



Analytical Synthesis of Planar Four-Bar Linkages for Coupler-Curve Path Generation Through Precision Points

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Abstract

This paper was performed using geometry and interactive motion (GIM) software, a learning and research software designed to facilitate the kinematic analysis and synthesis of planar mechanisms. This study presents an analytical and geometric approach to the synthesis of planar four-bar linkages for precise path generation using sixth order connecting coupler curves. Emphasis is placed on defining linkage configurations that enable a coupler point to pass through a set of three, four, or five user-defined precision points. The methodology integrates classical kinematic principles, Roberts' theorem, and focal circle theory to identify double points and optimize the trajectory of the coupler. Using complex number representations and algebraic formulations, the equations governing coupler motion are derived and applied to various linkage types, including crank-rocker, double-rocker, and limit mechanisms. Parallelogram constructions and focal circles are used to detect critical configurations and analyze the continuity and self-intersections of coupler paths. Computational simulations validate the synthesis approach across different configurations, highlighting trade-offs between accuracy and mechanical feasibility. The results demonstrate the applicability of the method in advanced motion control tasks such as robotic arms, dwell mechanisms, and function generation. This work contributes to the field of kinematic synthesis by offering a rigorous framework for linkage design with enhanced precision, reliability, and versatility.

Keywords: Path generation; Precision point synthesis; Coupler curve synthesis; Planar four-bar linkages; Linkage design.

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1. Introduction

Planar four-bar mechanisms are fundamental components in mechanical systems requiring precise motion control, including robotic manipulators,^[1] precision manufacturing devices, medical instruments, packaging machinery, and cam-follower systems. A central problem in designing such mechanisms is path generation-ensuring that a designated point on the coupler follows a prescribed trajectory.^[2] A common approach is to specify a set of precision points through which the coupler path must pass and then determine linkage parameters that reproduce this motion. Although the path generation of four-bar linkages has been extensively studied, analytical synthesis remains challenging due to the nonlinear geometry of the mechanism and the sixth-order nature of the coupler curve.^[3]

Classical works provide numerous geometric methods and

numerical strategies for linkage synthesis, including Burmester theory, cognate mechanisms, and homotopy-based approaches.^[4] However, many of these methods either rely on purely graphical constructions or involve heavy numerical optimization, making it difficult to derive explicit analytical relationships between geometric constraints, precision points, and resulting motion.^[5] Moreover, existing analytical formulations rarely integrate geometric insight-such as focal centers or double points-into the synthesis process, even though these features play a critical role in designing mechanisms with controlled dwell behavior or repeated positions.^[6]

To address this gap, this paper develops a fully analytical framework for coupler-curve synthesis of planar four-bar linkages based on up to five user-defined precision points.^[7] The method leverages explicit algebraic expressions of the sixth-order coupler curve and integrates geometric constructs such as Roberts' cognate mechanisms and the focal circle that passes through the fixed pivots.^[8] This combination enables a deeper characterization of the coupler path, including the ability to analytically determine double points-points that the coupler visits more than once-thereby expanding the design

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space for dwell mechanisms and oscillatory motion systems.^[9]

The proposed approach is intentionally focused on analytical synthesis, without reliance on numerical optimization or graphical interpolation. A set of computed simulations illustrates how varying the number of precision points (three, four, and five) affects feasible linkage configurations and the resulting coupler paths.^[10] These examples demonstrate not only the validity of the analytical formulation but also its practical usability for mechanism designers.

The approach combines complex number representation, geometric construction, and algebraic equation formulation to derive displacement relationships and evaluate the feasibility of linkage configurations.^[11,12] A series of computed simulations and plotted configurations validate the theoretical models, showcasing how different linkage types-including crank-rocker, double-rocker,^[13] and limit mechanisms-can be synthesized to satisfy motion constraints.

This paper aims to provide a consolidated analytical procedure for four-bar path generation that:

- Formulates the coupler curve explicitly in algebraic and complex-number form,
- Applies Roberts' theorem to generate cognate mechanisms that share the same coupler path,
- Uses focal-circle geometry to characterize and locate double points, and
- Validates the synthesized mechanisms using computed kinematic plots for multiple precision-point configurations.^[14]

To support practical implementation, the synthesized mechanisms are validated through computed plots, illustrating their motion across different configurations.^[15] While this paper focuses on the theoretical development and analytical formulation, the broader goal is to enable application in real-world systems such as.^[16] The methodology integrates geometric modeling, complex algebra, and analytical displacement equations into a single synthesis framework. By offering a systematic and repeatable process for path generation using precision points, this work contributes a practical tool for engineers working on motion-critical applications across robotics, manufacturing, and aerospace systems. By providing a structured and repeatable method for coupler-curve synthesis using precision points,^[17] this research contributes to the practical toolkit of engineers designing motion-critical mechanisms.^[18]

The software used in this paper (GIM), the initial steps of which were introduced in,^[19] is general-purpose software that can handle planar and spatial systems of any degree of freedom. It was developed by the COMPMECH research group of the Department of Mechanical Engineering of the University of the Basque Country UPV/EHU in Spain. It is designed for teaching and learning activities related to

important subjects in mechanical engineering, such as applied mechanics, mechanism and machine theory, computational kinematics and dynamics, mechanical design, and robotics.

2. Materials and methods

In this study, the term crank mechanisms refers specifically to those mechanisms that can be derived from the general class of articulated four-link mechanisms and their special configurations. These include common variants such as crank-rocker,^[13] double-crank,^[14] and double-rocker mechanisms. Among these, connecting coupler curves-the trajectories described by points on the coupler link-are of critical interest to designers due to their role in defining motion paths in practical applications. From a mathematical perspective, a connecting coupler curve is defined as the locus of a point on a rigid body (the coupler link), where two points of that body, labeled A and B , trace circular paths-denoted by circles α and β , respectively. These circles may, in special cases, degenerate into straight lines, depending on the configuration of the mechanism.^[15] Fig. 1 illustrates an articulated four-link mechanism with links a, b, c, d , commonly referred to as a crank-rocker mechanism.^[16] In this configuration, the dimensions satisfy the Grashof condition: $(a + d < b + c)$ a and d are the lengths of the shortest and longest links, b and c are the lengths of the coupler and the follower links, respectively.^[17,18] To analyze the coupler motion geometrically, auxiliary circles k_{b+a} and k_{b-a} are constructed with centers at A_0 and B_0 and with radii $(b + a)$ and $(b - a)$, respectively. These circles intersect the circular path β at specific points: B_a and B'_a for the outer circle k_{b+a} , B_i and B'_i for the inner circle k_{b-a} . The corresponding crank pin centers, representing the positions of points A_a, A'_a, A_i and A'_i , are then derived from these intersections. These geometrical constructs allow for precise determination of possible crank configurations and are essential for analyzing and synthesizing coupler curves in practice.

As shown in Fig. 1, the articulated four-bar linkage with link lengths a, b, c, d satisfies the Grashof condition. The coupler curves γ and γ' , traced by point C , illustrate two possible motion paths based on distinct coupler configurations. The auxiliary circles k_{b+a} and k_{b-a} are used to determine the intersection points with path β , corresponding to crank pin centers that influence coupler curve formation. The movement of the connecting coupler plane E , rigidly attached to link cb , can be achieved using two different configurations: the crank-rocker mechanisms A_0ABB_0 and $A_0A'B'B_0$, as illustrated in Fig. 1. In both cases, the coupler curve consists of two distinct segments, denoted γ and γ' , which correspond to the coupler paths generated by the respective triangle configuration ΔABC and $\Delta A'B'C'$, where the triangles are congruent. By transforming the shortest link a into a rack, as shown in Fig. 2, the mechanism becomes a two-crank mechanism.^[19] This configuration preserves the dual-segment coupler curve, again composed of paths γ, γ' . The transformation effectively allows continuous rotation of both input links while

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maintaining the same fundamental coupler motion.^[20]

The two-rocker mechanism shown in Fig. 3 includes a rack-type link c , which is positioned opposite to the shortest link. This configuration leads to a mechanism that satisfies the Grashof condition $a + d < b + c$, and allows analysis of multiple motion extremes. For each of the mechanisms B_0BAF and $BA'F'B_0$, the coupler curves again consist of two distinct parts, generated by two congruent coupler triangle configurations. Under these conditions, a total of eight dead

positions-mechanism configurations where motion is momentarily halted due to alignment of certain links-can be constructed. These are determined using the following auxiliary circles: k_{b+a} , and k_{b-a} , centered at B , k_{d+a} , and k_{d-a} , centered at B_0 . These constructions identify key positions such as F, A, B, C and their counterparts F', A', B', C' , revealing the mechanism's kinematic constraints and trajectory switching behavior. The coupler curves-shown in red-consist of the loci traced by point F in both configurations.

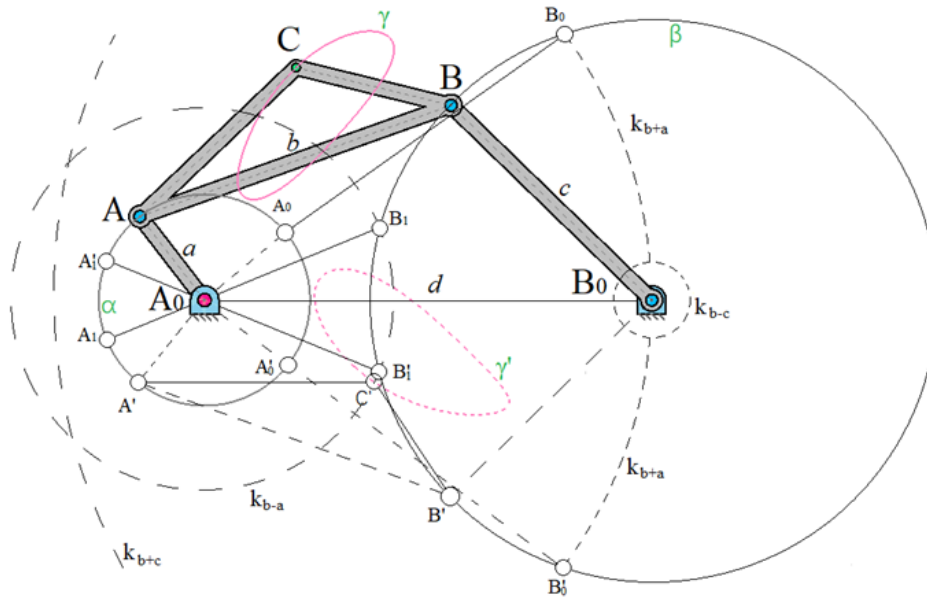


Fig. 1: Connecting coupler curve γ, γ' of a crank-rocker mechanism, composed of two segments. The mechanism consists of four rigid links a, b, c and d , forming a closed loop. Points A and B move along the circular paths α and β , respectively. Auxiliary circles k_{b+a} and k_{b-a} , centered at A_0 and B_0 , determine possible crank pin centers at intersections with circle β . The coupler curves γ and γ' , traced by point C on the coupler link, illustrate the path generation capabilities of the linkage for different configurations.

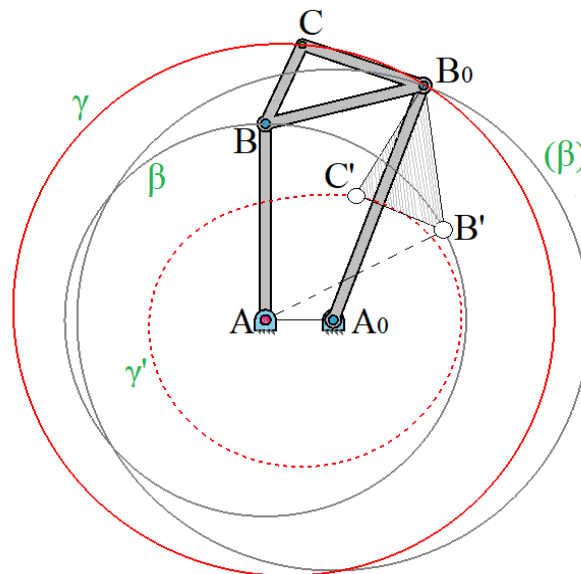


Fig. 2: Connecting coupler curve segments γ, γ' of a two-crank mechanism. The mechanism is derived from the configuration in Fig. 1 by converting the shortest link into a rack. As in the crank-rocker case, the coupler curve consists of two parts generated by two congruent coupler triangles ΔABC and $\Delta A'B'C'$, enabling a complete description of the coupler point trajectory along circles α and β .

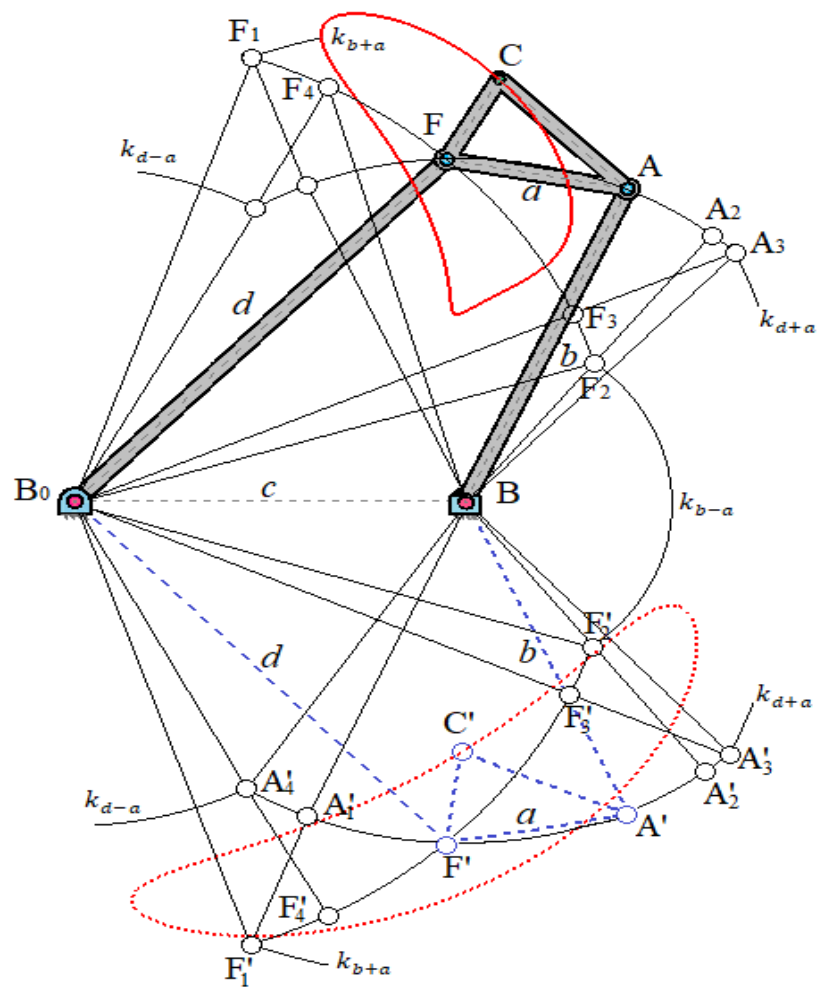


Fig. 3: Dead positions of a double-rocker mechanism. The mechanism features a rack-type link c , placed opposite the shortest link, producing two coupler curves (in red) composed of segments traced by coupler points F and F' . Eight dead positions are constructed using auxiliary circles k_{b+a} , k_{b-a} , k_{d+a} , and k_{d-a} . These identify external and internal limit configurations under Grashof condition constraints.

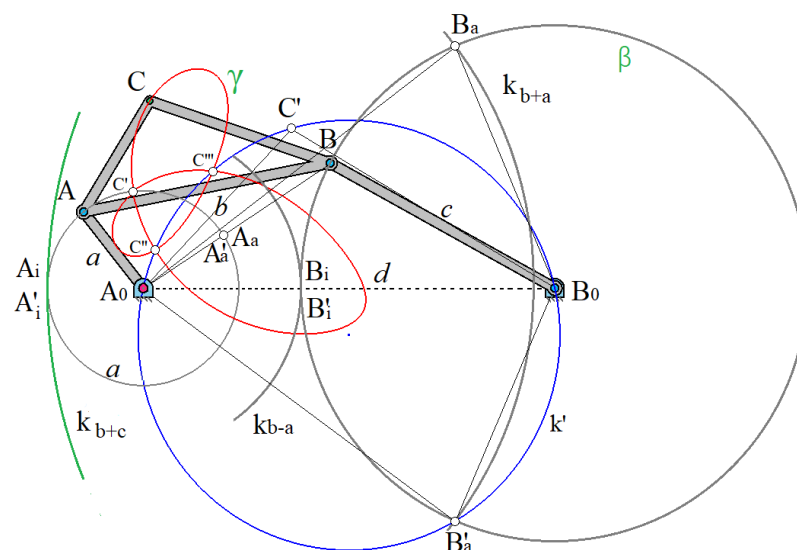


Fig. 4: Limit crank-rocker mechanism with coupler curve γ consisting of a single continuous segment. The mechanism satisfies the Grashof limit condition $a + d = b + c$. The coupler curve displays double points, where the path intersects itself, revealing key traits of the motion geometry. Circles k_{b+a} , k_{b-a} , and k' assist in identifying critical configurations and focal points that define trajectory behavior.

In Fig. 4, the hinged four-bar mechanism from Fig. 1 is reconfigured into a limit crank-rocker mechanism by placing the system on link d and adjusting the link dimensions such that: $a + d = b + c$. This equality marks the boundary condition of the Grashof theorem, resulting in a unique linkage configuration referred to as a limit mechanism. In this scenario, the coupler curve γ , traced by a point on the coupler link, forms a single continuous path rather than being composed of two distinct segments as seen in prior configurations. Notably, the curve exhibits double points, *i.e.*, self-intersections, which are of particular interest for applications requiring dwell behavior, repetitive paths, or velocity discontinuities. The construction involves auxiliary circles: k_{b+a} , k_{b-a} centered at B , k_{b+c} , k' centered at A_0 and B_0 , respectively, which intersect paths α and β at key positions $A_i = A'_i$, $B_i = B'_i$, and $B_a = B'_a$. These points determine the limits of motion and the geometry of the coupler curve.

The motion processes in the limit crank-rocker mechanism (Fig. 4) can proceed in two modes, depending on whether motion constraints are present.^[21] In the unconstrained case, if the crank a begins rotating clockwise from position A_a , proceeds through A' , and A_i , then point B (at the end of the coupler link b) reaches the intermediate position B_i along the segment A_0B_0 . If a mechanical stop is installed at point B_i , thereby halting the motion of coupler b , then link a continues its rotation independently. As a result, point B on the rocker arm c returns from B_i through B to the opposite limit position B_a . In the second case, where no stop is present at point B_i , the

rocker arm c undergoes oscillatory motion between the positions B_0B_a and $B_0B'_a$, while the crank a performs two full revolutions. In this scenario, the connecting coupler b transitions smoothly through all of its possible configurations, resulting in a continuous motion path. Therefore, the coupler curve γ traced by point C , consists of a single continuous branch. An important geometric observation is that the arc $B_aB_iB'_a$, described by the rocker pin, has its convex side facing the center of rotation of crank a . According to Burmester's classification, this structure is termed a convex limit crank-rocker mechanism. In configurations where the Grashof equality $d + c = a + b$ holds, transition through indefinite positions (*i.e.*, singularities or limit positions) can be managed by integrating gear engagements. In such cases, the centroids corresponding to the relative motion of links a and c are used for further kinematic analysis and design.

If, while maintaining the dimensions of links a , c , and d , the length of link b is reduced below the value shown in Fig. 4, the system becomes a two-rocker mechanism (as shown in Fig. 5). In this configuration, the inequality $a + d > b + c$ is satisfied, thus violating the Grashof condition,^[17] which precludes full rotation of any link and results in oscillatory motion of both input and output links.^[22]

Despite the restricted rotation, the coupler link b still traces a continuous trajectory. The resulting connecting coupler curve γ consists of a single, closed path, which is influenced by the constrained motion of the rocker links. In this configuration, the coupler curve passes through a complex

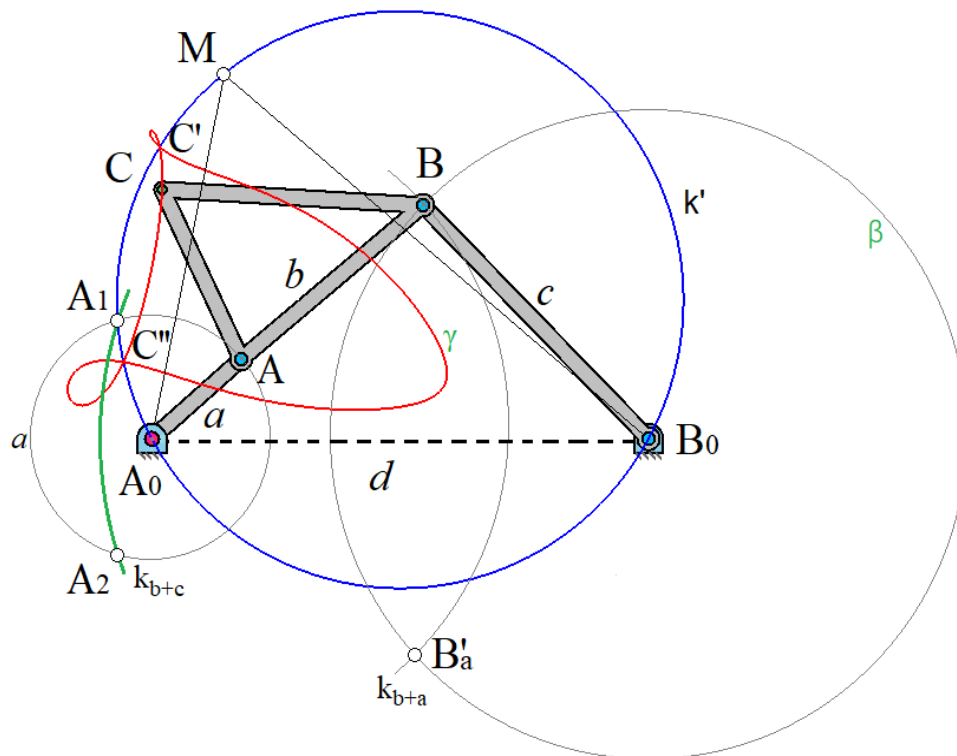


Fig. 5: Double-rocker mechanism where the Grashof condition $a + d > b + c$ is not satisfied. As a result, neither the input nor the output link can complete a full rotation. The coupler curve γ , shown in red, forms a single continuous loop traced by a point on the coupler link b . Circles k_{b+a} , k_{b+c} , and k' provide the geometric basis for determining key trajectory points and constraint boundaries.

sequence of positions, including multiple notable locations such as C , C' , and C'' , as visualized in red. This particular setup is useful in applications requiring reciprocating motion or symmetric path generation, such as certain press linkages, lifting mechanisms, or specialized automation systems. The two-rocker mechanism illustrated in Fig. 5 differs significantly from the configuration in Fig. 3. While the mechanism in Fig. 3 possesses eight dead positions, the double-rocker shown here has only four dead positions due to its non-Grashof nature. Despite this, the connecting coupler link AB traverses all possible positions continuously, without interruption. Consequently, the coupler curve γ traced by point C forms a single, closed branch. This observation leads us to a general conclusion regarding the structure of connecting coupler curves in four-bar mechanisms, formulated as the following theorem:

Theorem 1: Connecting coupler curves consisting of two branches can be obtained if and only if the sum of the shortest and longest links is less than the sum of the remaining two links. In all other cases, the resulting coupler curves take the form of a single, closed contour.

This theorem captures the relationship between link length proportions and curve continuity, and aligns with the implications of the Grashof condition and its boundary cases. This builds on the foundational result known as Roberts' Theorem, which states that for every coupler curve generated by a four-bar linkage, there exist two additional distinct four-bar linkages-called cognates-that generate the same coupler curve.^[23] According to the theorem introduced by Soni A.H. and Harrisberger, L., when we construct straight lines parallel to the sides of a triangle $A_0B_0M_0$ (as illustrated in Fig. 6) through an arbitrary point C in its plane, we obtain: three similar triangles $\triangle ABC \sim \triangle ECD \sim \triangle CFG$ and three hinged parallelograms A_0ACE , B_0BCF , M_0DCG . The resulting system forms what may be termed a hinged linkage structure, composed of interconnected parallelograms and triangles, all articulated at point C , which functions as a double hinge. This structure possesses two degrees of freedom, allowing it to be reconfigured by fixing one base link and actuating others-

typically via the links $\overline{AA_0}$ and $\overline{BB_0}$. By manipulating the configuration, we can obtain a new mechanism as shown in Fig. 7, which is a realization of Roberts' Theorem.^[23] This theorem confirms that a single coupler curve can be generated by three different four-bar mechanisms having similar coupler triangles. It provides a fundamental geometric basis for alternative synthesis solutions, enabling the designer to explore multiple linkage configurations that yield the same prescribed motion path.

The mechanism shown in Fig. 7 complies with Roberts' Theorem, as it demonstrates three cognate four-bar linkages that generate the same coupler curve.

If we consider the points A_0 , B_0 and M_0 as fixed hinges, we obtain a mechanism with $n = 10$ links and $g = 14$ hinges, and the hinges A_0 , B_0 , M_0 and C must be considered double, i.e. we obtain a mechanism with the number of degrees of freedom $F = 3(10 - 14 - 1) + 14 = -1$. The kinematic chain should therefore be a rigid body; but it is not such due to the existing geometric relationships, and belongs to mechanisms with passive connections.^[24]

Proof of Theorem 1: The proof of Roberts' theorem is based mainly on the following: the mechanism under consideration consists of a movable hinged parallelogram M_0GCD , articulated by a hinge M_0 with a stand and supplemented by similar triangles ECD and CFG ; then, if, for example, a point E moves along an arbitrary curve k_E relative to the stand, then the point F describes a similar curve k_F . Further, in the transition from Figs. 6, 7 we have similarity of triangles $\triangle A_0B_0M_0 \sim \triangle ABC$. If we disconnect the joint at point C , we will see that the mechanism in Fig. 7 breaks down into the following mechanisms: crank mechanism A_0ABB_0 with connecting rod triangle ABC ; crank mechanism A_0EDM_0 with connecting coupler triangle ECD ; crank mechanism B_0FGM_0 with connecting coupler triangle CFG , and the connecting coupler point C of each of these mechanisms describes the same connecting coupler curve γ . Thus, the following important theorem on the formation of the connecting coupler curve by three different four-link joints is true.

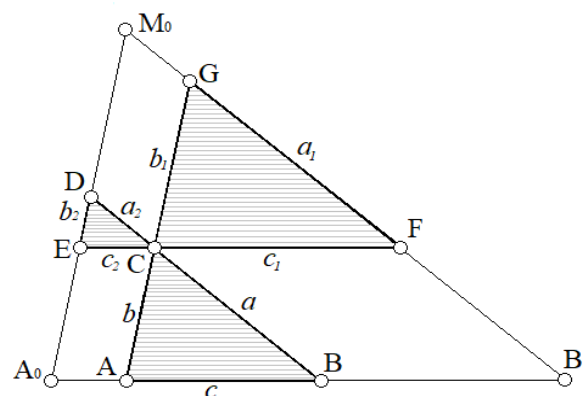


Fig. 6: Hinged geometric mechanism demonstrating the construction of three four-bar linkages according to Roberts' Theorem. The similar triangles $\triangle ABC$, $\triangle ECD$ and $\triangle CFG$ share a common point C , and the parallelograms A_0ACE , B_0BCF , and M_0DCG articulate the triangle vertices through hinge joints.

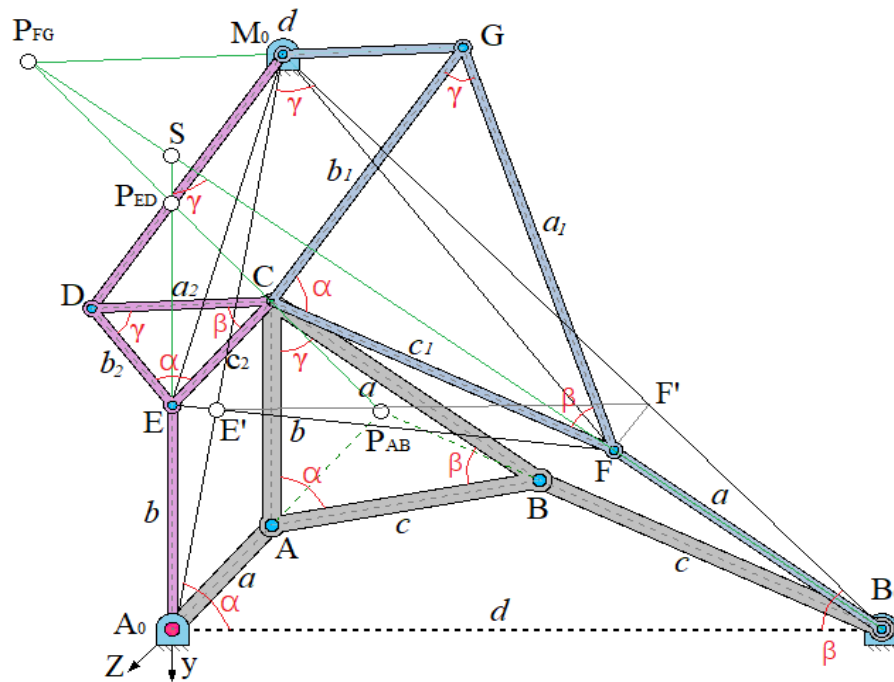


Fig. 7: Kinematic construction of a connecting coupler curve using three equivalent four-bar mechanisms derived from the Roberts configuration. Each mechanism (solid, dotted, and dot-dashed lines) shares the same coupler point C , which generates an identical coupler curve through different configurations of similar coupler triangles.

Theorem 2: The trajectory of point C can be obtained as the connecting coupler curve of three articulated four-link chains A_0ABB_0 , A_0EDM_0 , B_0FGM_0 with posts A_0B_0 , A_0M_0 , B_0M_0 , and their connecting coupler triangles ABC , ECD , and CFG are similar to each other and similar to triangle $A_0B_0M_0$.

Proof of Theorem 2: There are a number of proofs of Roberts' theorem,^[23] for example by Soni A.H.; and Harrisberger, L. If the mechanism A_0ABB_0 , shown in Fig. 7, is a limiting crank mechanism,^[25] then the other two mechanisms will be the same. The importance of Roberts' theorem for the synthesis of mechanisms can be explained by the following example: if for some problem, for example, for constructing a connecting coupler mechanism with stops, a hinged four-link mechanism A_0ABB_0 , is found, then, using Roberts' theorem,^[23] two other four-link mechanisms can be found with the help of which the same connecting coupler curve can be constructed.^[26] This circumstance is especially important in those cases when in the found hinged four-link mechanism we have, for example, unfavorable values of the transmission angles or if any difficulties arise in its drive and the possibility of changing the position of the hinges A_0 and B_0 , should be taken into account, and this change may prove expedient.

Example: Consider a working machine that uses a crank-rocker mechanism A_0ABB_0 to produce a coupler curve γ of point C , which serves a specific function—such as generating a dwell segment in the motion profile. If the machine is being reconstructed and it becomes necessary to change the drive (e.g., relocating motor placement or optimizing transmission), but it is critical to preserve the shape of curve γ , Roberts' Theorem ensures that two other mechanisms with identical

coupler curves can be found (see Fig. 8).^[25,26] These alternative mechanisms may provide improved performance or layout compatibility without altering the motion outcome.

Fig. 8 beautifully ties together the practical implications of Roberts' theorem, visually confirming that a single coupler curve γ can be generated by three distinct four-bar mechanisms, each based on a different pair of fixed pivots. This set of subfigures illustrates how three different four-bar mechanisms can generate the same coupler curve γ by employing similar coupler triangles based on Roberts' theorem.^[26]

Fig. 8a shows a crank-rocker mechanism A_0ABB_0 with a coupler triangle ΔABC . The coupler curve γ is traced by point C , and auxiliary points and lines (D, E, F, G, M_0) define the geometric basis of triangle similarity.

Fig. 8b shows geometric basis of the Roberts' transformation: triangle $\Delta A_0B_0M_0$ and the similar coupler triangle ΔABC , ΔECD , and ΔCFG .

Fig. 8c shows alternative four-bar mechanism A_0EDM_0 derived from triangle ΔECD . The same coupler curve γ is reproduced by point C , demonstrating functional equivalence.

Fig. 8d shows a crank-rocker mechanism A_0ABB_0 with a coupler triangle ΔABC . The coupler curve γ is traced by point C , and auxiliary points and lines A_0ABB_0 and $B_{II}B_0B_I B$ define the geometric basis of parallelogram similarity.

Fig. 8e shows another realization via crank mechanism B_0FGM_0 , corresponding to coupler triangle ΔCFG . Again, the point C follows the identical curve γ . A variation in configuration demonstrating the flexibility in linkage layout while preserving the kinematic function. Point C again traces the same curve using a different pair of fixed pivots B_0 and

M_0 .

Consider the four-link articulated mechanism A_0ABB_0 with a coupler triangle ΔABC , as shown in Fig. 8a. To synthesize two additional mechanisms generating the same coupler curve γ , we apply the construction method derived from Roberts' Theorem. The lines passing through A_0 and C parallel to CA and AA_0 , yield a point E at their intersection. Constructing triangle ECD (Fig. 8b), on side EC (Fig. 8a) we obtain point D ; from here we find the crank $\overline{M_0D}$ and thus obtain the desired crank mechanism, which is shown again in (Fig. 8c). Similarly, the lines drawn through B_0 and C parallel to CB and BB_0 , intersect at point F ; constructing triangle CFG on side $\overline{CF}$ we obtain G , from here we find the crank $\overline{M_0G}$; thus, we find the crank mechanism this mechanism is also

shown in (Fig. 8e). Thus, the required crank mechanisms can be derived by constructing and articulating the three-link chains: $\overline{A'_0E}$, $\overline{ECD}$, $\overline{DM'_0}$ and $\overline{B'_0F}$, $\overline{FGC}$, $\overline{GM'_0}$ (Fig. 8b) with points A_0 and M_0 and, accordingly, with B_0 and M_0 Fig. 8a. These chains must be placed on the correct side of the rack, such that the resulting connecting coupler curves γ consist of two branches, as dictated by the geometric configuration and Grashof condition. (Fig. 8c, 8d, 8e) together with the congruent connecting coupler curves γ plotted on them clearly illustrate Roberts' theorem,^[23] according to which one can find (Fig. 8e) another crank-rocker mechanism; the two-rocker mechanism in (Fig. 8c) is unsuitable for our purposes due to the difficulties arising in implementing its drive.

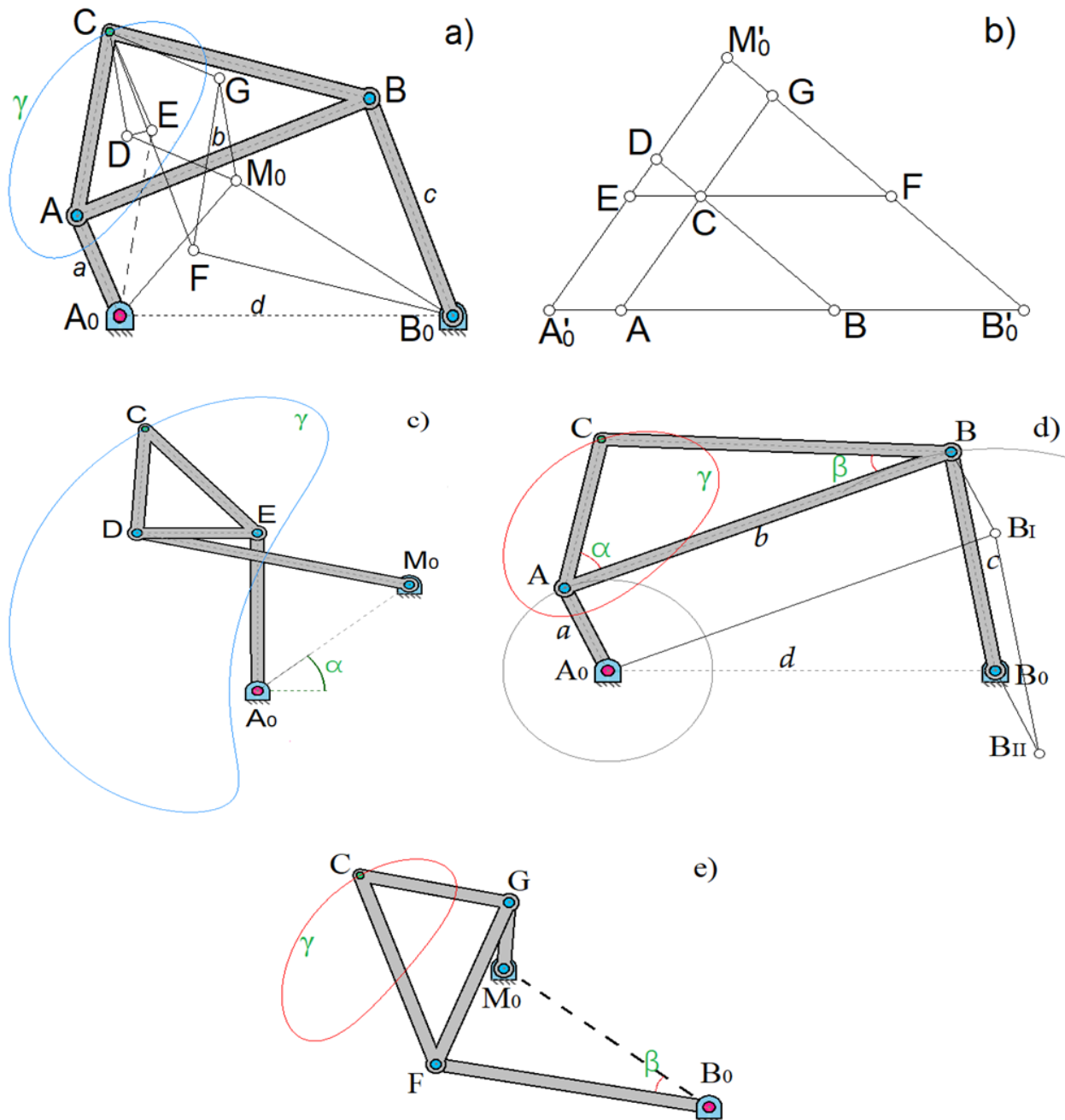


Fig. 8: Realization of Roberts' theorem – equivalent crank mechanisms generating the same coupler curve.

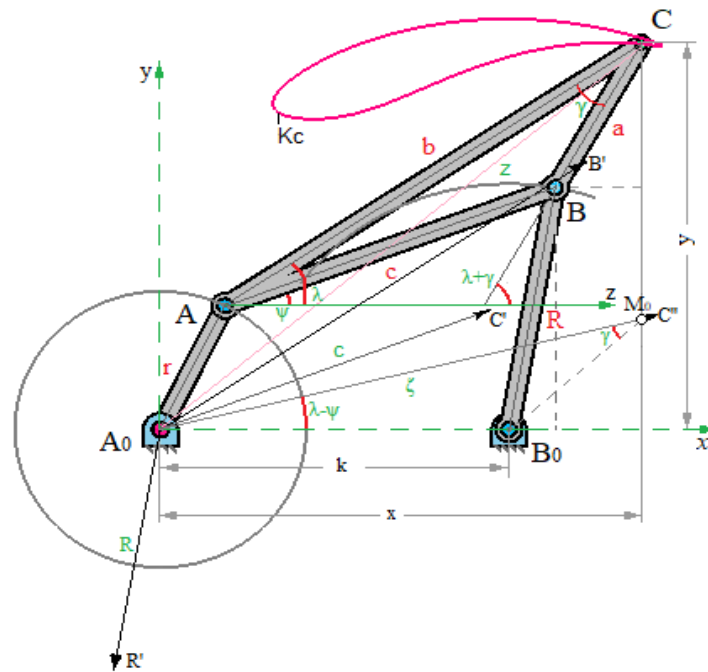


Fig. 9: Geometric and vector-based representation of a four-bar crank-rocker mechanism used to derive the equation of the connecting coupler curve K_C in rectangular coordinates and complex notation. The diagram shows all relevant link lengths, angles, and vectors necessary for formulating both analytical and numerical models of the coupler point motion.

3. Kinematic preliminaries

Let us consider a crank-rocker mechanism A_0ABB_0 , illustrated in Fig. 9, with a coupler triangle ΔABC . The mechanism has the following link lengths and geometric parameters: $\overline{A_0A} = \bar{r}$, $\overline{AB} = \bar{c}$, $\overline{BB_0} = \bar{R}$, $\overline{A_0B_0} = \bar{k}$, $\overline{AC} = b$, $\overline{BC} = a$, $\angle ACB = \gamma$, $\angle ZAC = \lambda$, $AZ \parallel A_0x$, let the vectors $\overline{A_0A} = \bar{r}$, $\overline{A_0C'} = \overline{AB} = \bar{c}$, $\overline{A_0B'} = \overline{AC} = b$, $\overline{A_0R'} = \overline{BB_0} = \bar{R}$, $\overline{A_0B_0} = \bar{k}$. Assuming the origin A_0 and the direction A_0x as the real axis of the complex plane xA_0y , we can associate complex numbers with the position vectors of key points in the mechanism $\bar{r}, \bar{c}, \bar{R}, \bar{k} \in \mathbb{C}$. Let us define the transformed points: $A_0C' \parallel AB$, $A_0B' \parallel AC$, $AR' \parallel BB_0$ and let $\bar{z} \in \mathbb{C}$ represent the complex position of the coupler point C . Then, we derive the following complex displacement equations from vector loop closures:

$$G_1 = \bar{r} + \bar{c} + \bar{R} - \bar{k} = 0 \quad (1)$$

$$G_2 = \bar{c} + \bar{R} + \bar{r} - \bar{k} = 0 \quad (2)$$

$$G_3 = \bar{c} + \bar{r} + \bar{R} - \bar{k} = 0 \quad (3)$$

$$G_4 = \bar{r} + \bar{R} + \bar{c} - \bar{k} = 0 \quad (4)$$

$$G_5 = \bar{R} + \bar{c} + \bar{r} - \bar{k} = 0 \quad (5)$$

$$G_6 = \bar{R} + \bar{r} + \bar{c} - \bar{k} = 0 \quad (6)$$

Each of these vector loop Eqs. (1-6) corresponds to a distinct transformation of the crank mechanism, as demonstrated by A.H. Soni,^[27] and is closely related to the Roberts' transformations discussed earlier.^[23] These

configurations represent kinematically equivalent four-bar linkages that trace the same coupler curve. To analytically describe the coupler point trajectory K_C , we introduce a complex number $\bar{z} \in \mathbb{C}$ representing the position of point C . Its motion, as the mechanism moves, can be written in the complex plane as:

$$\bar{z} = \bar{r} + \bar{\zeta}(\cos\psi + i\sin\psi) = \bar{r} + \bar{\zeta}e^{i\psi} \quad (7)$$

or, equivalently Eq. (7), in normalized form:

$$\bar{z} = \bar{r} + \bar{\zeta} \frac{\bar{c}}{|\bar{c}|} \quad (8)$$

where $\bar{r}$ is the position vector Eq. (8) of the crank A_0A ; $\bar{\zeta}$ is the vector from A and C ; ψ is the angular parameter describing the crank rotation.

This expression provides a parametric equation for the coupler curve in terms of link geometry and input crank angle ψ . It is foundational for developing computational simulations of coupler path generation and for incorporating position analysis into numerical optimization algorithms.

In the crank-rocker mechanism A_0ABB_0 , let the vector $\bar{\zeta} \in \mathbb{C}$ correspond to $\overline{A_0C''}$, when C'' lies on the extension of the coupler triangle, and $\angle C''A_0x = \angle CAB$. Given that $\Delta A_0B_0M_0 \sim \Delta ABC$, and $\overline{A_0C''} = \overline{AC}$, it follows that the vector $\bar{\zeta}$, upon rotation by an angle ψ about A_0 , becomes parallel to $\overline{AC}$ at any instant of motion. The unit complex number $e^{i\psi} = (\cos\psi + i\sin\psi)$ corresponds to a unit vector in the complex plane xA_0y , directed parallel to the vector $\overline{AB} = \overline{A_0C'}$. Therefore, we can represent this rotation using the normalized complex vector: $\frac{\bar{c}}{|\bar{c}|}$.

We now define a normalized constant in complex vector Eq. (9):

$$\frac{\bar{\zeta}}{|\bar{C}|} = \bar{v} \quad (9)$$

This leads to a simplified expression for the coupler curve K_C in complex form:

$$\bar{Z} = \bar{r} + \bar{v}\bar{c} \quad (10)$$

where $\bar{v}$ is a constant Eq. (10) complex scalar whose modulus equals $\frac{AC}{AB}$ and argument is $\angle CAB$.

Applying this complex representation to the transformed crank mechanisms G_2 and G_3 offers a clean analytical pathway for verifying Roberts' Theorem.^[23] As seen in Fig. 8d, in addition to the base mechanism A_0ABB_0 , the transformed four-bar linkages $A_0B'BB_0$ (denoted $bacd$) and $A_0B'B''B_0$ (denoted $bcad$) are illustrated Eq. (3) and Eq. (2). The $(bacd)$ configuration by the ratio $\overline{A_0M_0} : \overline{A_0B_0}$, and the second $(bcad)$ configuration by the ratio $\overline{B_0M_0} : \overline{A_0B_0}$, and applying the appropriate rotational transformations, we arrive at the mechanisms A_0EDM_0 (shown in Fig. 8c) and B_0FGM_0 (in Fig. 8e), both of which conform to the constructions derived from Roberts' Theorem. From this it becomes clear that both crank mechanisms constructed according to Roberts' theorem are similar to articulated four-link mechanisms transformed from the original one; in addition, the connecting coupler curves of the transformed crank mechanisms $A_0B'BB_0$ and $A_0B'B''B_0$ are similar to the connecting coupler curves of the original mechanism.^[28]

For a four-link articulated mechanism A_0ABB_0 (see Fig. 1), with link dimensions: $\overline{A_0A} = \bar{r}$, $\overline{AB} = \bar{c}$, $\overline{BB_0} = \bar{R}$, $\overline{A_0B_0} = \bar{k}$ we aim to derive the equation of the connecting coupler curve K_C traced by a coupler point C in rectangular coordinates x, y . Using the auxiliary angle $\lambda = \angle ZAC$, the coordinates of points $A(x_A, y_A)$ and $B(x_B, y_B)$ can be expressed as:

$$x_A = x - b \cdot \cos\lambda \quad (11)$$

$$y_A = y - b \cdot \sin\lambda \quad (12)$$

$$x_B = x - a \cdot \cos(\lambda + \gamma) \quad (13)$$

$$y_B = y - a \cdot \sin(\lambda + \gamma) \quad (14)$$

From the constraint that links maintain fixed lengths:

$$x_A^2 + y_A^2 - r^2 = 0 \quad (15)$$

$$(x_B - k)^2 + y_B^2 - R^2 = 0 \quad (16)$$

By substituting the coordinates from Eqs. (11–14) into Eqs. (15–16), and eliminating λ , we obtain:

$$2bx \cdot \cos\lambda + 2by \cdot \sin\lambda = x^2 + y^2 + b^2 - r^2 \quad (17)$$

$$2a(x - k)\cos(\lambda + \gamma) + 2ay \cdot \sin(\lambda + \gamma) = (x - k)^2 + y^2 + a^2 - R^2 \quad (18)$$

Using the angle-sum identities:

$$\cos(\lambda + \gamma) = \cos\lambda\cos\gamma - \sin\lambda\sin\gamma$$

$$\sin(\lambda + \gamma) = \sin\lambda\cos\gamma + \cos\lambda\sin\gamma$$

Eq. (18) expands into:

$$\begin{aligned} 2a[(x - k)(\cos\lambda\cos\gamma - \sin\lambda\sin\gamma) \\ + y(\sin\lambda\cos\gamma + \cos\lambda\sin\gamma)] \\ == (x - k)^2 + y^2 + a^2 - R^2 \end{aligned}$$

Collecting coefficients of $\cos\lambda$ and $\sin\lambda$, we obtain:

$$\begin{aligned} 2a[(x - k)\cos\gamma + y\sin\gamma]\cos\lambda \\ + 2a[y\cos\gamma \\ - (x - k)\sin\gamma]\sin\lambda = (x - k)^2 + y^2 \\ + a^2 - R^2 \end{aligned}$$

Now, Eq. (17) and Eq. (18) form a system of two linear equations in the unknowns $\cos\lambda$ and $\sin\lambda$. Solving this system eliminates λ and leads to three combined expressions. Multiplying Eq. (17) by:

$$a[(x - k)\cos\gamma + y\sin\gamma]$$

and Eq. (18) by:

$$bx$$

and subtracting yields the expressing U . Similarly, combining terms with the sine-related components yields V . The cross-term cancellation produces the quantity W , which arises naturally as:

$$W = 2ab \cdot \sin\gamma[x(x - k) + y^2 - ky \cdot \cot\gamma]$$

Thus the coupler curve satisfies the symmetric relation:

From these, the equation of the coupler curve K_C takes the symmetric form:

$$U^2 + V^2 = W^2 \quad (19)$$

where we put:

$$U = a[(x - k)\cos\gamma - y\sin\gamma](x^2 + y^2 + b^2 - r^2) - bx[(x - k)^2 + y^2 + a^2 - R^2] \quad (20)$$

$$V = a[(x - k)\sin\gamma - y\cos\gamma](x^2 + y^2 + b^2 - r^2) + by[(x - k)^2 + y^2 + a^2 - R^2] \quad (21)$$

$$W = 2ab \cdot \sin\gamma[x(x - k) + y^2 - ky \cdot \cot\gamma] \quad (22)$$

where U , V , and W are defined explicitly in Eq. (19) and Eqs. (20)–(22). This derivation follows the classical analytical development of coupler curves in Suh & Radcliffe (1978) and Sandor & Erdman (1984).

The expanded form of the coupler curve equation becomes:

$$\begin{aligned}
& a^2[(x-k)^2 + y^2](x^2 + y^2 + b^2 - r^2)^2 \\
& - 2ab[(x^2 + y^2 - kx)\cos\gamma + ky \\
& \cdot \sin\gamma](x^2 + y^2 + b^2 \\
& - r^2)[(x-k)^2 + y^2 + a^2 - R^2] \\
& + b^2(x^2 \\
& + y^2)[(x-k)^2 + y^2 + a^2 - R^2]^2 \\
& - 4a^2b^2[(x^2 + y^2 - kx)\sin\gamma \\
& - ky \cdot \cos\gamma]^2 = 0
\end{aligned} \quad (23)$$

Since Eq. (23) is of degree six in x and y , the curve is algebraic of the sixth order, confirming that it can intersect any straight line in up to six points. For a given line:

$$y = mx + n \quad (24)$$

substituting into Eq. (23) yields a sixth-degree polynomial in x :

$$a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \quad (25)$$

which confirms that the coupler curve can have up to six intersections with a given line.

Then for the coordinate x we obtain an Eq. (25) of the 6th degree, *i.e.* we will have six intersection points. If, in particular, $m = \pm i$ when $i = \sqrt{-1}$, then according to the equation Eq. (24) we obtain:

$$x^2 + y^2 = \pm 2inx + n^2 \quad (26)$$

where $i = \sqrt{-1}$, we obtain another form of the sixth-degree equation. This implies that the connecting coupler curve is a highly nonlinear trajectory that can exhibit loops, inflection points, and multiple branches.^[29] Then in the equation Eq. (23) the terms from the 6th to the 4th degree vanish; of the six roots of the equation, three become infinitely large. The imaginary cyclic points at infinity I and I' of the crank curve are triple points. In other words, it is a tricircular curve of the sixth order. The line at infinity has only two imaginary cyclic points in common with the crank curve.^[30]

The connecting coupler curve K_C has at each imaginary cyclic point three asymptotes which intersect the asymptotes corresponding to another cyclic point at three real points, at three focal centers (or foci) A_0 , B_0 and M_0 . Thus, for example, for the lines connecting the origin A_0 with I and I' . (due to the fact that $n = 0$), the equations have the form $y = \pm ix$, $x^2 + y^2 = 0$. For the intersection points of the lines A_0I and A_0I' in equation Eq. (26) the third-degree term with respect to x ; vanishes; the lines A_0I and A_0I' are thus tangent to the curve at the points I and I' , which proves that A_0 is a focal center of the connecting coupler curve.^[31] The same is true for the points B_0 and M_0 , and $\Delta A_0B_0M_0$ according to Roberts' theorem^[23] is similar to ΔABC and has the same direction of traverse. There is an extensive literature devoted to the study of connecting coupler curves. If a connecting coupler point C lies on the line AB between A and B , then we must set $\gamma = 180^\circ$, and $b + a = c$. We should pay attention to the special case when C is the midpoint of the segment AB and at the same time $R = r$. We

then obtain a lemniscate, *i.e.* a curve similar to a lemniscate; this curve is often used in practice and is also called the Watt curve. If point C lies on the line AB , but outside the segment AB , then we must set $\gamma = 0$ and $b - a = \pm c$. The equation of the connecting coupler curve in this case will have the following form:

$$U^2 + V^2 = 4a^2b^2k^2y^2 \quad (27)$$

moreover:

$$U = a(x-k)(x^2 + y^2 + b^2 - r^2) - bx((x-k)^2 + y^2 + a^2 - R^2) \quad (28)$$

$$V = -ay(x^2 + y^2 + b^2 - R^2) + by[(x-k)^2 + y^2 + a^2 - R^2] \quad (29)$$

For the case when C lies between A and B , it is necessary to change the sign in front of a in the Eq. (27) and Eq. (28) and Eq. (29).

Fig. 10 illustrates the double points on the connecting coupler curve K_C , corresponding to symmetrical positions of the coupler triangle ΔABC in a four-bar linkage A_0ABB_0 . These points occur when the mechanism passes through two congruent positions: $\Delta ABC = \Delta A'B'C'$, where A, A' lie on the input crank circle α , and B, B' on the output crank circle β . At the intersection point C of these two configurations: $\angle ACB = \angle A'CB' = \gamma$ thus $\angle ACA' = \angle BCB'$, leading to, lines A_0C and B_0C acting as angle bisectors, so $\angle A_0CB_0 = \angle ABC = \gamma$, which confirms the focal property. The focal triangle $\Delta A_0B_0M_0$, drawn in green and circumscribed by the auxiliary circle k' , is similar to the coupler triangle ΔABC . This triangle defines the centers of motion symmetry (foci) for the coupler curve and confirms that A_0 , B_0 , and M_0 are the real foci of the sixth-order curve K_C . The coupler curve K_C , shown in magenta, passes through its double points (*e.g.*, C, C', C'') and exhibits the expected behavior of a tricircular sixth-order algebraic curve-featuring loops, inflections, and multiple real branches.

As demonstrated in Fig. 10, the double point C of the coupler curve K_C lies on the circle k' , which is circumscribed around the triangle formed by the three focal centers A_0 , B_0 , and M_0 . The chord A_0B_0 subtends the inscribed angle γ at point C , confirming the angle relationship: $\angle A_0CB_0 = \gamma$. This same circle k' passes through a third hinge point M_0 which, according to Roberts' Theorem,^[32] acts as the fixed pivot for the two alternative crank mechanisms that generate the same coupler curve. The presence of double points on k' leads to the following theorem:

Theorem 3: The double points of the crank (coupler) curve K_C lie on the circle k' , which circumscribes the triangle formed by the three focal centers A_0 , B_0 , M_0 . Conversely, every intersection of K_C with the circle k' represents a double point of the curve.

Since the coupler curve K_C is a sixth-order algebraic curve, and has two imaginary triple cyclic points I and I' at infinity, it can additionally intersect any circle in the finite plane (such

as k') in up to six finite points. At each such intersection with k' , the curve passes through the same point twice, *i.e.*, it has a self-intersection or double point. This leads us to:

Theorem 4: The connecting coupler curve K_C has three double points on the circle k' , which passes through the three focal centers A_0, B_0, M_0 . Of these three, two may be complex conjugate or may coincide at a point of self-contact of the curve.

Each real double point is either a double point in the proper sense of the word, *i.e.* a nodal point, such as the points C', C'', C''' (Fig. 10), or a cusp point in those cases where it coincides with the instantaneous pole, or an isolated point to which no real positions of the crank can correspond.^[33,34] For points C , lying on the crank line AB , the crank curve K_C is symmetrical about the midline of the column A_0B_0 . In this case, the third focal center M_0 and the three double points lie on the lines passing through A_0 and B_0 . The Eq. (30) of the circle k' , passing through the three focal centers A_0, B_0, M_0 , has the following form:

$$x^2 - kx + y^2 - ky \cdot \cot \gamma = 0 \quad (30)$$

This form reflects the constraint geometry of the focal triangle and confirms the alignment of double points with symmetry axes of the crank configuration.

Assuming that $a \neq 0, b \neq 0, \sin \gamma \neq 0$, and taking into account the Eq. (22), we obtain:

$$W = 0 \quad (31)$$

Thus, an analytical study of the crank curve shows that the Eq. (31) $U = 0, V = 0, W = 0$, each of which, in accordance

with the Eq. (19), is a consequence of the other two, giving double points of the crank curve.

4. Results and discussion

The synthesis of planar linkages for specified precision points was performed using graphical and computational methods. The generated results show the movement trajectories of the synthesized mechanisms, demonstrating their capability to pass through defined positions while maintaining structural integrity. The presented Fig. 11 configurations include solutions with three, four, and five precision points, allowing a comparison of the effectiveness of different synthesis approaches.^[35] Each synthesized linkage was designed to pass through a distinct set of three, four, or five precision points. Although the coupler curves differ across configurations due to their unique input data, each mechanism successfully interpolates its respective precision points, demonstrating the flexibility of the analytical method for path generation.

The results obtained for the three-, four-, and five-point synthesis problems each correspond to distinct sets of user-defined coupler positions. Rather than approximating the same trajectory with increasing complexity, each case represents a separate design task aimed at verifying the flexibility and correctness of the analytical synthesis method.

In the three-point configuration, the analytical formulation yielded a simple and mechanically feasible four-bar linkage (see Fig. 11). The resulting coupler curve smoothly interpolated the prescribed points, with low curvature variation and relatively uniform link proportions. This confirms the validity of the base case and establishes a

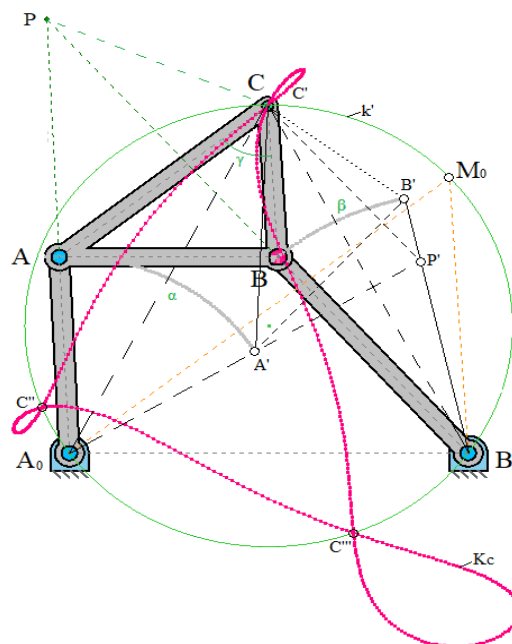


Fig. 10: Double points of the connecting coupler curve K_C located on the circle k' circumscribed about the focal triangle $\Delta A_0 B_0 M_0$, which is similar and congruent in orientation to the coupler triangle ΔABC . These double points occur at symmetric crank positions, where $\Delta ABC = \Delta A'B'C'$, and the lines A_0C and B_0C serve as angle bisectors. The foci A_0, B_0 , and M_0 define the triangle from which all such symmetrical configurations-and their shared coupler curve-can be analytically and geometrically derived.

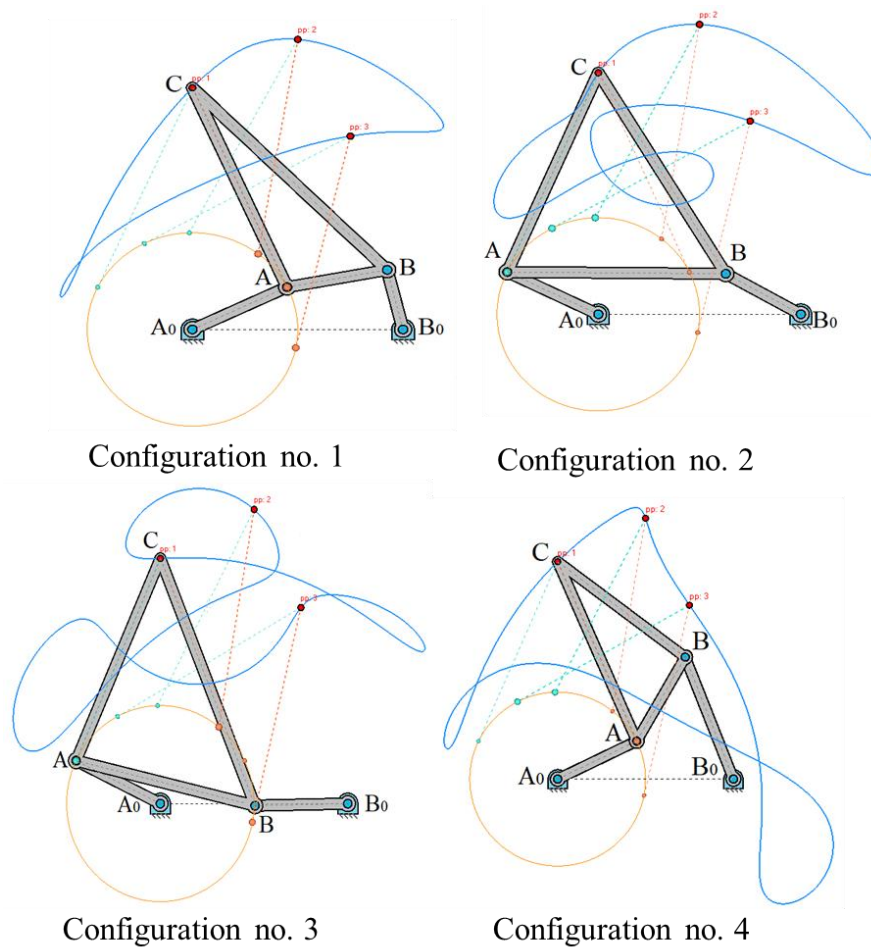


Fig. 11: Computed plot of synthesis mode: precision points (3 points).

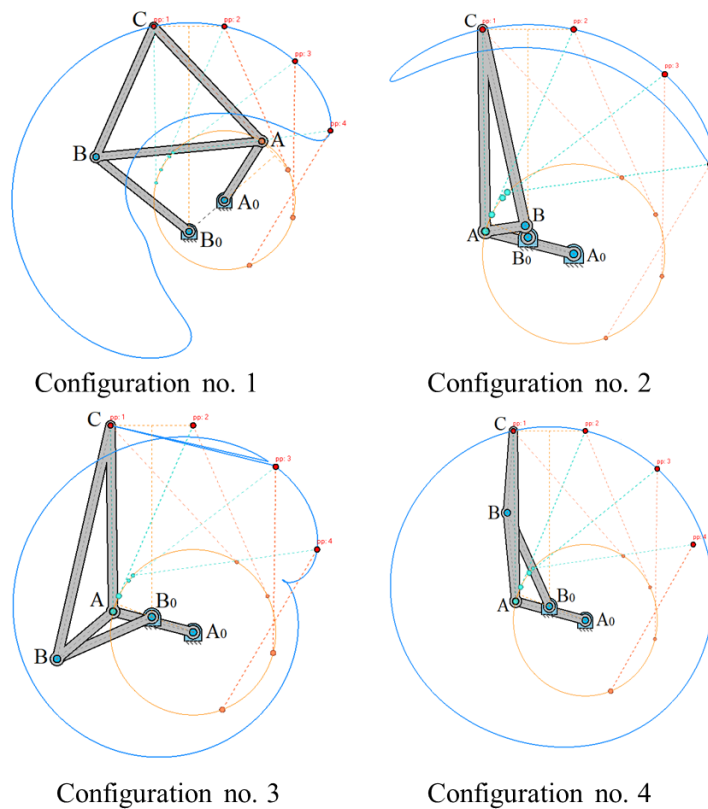


Fig. 12: Computed plot of synthesis mode: precision points (4 points).

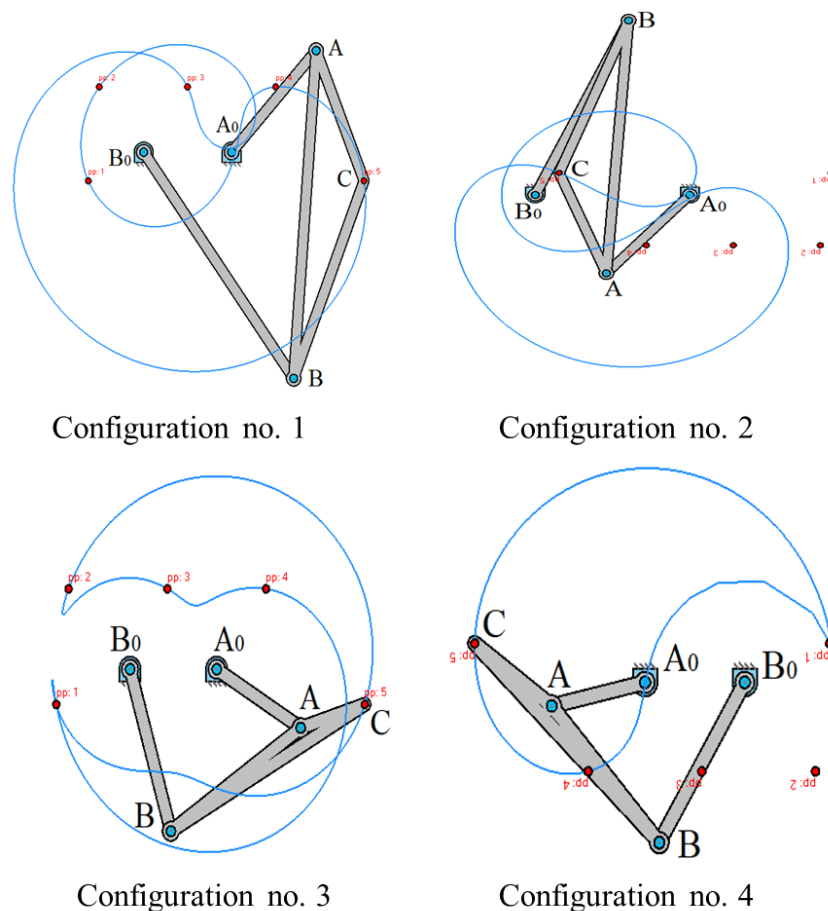


Fig. 13: Computed plot of synthesis mode: precision points (5 points).

reference for the method's baseline capability.

The four-point synthesis introduced an additional constraint, requiring the coupler curve to pass through a more complex set of target positions. The resulting linkage remained realizable, with no extreme link proportions or loss of mobility. The coupler curve continued to exhibit smoothness and fidelity to the input data, suggesting that the analytical method scales well under moderate geometric complexity. However, closer examination of the plotted trajectory shows increased sensitivity to crank rotation near singular configurations, which could impact dynamic performance.

In the five-point case, the synthesis required satisfying a more constrained geometric definition of the coupler curve. The resulting mechanism successfully interpolated all target points, but with noticeable geometric tightness in the linkage dimensions. One configuration exhibited very short coupler and rocker links relative to the ground span, which could pose practical issues related to manufacturing tolerances, bearing loads, and wear. Moreover, the coupler curve displayed increased curvature near the intermediate precision points, consistent with the expected geometric tension induced by over-specification.

Therefore, while the current method is valid for generating linkage dimensions that satisfy prescribed point sets analytically, it is recommended that future work integrate

post-synthesis checks such as transmission angle analysis or structural optimization to improve robustness.

The provided plots illustrate the synthesis mode for a four-bar linkage mechanism, focusing on precision point generation for coupler curve design.^[36] The synthesis aims to position the coupler point C, at three specified locations (marked as pp:1, pp:2, and pp:3) along a desired trajectory. The blue curve represents the coupler point's trajectory, showing how point C moves through the three prescribed precision points. The variations in the trajectory across different plots indicate different configurations and link lengths that satisfy the kinematic synthesis constraints. The mechanism undergoes reconfiguration as it attempts to pass through all three precision points. Some configurations show looping and additional crossings, which suggest alternative kinematic behaviors. The placement of the ground pivots (A_0 , B_0) and the link lengths significantly affect the shape of the coupler curve. In some plots, self-intersecting curves appear, which can indicate higher-order solutions or the presence of unintended dead positions. The ability to match three precision points ensures that the synthesized mechanism^[4] can achieve specific motion characteristics with minimal deviation. The similarity between the obtained coupler curves and classical Roberts-type solutions suggests that alternative four-bar mechanisms may generate similar paths. This allows for design flexibility where multiple linkage configurations can be

explored.

All four configurations demonstrate successful path generation through the specified precision points, validating the synthesis model based on the complex vector equation $\bar{Z} = \bar{r} + \bar{v}\bar{c}$. Despite all achieving the same positional constraints, the shape and nature of the coupler curves vary significantly across configurations:

- Some produce single-loop trajectories (e.g., Config. 1 and 4);
- Others exhibit self-intersections or loops, indicative of the inherent sixth-order algebraic nature of coupler curves (e.g., Config. 2 and 3).

A more constrained coupler path is generated, ensuring better adherence to the prescribed trajectory, as shown in Fig. 12. The additional precision point refines the shape of the coupler curve, making it more predictable and accurate. This configuration improves upon the three-point synthesis by reducing the error between specified positions. The trade-off in complexity and flexibility becomes evident, as more precision points introduce constraints on the linkage design. Computed trajectory plots for four synthesized four-bar linkage (configurations 1–4), each designed to ensure that the coupler point C passes through four prescribed precision points (pp:1–pp:4). The blue curve represents the traced path of point C, while the orange circle denotes the crank's circular motion at the grounded pivot A_0 .

The five-point synthesis significantly increases the accuracy of the coupler curve, as shown in Fig. 13, ensuring that it passes through all designated precision points with minimal deviation.^[13] The computed path exhibits a high level of fidelity to the intended motion.^[37] However, the increased number of precision points reduces the flexibility in choosing link dimensions, potentially leading to extreme link ratios or impractical designs. The computational complexity also increases, requiring more sophisticated numerical optimization techniques. The exact coordinates are provided in Table 1 and used as input for the trajectory synthesis

equations.

To evaluate the effectiveness of the proposed analytical method, three, four, and five-point synthesis configurations were simulated using the derived coupler curve equations.^[33] Four mechanism designs were created for each level of synthesis, and the generated paths were compared with the target precision points. Quantitative analysis was performed using root mean square error (RMSE) between the generated coupler trajectory and the desired precision point positions. Results showed that increasing the number of precision points improves path conformity, but also increases mechanical complexity and sensitivity to dimensional tolerances. Table 2 summarizes the error values for selected configurations. The 5-point synthesis showed the lowest RMSE (0.48 mm), indicating high accuracy in matching the desired trajectory. However, it also required longer or more constrained links, as evidenced by higher link ratios and reduced workspace flexibility.

Fig. 11 through Fig. 13 illustrates the motion paths for each synthesized linkage. While all configurations successfully pass through the designated precision points, only the higher-order configurations (4- and 5-point) generate curves with sufficient smoothness and velocity consistency for motion-sensitive applications, such as robotic arm trajectories. Additionally, the presence of double points was confirmed through focal circle construction, validating that the analytical equations capture self-intersections and repeated positions of the coupler, which are important for designing dwell or pause mechanisms. Overall, the results confirm that the proposed analytical method is effective for practical synthesis of four-bar mechanisms, offering trade-offs between accuracy, complexity, and feasibility.

The five-point synthesis exhibits higher accuracy but significantly reduced geometric flexibility because the sixth-order coupler curve must satisfy ten independent constraints associated with five precision points. This over-specifies the

Table 1: Key design parameters used in the simulations are.

Parameter	Symbol	Value (Example)
Crank length	a	25 mm
Coupler length	b	60 mm
Rocker length	c	40 mm
Fixed base	d	70 mm
Input angle range	θ_1	0°-360°
Point C location on coupler	(b', h)	b'=30 mm from A

Table 2: Coupler curve error analysis.

Configuration	Point	RMSE (mm)	Max Error (mm)	Remarks
Configuration 1	3	1.85	3.12	Simple, but moderate deviation
Configuration 2	4	1.02	2.08	Balanced accuracy and simplicity
Configuration 3	5	0.48	1.10	High precision, less flexibility

curve relative to its remaining degrees of freedom, forcing the algebraic solution toward extreme values of curvature and inflection. As a consequence, the resulting four-bar linkage parameters become ‘tight,’ often yielding near-singular or highly sensitive link dimensions. This behavior is consistent with known properties of sixth-order Burmester curves and the geometric limitations imposed when multiple precision points cluster near focal centers or double points. Thus, the increased accuracy of the five-point design comes at the cost of reduced manufacturability and mechanism robustness.

5. Conclusion

This study presented an analytical framework for the coupler-curve-based synthesis of planar four-bar linkages designed to interpolate a given set of precision points. The approach focused on deriving the sixth-order algebraic curve traced by a coupler point and determining appropriate linkage dimensions to generate accurate motion paths. A key contribution of this work is the integration of Roberts’ theorem and focal circle analysis to identify double points and construct alternative linkage configurations (cognates) that trace identical coupler curves. This not only supports design flexibility but also offers insight into trajectory self-intersections and motion repeatability-critical in dwell mechanisms and motion-sensitive applications.

The method was applied to synthesize coupler curves that passed through three, four, and five specified precision points. For each configuration:

Multiple mechanisms were evaluated graphically;

The trajectory of the coupler point was compared to the target precision points;

Error metrics (e.g., RMSE) were used to quantify the accuracy of the synthesized paths.

The results demonstrated a clear trade-off between path fidelity and mechanism complexity:

Three-point synthesis offers a simple solution with moderate path approximation;

Four-point synthesis strikes a balance between accuracy and design feasibility;

Five-point synthesis provides the most precise path tracking but introduces mechanical constraints and increased sensitivity to link dimensions.^[19]

While the current framework is analytical and geometric in nature, it sets the foundation for future integration with numerical optimization techniques. This would enable automated refinement of linkage parameters and extend the synthesis process to include velocity and acceleration constraints or workspace limitations.

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Conflict of Interest

There is no conflict of interest.

Supporting Information

Not applicable.

CRedit Statement

Zhastalap Abilkaiyr: Investigation, Writing - Original draft Preparation, Visualization, Project administration. **Algazy Zhauyt:** Conceptualization, Methodology, Software, Writing - Original draft preparation, Supervision, Project administration. **Kuanysh Alipbayev:** Validation, Writing-Original draft preparation, Funding acquisition. **Yerbol Kaliyev:** Resources, Writing - Review and editing, Funding acquisition. **Nazgul Kaliyeva and Nurgul Kurmangaliyeva:** Resources, Formal analysis, Writing - Original draft preparation, Project administration.

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