamics formulations (*e.g.*, floating frame of reference formulation (FFRF)/ANCF) absolute nodal coordinate formulation, which primarily target time-domain dynamic simulation, the proposed method is design-oriented and directly produces configuration-dependent engineering outputs-bending moments, axial forces, and longitudinal/transverse displacements-required for link cross-section sizing and stress assessment. The method is demonstrated on a six-bar planar mechanism to illustrate the influence of distributed inertia on internal force and deformation diagrams at low and high driving speeds.

Keywords: Finite element method; Rod mechanisms; Robotic manipulators; Stiffness analysis; Strength.

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1. Introduction

Research background: Rod mechanisms and robotic manipulators^[1] used in high-speed and high-precision applications must satisfy stringent strength and stiffness requirements.^[2] In practical design, internal force and displacement diagrams are evaluated over a motion cycle to identify critical sections and to size link cross-sections. FEM is the dominant framework for such calculations because it provides a direct relationship between nodal displacements and nodal forces through the global stiffness matrix.^[3]

Current research status: A large body of research addresses flexible manipulators and mechanisms using FEM-based link models and dynamic formulations, including experimental and numerical investigations of flexible-link manipulators, rotating beams, and load/trajectory limits under deformation.

In parallel, flexible multibody dynamics approaches such as the floating frame of reference formulation (FFRF) and the absolute nodal coordinate formulation (ANCF) have become standard tools for simulating flexible mechanisms undergoing large overall motion.^[4] These methods are highly effective for time-domain dynamic response, vibration and control studies.

Existing problems (gap): Despite these advances, a practical gap remains for design-oriented strength/stiffness assessment of moving rod mechanisms and manipulators.^[5] In many engineering-strength calculations, inertial effects are still introduced using resultant forces and moments applied at link centers of mass. This practice neglects the fact that both inertial and gravitational forces are distributed along the rod length and vary with configuration,^[6] which can significantly distort predicted bending moments, axial forces and deflections-especially at high speed. Moreover, while a distributed-inertia approach has been previously demonstrated for planar motion,^[7] a systematic extension to spatial rod models within an FEM-ready algorithm-including consistent load transformation and assembly procedures suitable for stiffness and stress evaluation-has not been sufficiently

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established in the literature.^[8]

The core contribution of this paper is the systematic generalization of the distributed-inertia concept from planar mechanisms to a formulation applicable to both planar and spatial rod models,^[9] implemented as an FEM-ready algorithm. The method:

1. performs kinematic analysis to obtain joint positions, link orientations, angular velocities/accelerations and pole accelerations;
2. constructs distributed inertial and weight load intensities along each rod in its local coordinate system;
3. converts these distributed loads into consistent equivalent nodal forces and moments;
4. assembles the global FEM equilibrium system to compute elastic displacements and to recover design-relevant diagrams $M_z(x)$, $N(x)$, $w_y(x)$, and $u(x)$ for each link.^[10]

Methodological distinction from ANCF/FFRF (why this is not redundant). While ANCF/FFRF are powerful for full flexible multibody simulation,^[11] they are not tailored to the common mechanism-design workflow where the primary deliverables are configuration-dependent internal force and displacement diagrams used for cross-section sizing and stress verification.^[12] In contrast, the proposed formulation integrates motion-induced distributed inertia directly into a structural FEM framework and provides closed-form consistent nodal load vectors and standard engineering outputs for each link and configuration. Therefore, the approach is computationally focused on stiffness/strength evaluation rather than full system-level dynamic simulation and control.^[13]

Finally, the method is demonstrated using a six-bar planar mechanism to verify the computational procedure and to show how distributed inertia alters internal force and deformation distributions under different driving speeds.^[14] The same algorithmic structure is directly transferable to spatial rod mechanisms by replacing the planar transformation matrices and element definitions with their spatial counterparts.^[15]

In summary, this work bridges kinematic inertia evaluation and design-oriented FEM strength/stiffness analysis by providing a unified distributed-inertia load construction and assembly procedure applicable to both planar and spatial rod mechanisms.^[16]

2. Materials and methods

The mechanism parameters are: link lengths $L_{AB}, L_{BC}, L_{CD}, L_{DG}, L_{CG}, L_{GE}$; coordinates of fixed hinges $A(X_A, Y_A)$ and $D(X_D, Y_D)$; ordinate of the slider guide Y_E ; and the prescribed input motion of the driving link Fig. 1a, given by the angular position $\theta_1 = 0.2\pi$ and angular velocity ω_1 .

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We perform the kinematic analysis in the global coordinate system (GCS) XAY using the method of.^[17] For each input position θ_1 , the procedure yields the coordinates of hinges B, C, G, E , the inclination angles of all links $\theta_1, \dots, \theta_6$, and the corresponding angular velocities ω_k and angular accelerations ε_k of the links, as well as the linear accelerations of hinges B, C, G, E . We now determine the distributed inertial and weight loads for the links assumed to have constant cross-sections in planar motion. $\theta_2, \theta_5, \theta_6$ are configuration-dependent through $B(\theta_1), C(\theta_1), G(\theta_1), E(\theta_1)$, which are obtained from the loop-closure equations:

Input crank gives point $B(\theta_1)$ with $A(X_A, Y_A)$ and L_{AB} :

$$X_B(\theta_1) = X_A + L_{AB} \cos \theta_1$$

$$Y_B(\theta_1) = Y_A + L_{AB} \sin \theta_1$$

Angle of link BC :

$$\theta_2(\theta_1) = \text{atan2}, (Y_C - Y_B(\theta_1), X_C - X_B(\theta_1))$$

Angle of link GE :

Since $E(X_E, Y_E)$ is the slider point (with $Y_E = \text{const}$):

$$\theta_5(\theta_1) = \text{atan2}, (Y_E - Y_G, X_E - X_G)$$

where X_E and G come from the position analysis at the same θ_1 .

Angle of link CG :

$$\theta_6(\theta_1) = \text{atan2}, (Y_G - Y_C, X_G - X_C)$$

$$\theta_2(\theta_1) = \text{atan2}, (Y_C - Y_A - L_{AB} \sin \theta_1, X_C - X_A - L_{AB} \cos \theta_1).$$

2.1 Distributed inertial and weight loads for a generic link

Consider link k Fig. 1b with local coordinate system $X'_k Y'_k$ attached to the link. The axis X'_k is directed from the pole P_k to the opposite joint, the link length is L_k , the cross-sectional area is A_k , and the specific weight of the material is γ_k . The inclination of the link with respect to the global X -axis is θ_k , while ω_k and ε_k are its angular velocity and angular acceleration, respectively. Let x'_k be the local coordinate Eq. (1) along X'_k (measured from the pole P_k). The projections of the resulting distributed load intensity (inertial + gravitational) per unit length, expressed in the local system, are taken as linear functions of x'_k .^[18]

$$q_y(x'_k) = a_{kq} + b_{kq} x'_k, \quad q_x(x'_k) = a_{kn} + b_{kn} x'_k \quad (1)$$

where q_y is the transverse component (across the link) and q_x is the longitudinal component (along the link).

For an arbitrary link k these coefficients are:

$$\begin{aligned} b_{kq} &= \frac{-\gamma_k A_k}{g} \varepsilon_k, \quad a_{kq} = -\gamma_k A_k \cos \theta_k - \frac{\gamma_k A_k}{g} \omega_{kp}^{y'k} \\ a_{kn} &= -\gamma_k A_k \sin \theta_k - \frac{\gamma_k A_k}{g} \omega_{kp}^{x'k}, \quad b_{kn} = \frac{-\gamma_k A_k}{g} \omega_k^2 \end{aligned} \quad (2)$$

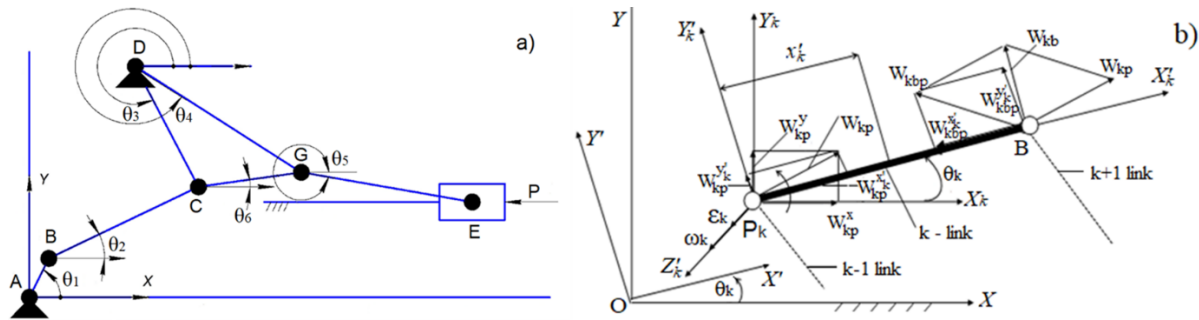


Fig. 1: Diagrams of a six-bar planar mechanism: a) geometry and b) the kinematic scheme for an arbitrary rod (“link k ”) with its local coordinate system.

where $g = 9.8 \text{ m/s}^2$ is the gravitational acceleration, $w_{kp}^{x'k}$ and $w_{kp}^{y'k}$ are the components of the acceleration of the pole P_k in the local system $X'_k Y'_k$.

The local components $w_{kp}^{x'k}$, $w_{kp}^{y'k}$ are obtained from the global acceleration components of the pole w_p^x , w_p^y by the standard rotation:

$$\begin{bmatrix} w_{kp}^{x'k} \\ w_{kp}^{y'k} \end{bmatrix} = \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} w_p^x \\ w_p^y \end{bmatrix} \quad (3)$$

Eq. (1) and Eq. (2) to Eq. (3) completely define the linear distribution of the combined inertial and weight load along each link.

2.2 Coefficients for the links of the six-bar mechanism

We now specify the poles P_k and the corresponding coefficients a_{kq} , b_{kq} , a_{kn} , b_{kn} for all links in Fig. 1a.

Link 1: AB

We choose the fixed point A as the pole, so $w_A^x = w_A^y = 0$. From Eq. (3):

$$\begin{bmatrix} w_{1A}^{x'1} \\ w_{1A}^{y'1} \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence:

$$\begin{aligned} a_{1q} &= -\gamma_1 A_1 \cos\theta_1, b_{1q} = -\frac{\gamma_1 A_1 \varepsilon_1}{g} = 0, a_{1n} = \\ &= -\gamma_1 A_1 \sin\theta_1; b_{1n} = -\frac{\gamma_1 A_1 \omega_1^2}{g} \end{aligned} \quad (4)$$

Link 2: BC

For link BC we take B as the pole. Its global acceleration components $(\ddot{X}_B, \ddot{Y}_B)$ are known from kinematic analysis. Transforming to the local system of link 2:

$$\begin{bmatrix} w_{2B}^{x'2} \\ w_{2B}^{y'2} \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \ddot{X}_B \\ \ddot{Y}_B \end{bmatrix} = \begin{bmatrix} \ddot{X}_B \cos\theta_2 + \ddot{Y}_B \sin\theta_2 \\ -\ddot{X}_B \sin\theta_2 + \ddot{Y}_B \cos\theta_2 \end{bmatrix}$$

Substituting into Eq. (2) gives:

$$a_{2q} = -\gamma_2 A_2 \cos\theta_2 - \frac{\gamma_2 A_2}{g} (-\ddot{X}_B \sin\theta_2 + \ddot{Y}_B \cos\theta_2),$$

$$\begin{aligned} b_{2q} &= -\frac{\gamma_2 A_2 \varepsilon_2}{g}, a_{2n} = -\gamma_2 A_2 \sin\theta_2 - \\ &= \frac{\gamma_2 A_2}{g} (\ddot{X}_B \sin\theta_2 + \ddot{Y}_B \cos\theta_2), b_{2n} = -\frac{\gamma_2 A_2 \omega_2^2}{g} \end{aligned} \quad (5)$$

Link 3: CD

For link 3 we take the fixed point D as the pole, so:

$$\begin{bmatrix} w_{3D}^{x'3} \\ w_{3D}^{y'3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore:

$$\begin{aligned} a_{3q} &= -\gamma_3 A_3 \cos\theta_3, b_{3q} = -\frac{\gamma_3 A_3 \varepsilon_3}{g}, a_{3n} = \\ &= -\gamma_3 A_3 \sin\theta_3, b_{3n} = -\frac{\gamma_3 A_3 \omega_3^2}{g} \end{aligned} \quad (6)$$

Link 4: DG

Link 4 is also hinged at D; its orientation is $\theta_3 + \lambda$, where λ is the constant offset angle between links 3 and 4. Since the pole is again the fixed point D, we have:

$$\begin{bmatrix} w_{4D}^{x'4} \\ w_{4D}^{y'4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} a_{4q} &= -\gamma_4 A_4 \cos(\theta_3 + \lambda), b_{4q} = -\frac{\gamma_4 A_4 \varepsilon_3}{g}, a_{4n} = \\ &= -\gamma_4 A_4 \sin(\theta_3 + \lambda), b_{4n} = -\frac{\gamma_4 A_4 \omega_3^2}{g} \end{aligned} \quad (7)$$

Link 5: GE

For link 5 we choose G as the pole. With the known global acceleration $(\ddot{X}_G, \ddot{Y}_G)$ we obtain:

$$\begin{bmatrix} w_{5G}^{x'5} \\ w_{5G}^{y'5} \end{bmatrix} = \begin{bmatrix} \cos\theta_5 & \sin\theta_5 \\ -\sin\theta_5 & \cos\theta_5 \end{bmatrix} \begin{bmatrix} \ddot{X}_G \\ \ddot{Y}_G \end{bmatrix} = \begin{bmatrix} \ddot{X}_G \cos\theta_5 + \ddot{Y}_G \sin\theta_5 \\ -\ddot{X}_G \sin\theta_5 + \ddot{Y}_G \cos\theta_5 \end{bmatrix}$$

Thus:

$$\begin{aligned} a_{5q} &= -\gamma_5 A_5 \cos\theta_5 - \frac{\gamma_5 A_5}{g} (-\ddot{X}_G \sin\theta_5 + \ddot{Y}_G \cos\theta_5), \\ b_{5q} &= -\frac{\gamma_5 A_5 \varepsilon_5}{g}, \end{aligned}$$

$$\begin{aligned} a_{5n} &= -\gamma_5 A_5 \sin\theta_5 - \frac{\gamma_5 A_5}{g} (\ddot{X}_G \cos\theta_5 + \ddot{Y}_G \sin\theta_5), \\ b_{5n} &= -\frac{\gamma_5 A_5 \omega_5^2}{g} \end{aligned} \quad (8)$$

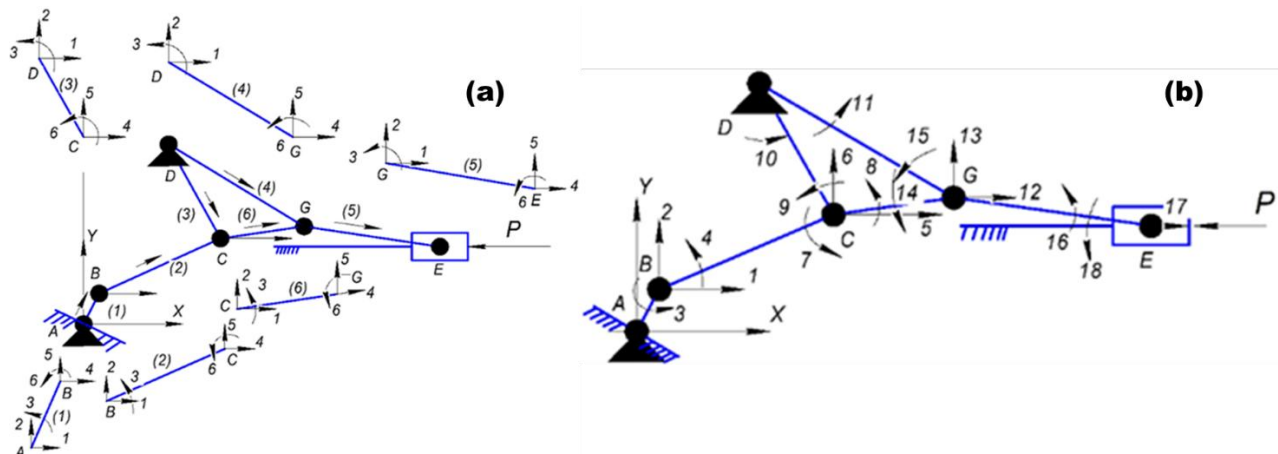


Fig. 2: a) the decomposition of the mechanism into six finite rod elements (FRE) and b) the correspondence between local and global degrees of freedom in the finite element model.

Table 1: Index correspondence between local and global DOF.

Element no.	Local DOF					
	Local DOF 1	Local DOF 2	Local DOF 3	Local DOF 4	Local DOF 5	Local DOF 6
Global DOF						
1	0	0	0	1	2	3
2	1	2	4	5	6	7
3	0	0	10	5	6	9
4	0	0	11	12	13	15
5	12	13	16	17	0	18
6	5	6	8	12	13	14

Link 6: CG

For link 6 we take C as the pole. With global acceleration $(\ddot{X}_C, \ddot{Y}_C)$:

$$\begin{bmatrix} w_{6C}^{x'} \\ w_{6C}^{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta_6 & \sin\theta_6 \\ -\sin\theta_6 & \cos\theta_6 \end{bmatrix} \begin{bmatrix} \ddot{X}_C \\ \ddot{Y}_C \end{bmatrix} = \begin{bmatrix} \ddot{X}_C \cos\theta_6 + \ddot{Y}_C \sin\theta_6 \\ -\ddot{X}_C \sin\theta_6 + \ddot{Y}_C \cos\theta_6 \end{bmatrix}$$

Hence:

$$\begin{aligned} a_{6q} &= -\gamma_6 A_6 \sin\theta_6 - \frac{\gamma_6 A_6}{g} (-\ddot{X}_C \sin\theta_6 + \ddot{Y}_C \cos\theta_6), \\ b_{6q} &= -\frac{\gamma_6 A_6 \varepsilon_6}{g}, \\ a_{6n} &= -\gamma_6 A_6 \sin\theta_6 - \frac{\gamma_6 A_6}{g} (\ddot{X}_C \cos\theta_6 + \ddot{Y}_C \sin\theta_6), \\ b_{6n} &= -\frac{\gamma_6 A_6 \omega_6^2}{g} \end{aligned} \quad (9)$$

Eq. (4), Eq. (5), Eq. (6), Eq. (7), Eq. (8) and Eq. (9) give the explicit expressions for the linear distributions $q_x(x'_k)$ and $q_y(x'_k)$ for all six links of the mechanism. These intensities are then used as equivalent distributed loads in the subsequent finite-element calculation of internal forces and elastic displacements.

3. Calculation model

The numerical data for the mechanism are taken from.^[19] The

fixed hinges have coordinates:

$$A(0, 0); D(0.25, 0.5)$$

The ordinate of the slider guide is:

$$Y_E = 0.2$$

and the constant offset angle between links 3 and 4 is:

$$\lambda = \pi/6$$

The link lengths are:

$$\begin{aligned} L_{AB} &= 0.095 \text{ m}, L_{BC} = 0.385 \text{ m}, L_{CD} = 0.095 \text{ m}, \\ L_{DG} &= 0.385 \text{ m}, L_{CG} = 0.075 \text{ m}, L_{GE} = 0.4 \text{ m} \end{aligned}$$

The specific weights of the materials are taken identical for all links:

$$\gamma_1 = \gamma_2 = \dots = \gamma_6 = 78 * 10^3 \text{ N/m}^3$$

and the gravitational acceleration is $g = 9.8 \text{ m/s}^2$.

The Young's moduli of the links are denoted by $E_1, \dots, E_6$ (for the numerical example they are taken according to),^[20] and an external force:

$$E_1 = E_2 = \dots = E_6 = 2 * 10^{11} \text{ N/m}^2, P = -104 \text{ N}$$

acts at point E along the slider direction (negative sign corresponds to the direction shown in Fig. 1a). The cross-sectional areas of the links are:

$$A_1 = 0.0001663533701 \text{ m}^2, A_2 = 0.0003513416567 \text{ m}^2,$$

$$A_3 = 0.0002907470432 \text{ m}^2, A_4 = 0.00007765800412 \text{ m}^2,$$

$$A_5 = 0.0002121774778 \text{ m}^2, A_6 = 0.0001790756909 \text{ m}^2,$$

Two characteristic regimes of the driving crank AB are considered:

$$\omega_1 = 1 \text{ rad/s to } \omega_1 = 100 \text{ rad/s}$$

For the finite element idealization, we divide the mechanism into six finite rod elements (FRE) Fig. 2a. Each two-joint link is represented by one FRE, which simplifies the numerical implementation and ensures direct correspondence with the model used in.^[21] The composite branch CDG is therefore modeled by three elements connected at joints C, D and G : elements CD, CG , and DG .

In general, any link could be subdivided into an arbitrary number of FRE, which is fully consistent with the finite element method. In this work, the mechanism is represented by six beam-type elements:

$$AB, BC, CD, CG, DG, GE$$

The mass of the slider is neglected, as in.^[13]

4. Finite element modeling

Fig. 2b shows the numbering of the generalized coordinates (degrees of freedom, DOF) associated with the elastic displacements and rotations in the finite element model of the mechanism. Each node of a planar beam element has three DOF (axial displacement, transverse displacement, and rotation), but some of them are fixed due to support and kinematic constraints. After applying the boundary conditions, the model has 18 active DOF.

To assemble the global stiffness matrix and global load vector, we introduce an index correspondence matrix that links the local DOF of each FRE to the global DOF numbers. Zero entries indicate degrees of freedom that are constrained (fixed). The mapping is given in Table 1. In this Table, each row corresponds to one finite rod element. The entries specify which global DOF numbers are associated with the six local DOF of that element. This mapping is used in the standard way to assemble the element stiffness matrices and equivalent nodal load vectors into the global system of equations for the mechanism.

4.1 Global stiffness matrix and nodal load vector

Using the DOF numbering in Fig. 2b and Table 1, the assembled global stiffness matrix of the mechanism has dimension [Supplementary Information file \(see Appendix A\)](#):

$$K \in \mathbb{R}^{18 \times 18}$$

As an example, the first diagonal Eq. (10) term K_{11} (corresponding to the horizontal displacement of node B) is obtained by summing the contributions from elements AB and BC :

$$K_{11} = \left(\frac{E_1 F_1}{L_1} \cos^2 \varphi_1 + \frac{12 E_1 J_1}{L_1^3} \sin^2 \varphi_1 \right) + \left(\frac{E_2 F_2}{L_2} \cos^2 \varphi_2 + \frac{12 E_2 J_2}{L_2^3} \sin^2 \varphi_2 \right) \quad (10)$$

where E_k is the Young's modulus, A_k the cross-sectional area, J_k the second moment of area, L_k the length, and φ_k the inclination angle of element k with respect to the global X -axis.

Another example is the coupling term between DOF 1 and DOF 3, which for element AB reads:

$$K_{13} = \frac{6 E_1 J_1}{L_1^2} \sin \varphi_1 \quad (11)$$

Near the right end of the mechanism Eq. (11) and Eq. (12), typical entries associated with element GE are:

$$K_{18,15} = 0, K_{18,16} = \frac{2 E_6 J_6}{L_6}, K_{18,17} = \frac{6 E_6 J_6}{L_6^2} \sin \theta_6, \quad K_{18,18} = \frac{4 E_5 J_5}{L_5} \quad (12)$$

where and below only representative coefficients are shown for brevity; all other entries of $\mathbf{K}$ are obtained in the standard way from the element stiffness matrices and the index correspondence matrix.

If only the external force at the slider is taken into account and distributed inertia is neglected Eq. (13), the global nodal force vector has the form:

$$\mathbf{F}^0 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -104, 0]^T \quad (13)$$

where the nonzero entry -104 N corresponds to the horizontal DOF at node E (DOF 17 in Fig. 2b).

The basic finite element equilibrium Eq. (14) are then:

$$\mathbf{K} \mathbf{u} = \mathbf{F}^0 \quad (14)$$

where $\mathbf{u}$ is the vector of global elastic displacements in the GCS. This system is solved using any suitable numerical method (e.g. direct Gaussian elimination or an iterative scheme).^[22]

4.2. Inclusion of distributed inertial loads

When the distributed inertia of the links in their planar motion is taken into account (for the present flat mechanism model), the same global stiffness matrix $\mathbf{K}$ is used, but the right-hand side is modified. Following,^[23] the global load vector Eq. (15) can be written as:

$$\mathbf{F} = \begin{bmatrix} f_1 \\ \vdots \\ f_8 \end{bmatrix} = \mathbf{P} - \sum_{(ij)} \mathbf{C}_i^{ji} \mathbf{Q}^{ji} + \mathbf{C}_i^{ij} \mathbf{Q}^{ij} \quad (15)$$

where $\mathbf{P}$ is the vector of external applied loads (including P at the slider), $\mathbf{C}_i^{ji}$ and $\mathbf{C}_i^{ij}$ are direction-cosine matrices

transforming local element forces to the global coordinate system at nodes i and j , $\mathbf{Q}^{ji}$ and $\mathbf{Q}^{ij}$ are 6×1 subvectors of equivalent nodal forces and moments at the ends j and i of the finite rod element (ij) in its local coordinate system (LCS). The subvectors $\mathbf{Q}^{ji}$ and $\mathbf{Q}^{ij}$ are computed using the previously derived linear distributions $q_x(x'_k)$ and $q_y(x'_k)$ along each link, i.e. they account for the longitudinal and transverse distributed inertial and gravitational loads acting along the degrees of freedom of element (ij) in its LCS.^[24] Their explicit component expressions follow directly from integrating the linearly distributed loads over the element length and are not repeated here for brevity.

With this modified right-hand side, the governing FEM system Eq. (16):

$$\mathbf{K}\mathbf{u} = \mathbf{F} \quad (16)$$

Provides the global elastic displacements that already include the effect of distributed inertia and weight of all mechanism links.

4.3 Equivalent nodal forces from distributed inertia

For each finite rod element (ij) we introduce a local coordinate system $x'y'$ aligned with the element axis. The transformation from local to global coordinates Eq. (17) is performed using the standard direction-cosine matrix. Following,^[25] we write the 3×3 matrix:

$$\mathbf{T} = \begin{bmatrix} L & -m & 0 \\ m & L & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (17)$$

where $L = \cos\varphi$ and $m = \sin\varphi$ are the direction cosines of the link with respect to the global X -axis (φ is the inclination angle of the element).

For a beam element with three DOF at each node (axial displacement, transverse displacement, rotation), the full 6×6 transformation matrix is block-diagonal:

$$[\mathbf{c}_i^{ij}] = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix}, [\mathbf{c}_i^{ji}] = [\mathbf{c}_i^{ij}] \quad (18)$$

Because the element is rectilinear and the same rotation applies at both ends. The equivalent nodal forces and moments at the ends of element (ij) in its local coordinate system Eq. (18) are collected in two subvectors $\mathbf{Q}^{ij}$ and $\mathbf{Q}^{ji}$. For a general link k of length L , with distributed inertial and gravitational load intensities:

$$q_x(x') = a_{kn} + b_{kn}x', q_y(x') = a_{kq} + b_{kq}x', 0 \leq x' \leq L$$

Integration along the element yields the local nodal resultants:^[12]

$$\mathbf{Q}^{ij} = \begin{Bmatrix} -\frac{a_{kn}L}{2} - \frac{b_{kn}L^2}{2} \\ -\frac{a_{kq}L}{2} - \frac{3b_{kq}L^2}{20} \\ -\frac{a_{kq}L^2}{12} - \frac{b_{kq}L^3}{30} \end{Bmatrix}, \mathbf{Q}^{ji} = \begin{Bmatrix} -\frac{a_{kn}L}{2} - \frac{b_{kn}L^2}{2} \\ -\frac{a_{kq}L}{2} - \frac{3b_{kq}L^2}{20} \\ -\frac{a_{kq}L^2}{12} - \frac{b_{kq}L^3}{30} \end{Bmatrix} \quad (19)$$

For this particular loading pattern, the end forces are symmetric; for more general cases $\mathbf{Q}^{ij}$ and $\mathbf{Q}^{ji}$ would differ Eq. (19).

The corresponding global equivalent nodal loads are obtained by:

$$\mathbf{F}_{glob}^{(ij)} = [\mathbf{c}_i^{ij}] \mathbf{Q}^{ij}, \mathbf{F}_{glob}^{(ji)} = [\mathbf{c}_i^{ji}] \mathbf{Q}^{ji} \quad (20)$$

These contributions enter the right-hand side of the global FEM Eq. (20) as:

$$-\sum_{j=1}^{i-1} [\mathbf{c}_i^{ji}] \mathbf{Q}^{ji} - \sum_{j=i+1}^{N_r} [\mathbf{c}_i^{ij}] \mathbf{Q}^{ij} \quad (21)$$

where N_r is the total number of rod elements. Eq. (21) expresses the resultant inertial influence of all adjacent elements on node i in global coordinates.

To clarify the mapping between local and global DOF, in the “formal” matrices below we show, in the last column, on the left the local DOF numbers (1, ..., 6) of each FRE and, on the right, the corresponding global DOF numbers of the mechanism model.

4.4 Example: element 1 (link AB)

For element 1, only the right node (at B) contributes inertial loads; at the left node A the acceleration is zero (fixed support). Therefore:

$$-\sum_{j=i+1}^{N_r} [\mathbf{c}_i^{ij}] \mathbf{Q}^{ij} = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 - DOF \rightarrow 0 \\ 2 - DOF \rightarrow 0 \\ 3 - DOF \rightarrow 0 \end{bmatrix}$$

At the right end of element 1 (node B) the equivalent local nodal results are:

$$\mathbf{Q}^{ji} = \begin{Bmatrix} \frac{a_{n1}L_1}{2} + \frac{b_{n1}L_1^2}{2} \\ \frac{a_{q1}L_1}{2} + \frac{7b_{q1}L_1^2}{20} \\ -\frac{a_{q1}L_1^2}{12} + \frac{b_{q1}L_1^3}{30} \end{Bmatrix}$$

So that:

$$-\sum_{j=1}^{i-1} [\mathbf{c}_i^{ji}] \mathbf{Q}^{ji} = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \mathbf{Q}^{ji} = \begin{Bmatrix} c\theta_1 \left(\frac{a_{n1}L_1}{2} + \frac{b_{n1}L_1^2}{2} \right) - s\theta_1 \left(\frac{a_{q1}L_1}{2} + \frac{7b_{q1}L_1^2}{20} \right) \\ s\theta_1 \left(\frac{a_{n1}L_1}{2} + \frac{b_{n1}L_1^2}{2} \right) + c\theta_1 \left(\frac{a_{q1}L_1}{2} + \frac{7b_{q1}L_1^2}{20} \right) \\ -\frac{a_{q1}L_1^2}{12} - \frac{b_{q1}L_1^3}{30} \end{Bmatrix} \Rightarrow \begin{bmatrix} 4 - DOF \rightarrow 1 \\ 5 - DOF \rightarrow 2 \\ 6 - DOF \rightarrow 3 \end{bmatrix} \quad (22)$$

where the triplets $[1 \ 2 \ 3]^T$ and $[4 \ 5 \ 6]^T$ (left side of the last “formal” columns) are the local DOF numbers of the element, while the numbers $[0 \ 0 \ 0]^T$ and $[1 \ 2 \ 3]^T$ on the right are the corresponding global DOF numbers of the whole model.

For elements 2–6 the vectors $\mathbf{Q}^{ij}$, $\mathbf{Q}^{ji}$ and their transformations into the global system are obtained in exactly

the same way, using the appropriate coefficients $a_{kn}, b_{kn}, a_{kq}, b_{kq}$, lengths L_k , and orientation angles θ_k . For brevity, the explicit formulas for these elements are omitted, but they follow directly from Eq. (17) to Eq. (22).

4.5. Inertial nodal loads for individual finite rod elements

Using the general expressions Eq. (17) to Eq. (22), the equivalent nodal forces and moments from the distributed inertial and gravitational loads are obtained for each finite rod element (FRE) and then transformed into the global coordinate system. Below we list the final global nodal load vectors for FRE 2–6, together with the correspondence between local and global degrees of freedom (DOF).

For compactness, we use the notation:

$$c\theta_k = \cos\theta_k, s\theta_k = \sin\theta_k$$

FRE 2 (link BC)

Node B (local DOF 1-3, global DOF 1, 2, 4):

$$\mathbf{F}_B^{(2)} = \begin{Bmatrix} c\theta_2 \left(\frac{a_{n2}L_2}{2} + \frac{b_{n2}L_2^2}{2} \right) - s\theta_2 \left(\frac{a_{q2}L_2}{2} + \frac{3b_{q2}L_2^2}{20} \right) \\ s\theta_2 \left(\frac{a_{n2}L_2}{2} + \frac{b_{n2}L_2^2}{2} \right) + c\theta_2 \left(\frac{a_{q2}L_2}{2} + \frac{3b_{q2}L_2^2}{20} \right) \\ \frac{a_{q2}L_2^2}{12} - \frac{b_{q2}L_2^3}{30} \end{Bmatrix} \Rightarrow [\text{local (1, 2, 3)} \rightarrow \text{global (1, 2, 4)}]$$

Node C (local DOF 4-6, global DOF 5, 6, 7):

$$\mathbf{F}_C^{(2)} = \begin{Bmatrix} c\theta_2 \left(\frac{a_{n2}L_2}{2} + \frac{b_{n2}L_2^2}{2} \right) - s\theta_2 \left(\frac{a_{q2}L_2}{2} + \frac{7b_{q2}L_2^2}{20} \right) \\ s\theta_2 \left(\frac{a_{n2}L_2}{2} + \frac{b_{n2}L_2^2}{2} \right) + c\theta_2 \left(\frac{a_{q2}L_2}{2} + \frac{7b_{q2}L_2^2}{20} \right) \\ -\frac{a_{q2}L_2^2}{12} - \frac{b_{q2}L_2^3}{30} \end{Bmatrix} \Rightarrow [\text{local (4, 5, 6)} \rightarrow \text{global (5, 6, 7)}]$$

FRE 3 (link CD)

Node C (local DOF 1-3, global DOF 0, 0, 10):

$$\text{For FRE 3: } \mathbf{F}_C^{(3)} = \begin{Bmatrix} 0 \\ 0 \\ \frac{a_{q3}L_3^2}{12} + \frac{b_{q3}L_3^3}{30} \end{Bmatrix} \Rightarrow [\text{local (1, 2, 3)} \rightarrow \text{global (0, 0, 10)}]$$

Node D (local DOF 4-6, global DOF 5, 6, 9):

$$\mathbf{F}_D^{(3)} = \begin{Bmatrix} c\theta_3 \left(\frac{a_{n3}L_3}{2} + \frac{b_{n3}L_3^2}{2} \right) - s\theta_3 \left(\frac{a_{q3}L_3}{2} + \frac{7b_{q3}L_3^2}{20} \right) \\ s\theta_3 \left(\frac{a_{n3}L_3}{2} + \frac{b_{n3}L_3^2}{2} \right) + c\theta_3 \left(\frac{a_{q3}L_3}{2} + \frac{7b_{q3}L_3^2}{20} \right) \\ -\frac{a_{q3}L_3^2}{12} - \frac{b_{q3}L_3^3}{30} \end{Bmatrix} \Rightarrow [\text{local (4, 5, 6)} \rightarrow \text{global (5, 6, 9)}]$$

For FRE 4 (link DG, angle $\theta_3 + \alpha$)

Node D (local DOF 1-3, global DOF 0, 0, 11):

$$\mathbf{F}_D^{(4)} = \begin{Bmatrix} 0 \\ 0 \\ \frac{a_{q4}L_4^2}{12} + \frac{b_{q4}L_4^3}{30} \end{Bmatrix} \Rightarrow [\text{local (1, 2, 3)} \rightarrow \text{global (0, 0, 11)}]$$

Node G (local DOF 4-6, global DOF 12, 13, 15):

$$\mathbf{F}_G^{(4)} = \begin{Bmatrix} c(\theta_3 + \alpha) \left(\frac{a_{n4}L_4}{2} + \frac{b_{n4}L_4^2}{2} \right) - s(\theta_3 + \alpha) \left(\frac{a_{q4}L_4}{2} + \frac{7b_{q4}L_4^2}{20} \right) \\ s(\theta_3 + \alpha) \left(\frac{a_{n4}L_4}{2} + \frac{b_{n4}L_4^2}{2} \right) + c(\theta_3 + \alpha) \left(\frac{a_{q4}L_4}{2} + \frac{7b_{q4}L_4^2}{20} \right) \\ -\frac{a_{q4}L_4^2}{12} - \frac{b_{q4}L_4^3}{30} \end{Bmatrix} \Rightarrow [\text{local (4, 5, 6)} \rightarrow \text{global (12, 13, 15)}]$$

For FRE 5 (link GE)

Node G (local DOF 1-3, global DOF 12, 13, 16):

$$\mathbf{F}_G^{(5)} = \begin{Bmatrix} c\theta_5 \left(\frac{a_{n5}L_5}{2} + \frac{b_{n5}L_5^2}{2} \right) - s\theta_5 \left(\frac{a_{q5}L_5}{2} + \frac{7b_{q5}L_5^2}{20} \right) \\ s\theta_5 \left(\frac{a_{n5}L_5}{2} + \frac{b_{n5}L_5^2}{2} \right) + c\theta_5 \left(\frac{a_{q5}L_5}{2} + \frac{7b_{q5}L_5^2}{20} \right) \\ \frac{a_{q5}L_5^2}{12} + \frac{b_{q5}L_5^3}{30} \end{Bmatrix} \Rightarrow [\text{local (1, 2, 3)} \rightarrow \text{global (12, 13, 16)}]$$

Node E (local DOF 4-6, global DOF 17, 0, 18):

$$\mathbf{F}_E^{(5)} = \begin{Bmatrix} c\theta_5 \left(\frac{a_{n5}L_5}{2} + \frac{b_{n5}L_5^2}{2} \right) - s\theta_5 \left(\frac{a_{q5}L_5}{2} + \frac{7b_{q5}L_5^2}{20} \right) \\ s\theta_5 \left(\frac{a_{n5}L_5}{2} + \frac{b_{n5}L_5^2}{2} \right) + c\theta_5 \left(\frac{a_{q5}L_5}{2} + \frac{7b_{q5}L_5^2}{20} \right) \\ -\frac{a_{q5}L_5^2}{12} - \frac{b_{q5}L_5^3}{30} \end{Bmatrix} \Rightarrow [\text{local (4, 5, 6)} \rightarrow \text{global (17, 0, 18)}]$$

For FRE 6 (link CG)

Node C (local DOF 1-3, global DOF 5, 6, 8):

$$\mathbf{F}_C^{(6)} = \begin{Bmatrix} c\theta_6 \left(\frac{a_{n6}L_6}{2} + \frac{b_{n6}L_6^2}{2} \right) - s\theta_6 \left(\frac{a_{q6}L_6}{2} + \frac{3b_{q6}L_6^2}{20} \right) \\ s\theta_6 \left(\frac{a_{n6}L_6}{2} + \frac{b_{n6}L_6^2}{2} \right) + c\theta_6 \left(\frac{a_{q6}L_6}{2} + \frac{3b_{q6}L_6^2}{20} \right) \\ \frac{a_{q6}L_6^2}{12} + \frac{b_{q6}L_6^3}{30} \end{Bmatrix} \Rightarrow [\text{local (1, 2, 3)} \rightarrow \text{global (5, 6, 8)}]$$

Node G (local DOF 4-6, global DOF 12, 13, 14):

$$\mathbf{F}_G^{(6)} = \begin{Bmatrix} c\theta_6 \left(\frac{a_{n6}L_6}{2} + \frac{b_{n6}L_6^2}{2} \right) - s\theta_6 \left(\frac{a_{q6}L_6}{2} + \frac{3b_{q6}L_6^2}{20} \right) \\ s\theta_6 \left(\frac{a_{n6}L_6}{2} + \frac{b_{n6}L_6^2}{2} \right) + c\theta_6 \left(\frac{a_{q6}L_6}{2} + \frac{3b_{q6}L_6^2}{20} \right) \\ -\frac{a_{q6}L_6^2}{12} - \frac{b_{q6}L_6^3}{30} \end{Bmatrix} \Rightarrow [\text{local (4, 5, 6)} \rightarrow \text{global (12, 13, 14)}]$$

All these vectors are added, with appropriate sign, to the

global right-hand side of the FEM system Eq. (16). They represent the equivalent nodal forces and moments due to the distributed inertial and gravitational loads of each link, already transformed into the global coordinate system and matched to the global DOF numbering.

4.6 Assembly of the global load vector and displacement transformation

Collecting all inertial components obtained for the individual FRE and arranging them according to the global DOF numbering (Fig. 2b, Table 1), the global right-hand side of the FEM system can be written as:

$$\{F\} = [f_1 \cdots f_{18}]^T$$

As an example, the first, seventeenth and eighteenth components are:

$$\begin{aligned} f_1 &= \cos\theta_1 \left(\frac{a_{1n}L_1}{2} + \frac{b_{1n}L_1^2}{2} \right) - \sin\theta_1 \left(\frac{a_{1q}L_1}{2} + \frac{7b_{1q}L_1^2}{20} \right) + \\ &\quad \cos\theta_2 \left(\frac{a_{n2}L_2}{2} + \frac{b_{n2}L_2^2}{2} \right) - \sin\theta_2 \left(\frac{a_{q2}L_2}{2} + \frac{3b_{q2}L_2^2}{20} \right) \\ f_{17} &= \cos\theta_5 \left(\frac{a_{n5}L_5}{2} + \frac{b_{n5}L_5^2}{2} \right) - \sin\theta_5 \left(\frac{a_{q5}L_5}{2} + \frac{3b_{q5}L_5^2}{20} \right) - \\ &\quad 104 \\ f_{18} &= -\frac{a_{q5}L_5^2}{12} - \frac{b_{q5}L_5^3}{30}. \end{aligned}$$

Thus, the governing FEM equilibrium equations with distributed inertia included are:

$$[K]\{\Delta\} = \{F\} \quad (23)$$

where $\{\Delta\}$ is the vector of global nodal elastic displacements (DOF 1–18). Solving Eq. (23) with any suitable numerical method^[17] yields the global displacement field of the mechanism.

4.7 Transformation of displacements from GCS to local element systems

For stress and internal-force evaluation it is convenient to express the element deformations in the local coordinate system rigidly attached to each FRE. For element $e = (ij)$ this is done by the standard transformation:

$$[\Delta^e] = [C_i^{ij}] \cdot \{\Delta^*\} \quad (24)$$

where $\{\Delta^*\}$ is the vector of global DOF of that element (extracted from $\{\Delta\}$ using Table 1), $[\Delta^e]$ is the vector of local nodal displacements in the element's LCS, $[C_i^{ij}]$ is the 6×6 transformation matrix defined in Eq. (24).

Using the shorthand $c\theta = \cos\theta$, $s\theta = \sin\theta$, the transformations for all six FRE are:

FRE 1 (link AB)

Global DOF vector for the element:

$$\{\Delta_1^*\} = \{0, 0, 0, u_1, u_2, u_3\}^T$$

Local displacements:

$$\{\Delta^1\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_1 \cdot c\theta_1 + u_2 \cdot s\theta_1 \\ -u_1 \cdot s\theta_1 + u_2 \cdot c\theta_1 \\ u_3 \end{bmatrix} \quad (25)$$

FRE 2 (link BC)

$$\begin{aligned} \{\Delta_2^*\} &= \{u_1, u_2, u_4, u_5, u_6, u_7\}^T \\ \{\Delta^2\} &= \begin{bmatrix} u_1 \cdot c\theta_2 + u_2 \cdot s\theta_2 \\ -u_1 \cdot s\theta_2 + u_2 \cdot c\theta_2 \\ u_4 \\ u_5 \cdot c\theta_2 + u_6 \cdot s\theta_2 \\ -u_5 \cdot s\theta_2 + u_6 \cdot c\theta_2 \\ u_7 \end{bmatrix} \end{aligned} \quad (26)$$

FRE 3 (link CD)

$$\begin{aligned} \{\Delta_3^*\} &= \{0, 0, u_{10}, u_5, u_6, u_9\}^T \\ \{\Delta^3\} &= \begin{bmatrix} 0 \\ 0 \\ u_{10} \\ u_5 \cdot c\theta_3 + u_6 \cdot s\theta_3 \\ -u_5 \cdot s\theta_3 + u_6 \cdot c\theta_3 \\ u_9 \end{bmatrix} \end{aligned} \quad (27)$$

FRE 4 (link DG, angle $\theta_3 + \alpha$)

$$\begin{aligned} \{\Delta_4^*\} &= \{0, 0, u_{11}, u_{12}, u_{13}, u_{15}\}^T \\ \{\Delta^4\} &= \begin{bmatrix} 0 \\ 0 \\ u_{11} \\ u_{12} \cdot c(\theta_3 + \alpha) + u_{13} \cdot s(\theta_3 + \alpha) \\ -u_{12} \cdot s(\theta_3 + \alpha) + u_{13} \cdot c(\theta_3 + \alpha) \\ u_{15} \end{bmatrix} \end{aligned} \quad (28)$$

FRE 5 (link GE)

$$\begin{aligned} \{\Delta_5^*\} &= \{u_{12}, u_{13}, u_{16}, u_{17}, 0, u_{18}\}^T \\ \{\Delta^5\} &= \begin{bmatrix} u_{12} \cdot c\theta_5 + u_{13} \cdot s\theta_5 \\ -u_{12} \cdot s\theta_5 + u_{13} \cdot c\theta_5 \\ u_{16} \\ u_{17} \cdot c\theta_5 \\ -u_{17} \cdot s\theta_5 \\ u_{18} \end{bmatrix} \end{aligned}$$

FRE 6 (link CG)

$$\begin{aligned} \{\Delta_6^*\} &= \{u_5, u_6, u_8, u_{12}, u_{13}, u_{14}\}^T \\ \{\Delta^6\} &= \begin{bmatrix} u_5 \cdot c\theta_6 + u_6 \cdot s\theta_6 \\ -u_5 \cdot s\theta_6 + u_6 \cdot c\theta_6 \\ u_8 \\ u_{12} \cdot c\theta_6 + u_{13} \cdot s\theta_6 \\ -u_{12} \cdot s\theta_6 + u_{13} \cdot c\theta_6 \\ u_{14} \end{bmatrix} \end{aligned} \quad (30)$$

These local displacement vectors $\{\Delta^e\}$ are then used to compute internal forces, bending moments and axial forces in

each link according to the standard beam-element relations Eq. (25), Eq. (26), Eq. (27), Eq. (28), Eq. (29) and Eq. (30), now under the combined action of the external load at the slider and the distributed inertial and gravitational loads of the moving mechanism.

4.8 Element reactions and internal forces in the local system

For each planar finite rod element (FRE) in its local coordinate system (LCS), the basic FEM relation is satisfied:^[17]

$$[K^e]\{\Delta^e\} = \{R^e\} \quad (31)$$

where $[K^e]$ is the 6×6 element stiffness matrix in the LCS, $\{\Delta^e\}$ is the 6×1 vector of local nodal elastic displacements, $\{R^e\}$ is the 6×1 vector of nodal reactions (forces and bending moments) of the FRE in the LCS.

The i -th component of the reaction vector for element e is obtained from:

$$\{R_i^e\} = K_{i1}^e \cdot \Delta_1^e + K_{i2}^e \cdot \Delta_2^e + \dots + K_{i6}^e \cdot \Delta_6^e, \quad (i = 1, \dots, 6) \quad (32)$$

where K_{ij}^e are the entries of $[K^e]$ and Δ_j^e are the local nodal displacement at the element ends. After transforming the global displacements $\{\Delta\}$ to local element displacements $\{\Delta^e\}$ as described in Section 4.7, we compute the nodal reaction forces and moments $\{R^e\}$ for each element $e = 1, \dots, 6$ from Eq. (31) and Eq. (32).

Using these nodal quantities and the distributed inertial and gravitational load coefficients a_n , b_n , a_q^y , b_q^y introduced earlier, we can write the longitudinal displacement, transverse displacement, rotation, and internal forces at any intermediate point x of an element $0 \leq x \leq l$ in its LCS,^[18] taking into account the distributed inertia. Let l be the element length, u_0 , u_l the longitudinal displacements at the left and right nodes, w_{y0} , w_{yl} the transverse nodal displacements, and φ_{z0} , φ_{zl} the nodal rotations about the z -axis. Denote by E the Young's modulus, F the cross-sectional area, and J_z the second moment of area about the z -axis.

4.9 Displacements and rotation along the element

Longitudinal displacement $u(x)$:

$$u(x) = u_0 + (u_l - u_0) \frac{x}{l} + a_n \frac{x(x-l)}{2EF} - b_n \frac{x(x^2-l^2)}{6EF} \quad (33)$$

Transverse displacement $w_y(x)$:

$$w_y(x) = (1 + 2\frac{x^3}{l^3} - 3\frac{x^2}{l^2})w_{y0} + \frac{x^2}{l^2}(3 - 2\frac{x}{l})w_{yl} + x(1 - \frac{x}{l})^2\varphi_{z0} + x(\frac{x}{l} - 1)\frac{x^2}{l}\varphi_{zl} + a_q^y \frac{x^2(1-x)^2}{24EJ_z} + b_q^y \frac{x^2(2l^3 - 3l^2x + x^3)}{120EJ_z} \quad (34)$$

$$\varphi_z(x) = 6\frac{x}{l^2}(\frac{x}{l} - 1)w_{y0} + 6\frac{x}{l^2}(1 - \frac{x}{l})w_{yl} + (3\frac{x^2}{l^2} - 4\frac{x}{l} + 1)\varphi_{z0} + \frac{x}{l}(3\frac{x}{l} - 2)\varphi_{zl} + \quad (35)$$

$$+ a_k \frac{x(2l^2 - 6xl + 3x^2)}{24EJ_z} + b_k \frac{x(5x^3 - 9xl^2 + 4l^3)}{120EJ_z}$$

where a_n , b_n correspond to the axial distributed load coefficients, a_q^y , b_q^y and a_k , b_k are the coefficients associated with the transverse distributed load and its inertial part Eq. (33), Eq. (34) and Eq. (35), as obtained in Sect. 4.

4.10 Internal axial force and bending moment

The axial force $N(x)$ in the element section is:

$$N(x) = EF \left\{ \frac{(u_l - u_0)}{l} + a_n \left(\frac{l}{2} - x \right) + b_n \frac{(l^2 - 3x^2)}{6} \right\} \quad (36)$$

and the bending moment $M_z(x)$ about the z -axis is:

$$M_z(x) = -EJ_z \left[\left(\frac{12x}{l^3} - \frac{6}{l^2} \right) w_{y0} + \left(\frac{6}{l^2} - \frac{12x}{l^3} \right) w_{yl} + \left(\frac{6x}{l^2} - \frac{4}{l} \right) \varphi_{y0} + \left(\frac{6x}{l^2} - \frac{2}{l} \right) \varphi_{yl} \right] - a_k^y \frac{(2l^2 - 12lx + 9x^2)}{24EJ_z} + b_k^y \frac{(4l^3 - 18l^2x + 20x^3)}{120EJ_z} \quad (37)$$

All quantities in Eq. (36) and Eq. (37) are expressed in the local coordinate system of the FRE. These relations are used to construct diagrams of longitudinal force $N(x)$, bending moment $M_z(x)$, and elastic displacements $u(x)$, $w_y(x)$ for each element, fully accounting for the distributed inertial and gravitational effects in the moving mechanism.

5. Results and discussion

Figs. 3–6 present the distributions of internal forces and elastic displacements obtained by the proposed finite rod element formulation with distributed inertial loading. The diagrams are plotted directly on the mechanism geometry for selected input positions θ_1 and two driving speeds, $\omega_1 = 1$ rad/s $\omega_1 = 100$ rad/s. Because the magnitudes differ strongly between the low- and high-speed cases, the diagrams are displayed with scale factors (division for forces/moments and multiplication for displacements), as indicated in each caption. Figs. 3-6 are presented primarily for qualitative visualization of the distribution and location of extrema. Because the diagrams are plotted with indicated scale factors $M_z/10^3$, $M_z/10^6$, $N/10^5$, $N/10^6$, $10w_y$, $3000u$, $1000u$, approximate values can be read from the plots; however, accurate peak magnitudes and critical locations are obtained from the numerical post-processing of $M_z(t)$, $N(t)$, $w_y(t)$ and $u(t)$ along each element.

Fig. 3a shows that the bending response is highly configuration-dependent and is amplified significantly by speed. At $\theta_1 = 210^\circ$ the mechanism is in a posture where several links experience strong transverse inertial components, producing large bending moments in the central region of the structure, particularly around joints C and G, where multiple elements meet and load paths change direction. These joints act as bending “hot spots” because: (i) they transmit forces between multiple members, and (ii) small variations in kinematic acceleration of one link are redistributed into transverse loading of adjacent links. A key observation from Fig. 3b is the dramatic growth of bending moments when ω_1

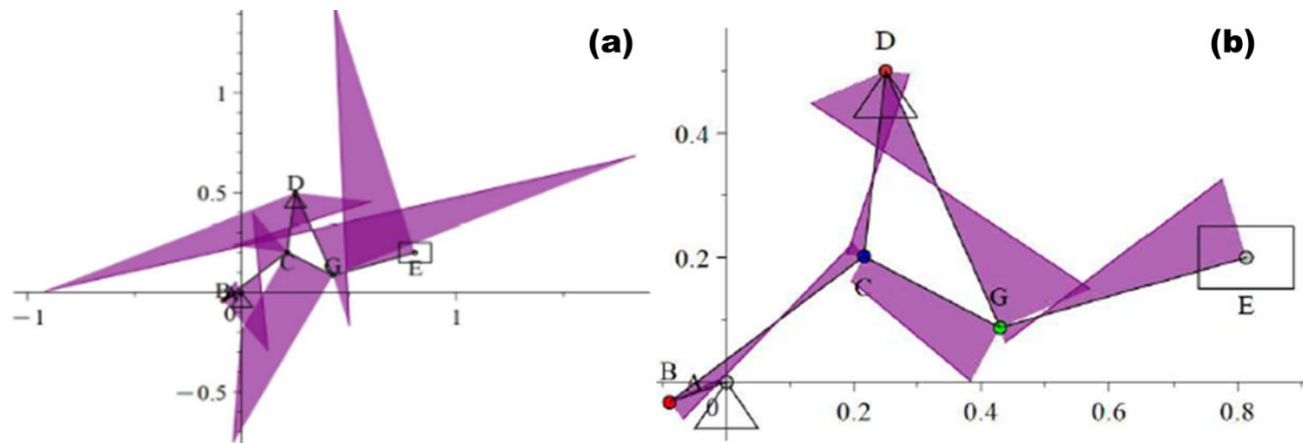


Fig. 3: Bending moment diagrams $M_z(x)$ of the mechanism at $\theta_1 = 210^\circ$: a) $\omega_1 = 1$ rad/s, M_z values scaled by 10^3 ; b) $\omega_1 = 100$ rad/s, M_z values scaled by 10^6 .

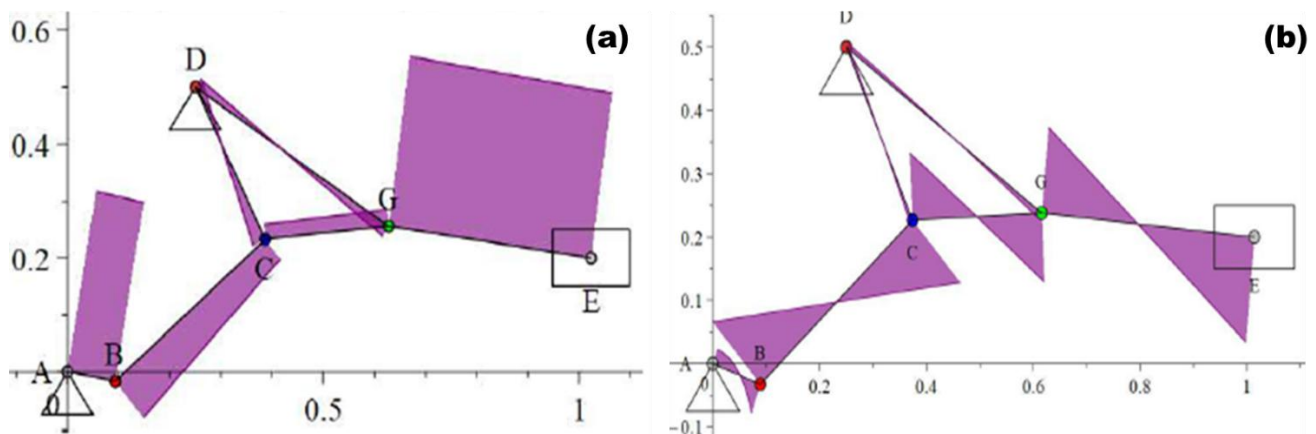


Fig. 4: Longitudinal (axial) force diagrams $N(x)$ of the mechanism at $\theta_1 = 350^\circ$: a) $\omega_1 = 1$ rad/s, N values scaled by 10^5 ; b) $\omega_1 = 100$ rad/s, N values scaled by 10^6 .

increases from 1 to 100 rad/s. This is consistent with the physics of distributed inertia: the dominant inertia term in the longitudinal component is proportional to ω^2 , while transverse components depend on both pole accelerations and angular acceleration terms. Therefore, even if ε_k is small, the ω^2 dependent inertial loading increases sharply with speed, causing bending moment envelopes to expand and to shift along the links. In practical terms, the high-speed case implies a much stricter requirement on cross-section design, because the peak bending moments determine maximum bending stress $\sigma_b \sim M_z c / J_z$.

Engineering implication: the critical design sections (maximum $|M_z|$) occur near the multi-link joints and in members aligned such that their local normal direction aligns with the dominant inertia direction. These results justify using distributed inertia rather than a single resultant at the center of mass, because bending moments are sensitive to the spatial distribution of load along the rod.

Fig. 4a presents axial force diagrams at $\theta_1 = 350^\circ$. Compared with bending moments, axial forces are governed mainly by the projection of inertia along the link axis (centripetal-type term $\propto \omega^2$) and the geometric transmission of forces through the kinematic chain. The diagrams indicate that some links enter high tensile or compressive regimes

depending on posture. This is expected, since in a multi-loop mechanism the same external slider force P and inertial forces can produce alternating tension/compression in different branches. In the high-speed case, axial forces also increase markedly, but their spatial distribution is typically smoother than bending moments because axial response is less sensitive to local curvature of deflection and more controlled by global equilibrium of the chain. Nevertheless, the axial forces strongly affect the combined stress state: $\sigma(x) \approx \frac{N(x)}{A} \pm \frac{M_z(x)c}{J_z}$, meaning that even moderate bending, when combined with large axial compression, may lead to buckling risk for slender rods.

Engineering implication: Fig. 4b suggests that for high-speed operation it is not sufficient to design solely by bending; combined axial–bending criteria and stability checks (Euler buckling for compressive members) should be considered for the most loaded links.

Fig. 5a and Fig. 5b shows transverse displacement diagrams at $\theta_1 = 130^\circ$. The displacement fields demonstrate clear growth of lateral deflections with speed and reveal where the mechanism becomes elastically “soft” in the current configuration. In multi-link mechanisms, transverse displacements tend to concentrate in links experiencing large

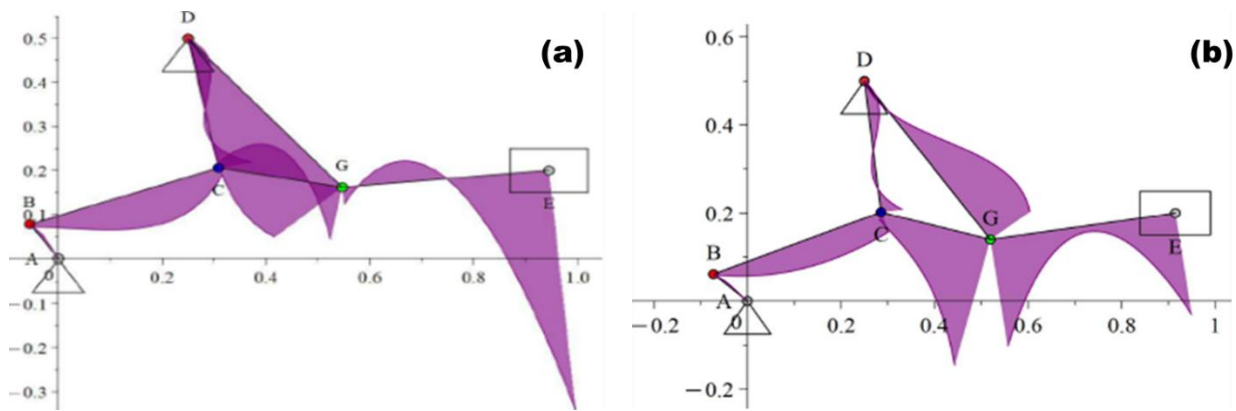


Fig. 5: Transverse displacement diagrams $w_y(x)$ of the mechanism at $\theta_1 = 130^\circ$: a) $\omega_1 = 1$ rad/s, b) $\omega_1 = 100$ rad/s, w_y values scaled by a factor of 10 (multiplied by 10).

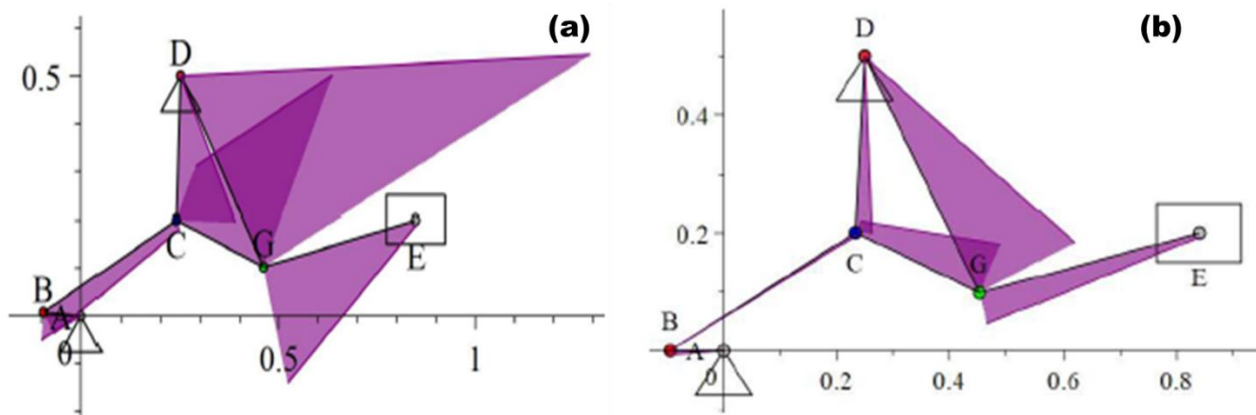


Fig. 6: Longitudinal displacement diagrams $u(x)$ of the mechanism at $\theta_1 = 175^\circ$: a) $\omega_1 = 1$ rad/s, u values scaled by a factor of 3000 (multiplied by 3000); increase in 3000; b) $\omega_1 = 100$ rad/s, values scaled by a factor of 1000 (multiplied by 1000).

bending moments, but the displacement maxima also depend on local bending stiffness EJ_z and on boundary constraints introduced by adjacent links and joints.

The high-speed case exhibits larger and more spatially varying transverse deflection shapes, indicating that distributed inertia excites bending deformation modes more strongly as ω increases. Importantly, these deflection shapes directly influence functional accuracy: deviations of hinge positions from their ideal trajectories perturb the closure constraints and can lead to positioning errors at the output (slider point E) and to increased contact forces in joints.

Engineering implication: transverse displacement diagrams provide a direct indicator of pose-dependent accuracy degradation. For precision manipulators, these results motivate either increasing EJ_z of the most compliant links, reducing operating speed, or modifying geometry to decrease transverse inertial components.

Fig. 6a and Fig. 6b reports longitudinal displacement diagrams at $\theta_1 = 175^\circ$. Compared to transverse deflections, axial elongations are usually smaller in absolute magnitude, which is why the diagrams are plotted with higher magnification factors. Even so, the distributions show that axial deformations can become non-negligible in high-speed regimes because axial inertia and transmitted forces increase with ω^2 . Figs. 3–6 present response quantities (internal forces,

moments, and elastic displacements) and are scaled for visualization; therefore, only approximate values can be visually estimated, while accurate maxima are obtained from numerical post-processing. In contrast, the section second moments J_z (and, if used, mass moments of inertia J_G) are model input parameters computed from link geometry and mass distribution and are not extracted from the response diagrams; these properties are specified separately in the calculation model.

Although axial displacements do not directly create large bending stress, they affect the closure of kinematic loops and therefore can alter internal load redistribution. In over constrained or nearly singular configurations, small axial changes can shift joint reaction forces significantly, indirectly influencing bending moment peaks in neighboring links.

Engineering implication: for high-speed mechanisms with stiff but slender links, axial deformations may still contribute to output error and internal force redistribution; therefore, both EA and EJ_z must be accounted for in stiffness design.

Taken together, Figs. 3–6 confirm three general trends:

1. Speed amplification is strong and nonlinear. The diagrams expand substantially from $\omega_1 = 1$ to 100 rad/s, consistent with inertia terms dominated by ω^2 . This confirms that high-speed regimes are governed more by inertia than by external load P alone.

2. Critical zones occur near branching joints. The largest bending moments and deflections concentrate around joints where the load path changes direction and where multiple elements meet (notably around C and G). These are the most critical locations for strength design and for joint durability.

3. Distributed inertia changes not only magnitudes but also shapes. The diagrams are not simple scaled copies between speeds; their shapes differ because pole accelerations, angular accelerations, and geometric projections vary across the mechanism. This supports the main methodological point of the paper: representing inertia as a distributed load yields more realistic internal force and displacement distributions than applying resultant loads at link centers of mass.

Design consequence: the proposed FEM formulation is suitable for selecting cross-sections and materials of links in a configuration-dependent manner, identifying critical sections for fatigue and yielding, and evaluating elastic accuracy of the mechanism under realistic operating speeds. The results also suggest that reducing mass (or redistributing mass), increasing bending stiffness EJ_z of the most loaded links, and avoiding high-speed operation near unfavorable configurations are effective strategies to mitigate inertial-induced stresses and deformations.

6. Limitations and future work

The present study is primarily methodological and is demonstrated on a benchmark numerical example of a six-bar planar mechanism. A physical experimental campaign is not included, which constitutes a limitation of the current manuscript. Nevertheless, the aim of this work is to establish a consistent FEM-ready procedure for incorporating distributed inertial and gravitational loading into strength and stiffness evaluation of moving rod mechanisms and manipulators. The benchmark example is used to verify implementation consistency and to illustrate the influence of distributed inertia on the resulting diagrams $M_z(x)$, $N(x)$, $w_y(x)$, and $u(x)$ under low- and high-speed operating conditions.

Although experimental results are absent, the proposed formulation is designed to be directly testable using standard measurement techniques. A practical validation protocol for future work would include:

1. Prototype and instrumentation. A scaled planar mechanism (or a single representative subchain) can be fabricated using aluminum or steel links with known cross-section geometry. Distributed inertia effects become observable at higher speeds; therefore, a servomotor drive with controlled ω_1 and ε_1 is required. Strain gauges placed at the predicted critical sections x^* (from the computed $\max |M_z(x)|$ locations) can provide bending strain histories, while an additional gauge configuration can separate axial and bending contributions.
2. Kinematic tracking and joint accelerations. Joint coordinates and velocities can be recorded using high-speed video tracking or an optical motion capture system. This provides an independent estimate of kinematic accelerations

of joints B, C, G and the slider point E , which are the inputs to the distributed inertia load model. The measured kinematics also confirm that the prescribed-motion assumption used in the present computations is satisfied.

3. Comparison metrics. The most direct comparison between experiment and the proposed model is performed at:

- (i) peak bending strains at the instrumented sections (mapped to M_z via $\sigma_b \approx M_z c / J_z$);
- (ii) deflections of selected joints, especially near the output point E ;
- (iii) speed sensitivity, i.e., how maxima scale when ω_1 is increased.

These metrics correspond closely to the method's design outputs and can distinguish distributed-load predictions from center-of-mass lumped inertia approximations.

4. Modeling assumptions. The present approach uses a one-way coupling strategy: rigid-body kinematics provide acceleration inputs for distributed inertia loads, which are then applied to the structural FEM model. This is appropriate when elastic deformations are small relative to link lengths and do not significantly alter the kinematics. In experimental validation, this assumption can be checked by comparing measured deflections to link lengths and by evaluating whether kinematic deviations remain within a small-strain regime. If required, the algorithm can be extended to an iterative coupling scheme (update kinematics using deformed geometry) as part of future developments.

In summary, while the manuscript currently provides a benchmark numerical demonstration rather than physical experimental validation, the proposed method has clear and feasible experimental verification routes using standard strain and displacement measurements. The absence of experiments is therefore acknowledged as a limitation, and performing a dedicated prototype validation (including comparison against lumped inertia baselines and/or multibody simulation) is identified as a priority direction for future work.

7. Conclusion

This work developed a finite rod element (FRE) formulation within an FEM framework for strength and stiffness assessment of rod mechanisms and robotic manipulators while explicitly accounting for the distributed inertia of motion. In contrast to the common engineering simplification where inertia is represented by equivalent resultant forces and moments applied at the link center of mass, the proposed method treats inertia and gravity as distributed loads along each moving link, computed from configuration-dependent kinematic quantities and introduced into the FEM system through consistent equivalent nodal loads. The approach provides a design-oriented workflow that directly outputs internal-force and deformation diagrams-bending moments $M_z(x)$, axial forces $N(x)$, transverse deflections $w_y(x)$, and longitudinal displacements $u(x)$ -for each link and configuration.

Practical implications for real systems (experimental

relevance)

The method is particularly important for high-speed mechanisms, where inertial loads dominate and scale strongly with angular velocity. A realistic application scenario is a high-speed industrial pick-and-place robot (e.g., packaging, electronics assembly) operating at high cycle rates, where even a small underestimation of bending moments in lightweight links can lead to noticeable end-effector positioning errors, increased vibration, and accelerated joint wear. Similarly, in precision medical manipulators (e.g., minimally invasive instruments or imaging-guided devices), small elastic deviations induced by distributed inertia can degrade accuracy and repeatability, which directly affects safety margins and performance. In such cases, representing inertia as a distributed load rather than a concentrated center-of-mass resultant is critical to correctly predicting the peak bending moment zones and the configuration-dependent deformation patterns that govern tracking accuracy.

From an experimental standpoint, the proposed framework supports validation using standard measurements: strain gauges placed near predicted critical sections (identified by x^* where $|M_z(x)|$ is maximal) can be used to compare measured bending stresses with model predictions, while laser displacement sensors or vision-based tracking can measure deflection at selected joints (e.g., near C , G , and the output point E) across operating speeds. Because the method produces explicit $M_z(x)$, $N(x)$, $w_y(x)$, and $u(x)$ distributions, it enables direct mapping between measured quantities and predicted link-level responses.

The output diagrams provide immediate, actionable guidance for structural design:

Cross-section sizing: Engineers use the maximum values and locations of $|M_z(x)|$ and $|N(x)|$ to determine the required section modulus and area along the link, and to select safety factors for yielding and fatigue. For example, bending stress can be estimated as $\sigma_b(x) \approx M_z(x)c/J_z$ and axial stress as $\sigma_a(x) \approx N(x)/A$, enabling combined stress checks at the critical section x^* .

Material selection: If inertia-induced bending dominates at high ω , increasing E and improving stiffness-to-mass ratio becomes beneficial. The diagrams allow a designer to distinguish whether performance is limited by bending stiffness (EJ_z) or axial stiffness (EA) and to choose materials accordingly (e.g., high-modulus alloys or composites for lightweight high-speed links).

Geometry modification to mitigate distributed inertia: Since inertia loads grow rapidly with ω , mitigation is often achieved by increasing bending stiffness without excessive mass increase. The diagrams indicate which links and which regions require reinforcement. Typical design actions include switching from solid to hollow sections (tubes) to increase J_z efficiently, adding local stiffeners near joints where $|M_z|$ peaks, reducing mass in segments that contribute most to distributed inertia, or redistributing mass toward the base to reduce dynamic loading in distal links.

Speed–configuration constraints: By evaluating peak responses across configurations and speeds, the designer can define safe operating regimes (avoidance of configurations producing extreme inertia-induced moments) and set limits on maximum allowable speed or acceleration for a given design.

Overall, the proposed FEM-based distributed inertia formulation offers a practical and scalable tool for engineering design of mechanisms and manipulators, particularly in high-speed and precision applications where the traditional center-of-mass lumping of inertia can lead to nonconservative estimates of internal forces, bending moments, and deformation-induced positioning error.

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Conflict of Interest

There is no conflict of interest.

Supporting Information

Applicable.

CRedit Statement

Yerbol Temirbekov: Investigation, Writing - Original draft Preparation, Visualization, Project administration. **Muratulla Utenov:** Conceptualization, Methodology, Software, Writing - Original draft preparation, Supervision, Project administration. **Seitbek Zhunisbekov:** Validation, Writing - Original draft preparation, Funding acquisition. **Algazy Zhauyt:** Formal analysis, Methodology, Writing - Review and editing, Funding acquisition. **Saltanat Zhilkibayeva:** Resources, Formal analysis, Writing - Original draft preparation, Project administration. **Bairon Karassayev:** Resources, Writing - Original draft preparation, Project administration.

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