PROBLEMS AND SOLUTIONS IN MODERN PHYSICS

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Сборник задач предназначен для самостоятельного овладения методами решения типовых задач по заключительным разделам курса общей физики для будущих бакалавров техники и технологий: специальной теории относительности, основам квантовой теории, атомной физики, физики молекул и конденсированного состояния, физики ядра и элементарных частиц, астрофизики и космологии. Книга содержит свыше 700 задач, разнообразных как по содержанию, так и по трудности. Наряду с задачами, иллюстрирующими основные понятия и законы и относящимися к основному обязательному курсу, в сборник включены и более сложные задачи, помогающие углубленному изучению предмета.

Учебное пособие составлено с учетом Государственных стандартов технических специальностей бакалавриата и рассчитано на студентов технических вузов и преподавателей общей физики.

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Cover picture: Computer-simulated production of a black hole in ATLAS. This track is an example of simulated data modeled for the ATLAS detector on the Large Hadron Collider at CERN, which began taking data in 2009. These tracks would be produced if a miniature black hole was created in the proton-proton collision. Such a small black hole would decay instantly to various particles via a process known as Hawking radiation. (*Courtesy of CERN*)

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Preface

Problems worthy of attack,
prove their worth by hitting back.

Piet Hein,
Danish inventor (1905–1996)

Learning how to approach and solve problems is a basic part of any physics course. It is a highly useful skill in itself, but it is also important because the process helps bring understanding of the physics.

This book is intended as a complement to any standard calculus-based general physics course for undergraduate science and engineering majors. It is not just a collection of problems. Neither it is a kind of a textbook. The purpose of this book is to show you how to do physics problems. It will show you how to work the most typical problems and explain why they are being done the way they are.

The book covers the material that is traditionally called "modern" physics in English-speaking community—the topics centered on relativity and quantum physics. These topics usually constitute closing chapters of most general physics courses. It hardly seems possible to underestimate these topics in view of their importance for today's technology and, more critically, for understanding contemporary world. Many of the ideas of modern physics are not mathematically difficult. However, they can be nonintuitive, and we think it is important that students begin to develop intuition about this material as early as possible.

Each chapter begins with a concise theoretical introduction intended to remind fundamental concepts and the relevant formulae to the reader. (By no means this brief discussion of basic ideas is substitute of a regular text!) Theory is been interwoven with concrete worked Examples. The Examples follow the development of the theory. They address the needs of any student who has ever felt that they "understand the concept but cannot do the problems." The Examples help students to see how to begin with a seemingly complex situation, identify the relevant physical concepts, decide what tools are needed to solve the problem, carry out the solution, and then evaluate whether the result makes sense.

The Problems are separated into two categories. The first group of Problems is organized by section. Section implies a specific topic. Within each section Problems are arranged in the order of increasing difficulty. First come Problems that are rather simple, single-concept, designed to give students confidence. Then follow problems providing more of a challenge and often the combination of two different concepts, and the most complex, typically demanding significant synthesis of concepts in the text. The second group, called General Problems, resembles the situations that are met in real-life science and engineering. They also help to develop the student's appreciation of the links that exist throughout physics as well as how to approach problems for which the clues may be more obscure. General Problems also have gradation in difficulty and most difficult of them are intended to challenge the strongest students.

A few words about uncertainty of physical quantities and significant figures. No measurement is absolutely precise. There is an uncertainty associated with every measurement. Uncertainty arises from different sources. Among the most important, other than blunders, are the limited accuracy of every measuring instrument and the inability to read an instrument beyond some fraction of the smallest division shown. Physical data are reported with the associated precision, or estimated uncertainty. For example, the width of a board might be written as 8.8 ± 0.1 cm. The ± 0.1 cm represents the estimated uncertainty in the measurement, so that the actual width most likely lies between 8.7 and 8.9 cm.

Often the uncertainty in a measured value is not specified explicitly. In such cases, the uncertainty is generally assumed to be one unit in the last digit specified. For example if a length is given as 8.8 cm, the uncertainty is assumed to be about 0.1 cm.

The number of reliably known digits in a number is called the number of *significant figures*. Thus there are four significant figures in the number 23.21 cm and two in the number 0.062 cm (the zeros in the latter are merely place holders that show where the decimal point goes). The number of significant figures may not always be clear. Take, for example, the number 80. Are there one or two significant figures? If we say it is about 80 km between two cities, there is only one significant figure (the 8) since the zero is merely a place holder. If it is exactly 80 km, within an accuracy

of ± 1 km, then the 80 has two significant figures.[1] If it is precisely 80 km, to within ± 0.1 km, then we write 80.0 km.

When making measurements, or when doing calculations, you should avoid the temptation to keep more digits in the final answer than is justified. For example, to calculate the area of a rectangle 11.3 cm by 6.8 cm, the result of multiplication would be 76.84 cm^2. But this answer is clearly not accurate to ± 0.01 cm^2, since (using the outer limits of the assumed uncertainty for each measurement) the result could be between $11.2 \times 6.7 = 75.04$ cm^2 and $11.4 \times 6.9 = 78.66$ cm^2. At best, we can quote the answer as 77 cm^2, which implies an uncertainty of about ± 1 cm^2. The other two digits (8 and 4 in the number 76.84 cm^2) must be dropped since they are not significant. As a rough general rule, (i. e., in the absence of a detailed consideration of uncertainties), we can say that: *the final result of a multiplication or division should have only as many digits as the number with the least number of significant figures used in the calculation.* In our example, 6.8 cm has the least number of significant figures, namely two. Thus the result 76.84 cm^2 needs to be rounded off to 77 cm^2.

When adding or subtracting numbers, it is the location of the decimal point that matters, not the number of significant figures. For example, the result of adding 123.62 to 8.9 is 132.5 (and not 132.52). Although 123.62 has an uncertainty of about 0.01, 8.9 has an uncertainty of about 0.1. So their sum has an uncertainty of about 0.1 and should be written as 132.5.

Sometimes, quantities are assumed to be known exactly, i. e. with an infinite number of significant figures. For instance, the numbers as 100 centimeters in 1 meter, or 60 minutes in 1 hour are exact quantities by virtue of their definition. Another examples are mathematical constants like $\pi = 3.1415926535\ldots$ and $e = 2.718281828\ldots$. Such quantities do not affect the accuracy of your calculations that is determined by the quantities with finite number of significant figures.

Keep in mind when you use a calculator that all the digits it produces may not be significant. When you divide 2.0 by 3.0, the proper answer is 0.67, and not some such thing as 0.66666666. Digits should not be quoted

[1] If the 80 has two significant figures, some people prefer to write it 80., with a decimal point. This is not always (or even usually) done, so the number of significant figures in 80 can be ambiguous unless something is said about it, such as "about" or "precisely".

(or written down) in a result, unless they are truly significant figures. However, to obtain the most accurate result, you should normally keep an extra significant figure or two throughout a calculation, and round off only in the final result. Note also that calculators sometimes give too few significant figures. For example, when you multiply 2.5×3.2, a calculator may give the answer as simply 8. But the answer is good to two significant figures, so the proper answer is 8.0.

We commonly write numbers in "powers-of-ten," or "scientific" notation —for instance 36 900 as 3.69×10^4, or 0.0021 as 2.1×10^{-3}. One advantage of scientific notation is that it allows the number of significant figures to be clearly expressed. For example, it is not clear whether 36 900 has three, four, or five significant figures. With powers of ten notation the ambiguity can be avoided: if the number is known to an accuracy of three significant figures, we write 3.69×10^4, but if it is known to four, we write 3.690×10^4.

We are sometimes interested only in an approximate value for a quantity. This might be because an accurate calculation would take more time than it is worth or would require additional data that are not available. In other cases, we may want to make a rough estimate in order to check an accurate calculation made on a calculator, to make sure that no blunders were made when entering the numbers. A rough estimate is made by rounding off all numbers to one significant figure and its power of 10, and after the calculation is made, again, only one significant figure is kept. Such an estimate is called an *order-of-magnitude estimate* and can be accurate within a factor of 10, and often better. In fact, the phrase "order of magnitude" is sometimes used to refer simply to the power of 10.

It is the sincere desire of the author that this book help you to better understand physical concepts and work the associated problems. Comments and corrections are welcome.

A. Mustafin.

1 Special Theory of Relativity

1.1 Review

An *inertial reference frame* is one in which Newton's law of inertia holds. Inertial reference frames can move at constant velocity relative to one another; accelerating reference frames are noninertial.

The *special theory of relativity* is based on two principles: the *relativity principle*, which states that the laws of physics are the same in all inertial reference frames, and the principle of the *constancy of the speed of light*, which states that the speed of light in empty space has the same value in all inertial reference frames.

One consequence of relativity theory is that two events that are simultaneous in one reference frame may not be simultaneous in another. Other effects are *time dilation*: moving clocks are measured to run slow; and *length contraction*: the length of a moving object is measured to be shorter (in its direction of motion) than when it is at rest. Quantitatively,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}\,; \tag{1.1}$$

$$L = L_0\sqrt{1 - v^2/c^2}\,, \tag{1.2}$$

where Δt and L are the time interval and length of objects (or events) observed as they move by at the speed v, Δt_0 and L_0 are the *proper time* and *proper length*—that is, the same quantities as measured in the rest frame of the objects or events.

The quantities v/c and $1/\sqrt{1 - v^2/c^2}$ in Eqs. 1.1 and 1.2 appear so often in relativity that they are given their own symbols β and γ respectively:

$$\beta = \frac{v}{c}\,; \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{1.3}$$

In terms of these symbols, we can express Eqs. 1.1 and 1.2 as

$$\Delta t = \gamma \Delta t_0 \quad \text{and} \quad L = \frac{L_0}{\gamma}.$$

Example 1–1. (a) What will be the mean lifetime of a muon as measured in the laboratory if it is traveling at $v = 0.60\,c = 1.8 \times 10^8$ m/s with respect to the laboratory? Its mean lifetime at rest is 2.2×10^{-6} s. (b) How far does a muon travel in the laboratory, on average, before decaying?

SOLUTION. (a) If an observer were to move along with the muon (the muon would be at rest with this observer), the muon would have a mean life of 2.2×10^{-6} s. To an observer in the lab, the muon lives longer because of the time dilation. From Eq. 1.1 with $v = 0.60\,c$, we have

$$\Delta t = \gamma \Delta t_0 = \frac{2.2 \times 10^{-6}\ \text{s}}{\sqrt{1 - (0.36\,c^2)/c^2}} = \frac{2.2 \times 10^{-6}\ \text{s}}{\sqrt{0.64}} = 2.8 \times 10^{-6}\ \text{s}.$$

(b) At a speed of 1.8×10^8 m/s, classical physics would tell us that with a men life of 2.2 μs, an average muon would travel $d = vt = (1.8 \times 10^8\ \text{m/s})(2.2 \times 10^{-6}\ \text{s}) = 400$ m. But relativity predicts an average distance of $d = vt = (1.8 \times 10^8\ \text{m/s})(2.8 \times 10^{-6}\ \text{s}) = 500$ m, and it is this longer distance that is measured experimentally.

Example 1–2. Let's check time dilation for everyday speeds. A car traveling 100 km/h covers a certain distance in 10.0 s according to the driver's watch. What does an observer on Earth measure for the time interval?

SOLUTION. The car's speed relative to Earth is 100 km/h = $(1.00 \times 10^5\ \text{m/s})(3\,600\ \text{s}) = 27.8$ m/s. We set $\Delta t_0 = 10.00$ s in the time-dilation formula (the driver is at rest in the reference frame of the car), and then Δt is

$$\begin{aligned} \Delta t &= \gamma \Delta t_0 = \frac{10.00\ \text{s}}{\sqrt{1 - (27.8\ \text{m/s})^2/(3.00 \times 10^8\ \text{m/s})^2}} \\ &= \frac{10.00\ \text{s}}{\sqrt{1 - 8.59 \times 10^{-15}}}\,. \end{aligned}$$

If you put these numbers into a calculator, you will obtain $\Delta t = 10.00$ s, since the denominator differs from 1 by such a tiny amount. Indeed, the time measured by an observer on Earth would be no different from that measured by the driver, even with the best of today's instruments. A computer that could calculate to a large number of decimal places could reveal a difference between Δt and Δt_0. But we can estimate the difference

quite easily using the binomial expansion,

$$(1 \pm x)^n \approx 1 \pm nx,$$

valid for $x \ll 1$. In our time-dilation formula, we have factor $\left(1-\beta^2\right)^{-1/2}$. Thus

$$\begin{aligned} \Delta t &= \Delta t_0 \left(1-\beta^2\right)^{-1/2} \approx \Delta t_0 \left(1+\tfrac{1}{2}\beta^2\right) \\ &= (10.00 \text{ s}) \left(1 + \frac{1}{2}\left(\frac{27.8 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}}\right)^2\right) \\ &\approx 10.00 \text{ s} + 4 \times 10^{-15} \text{ s}. \end{aligned}$$

So the difference between Δt and Δt_0 is predicted to be 4×10^{-15} s, an extremely small amount.

Example 1–3. A rectangular painting measures 1.00 m tall and 1.50 m wide. It is hung on the side wall of a spaceship which is moving past the Earth at a speed of $0.90\,c$. (a) What are the dimensions of the picture according to the captain of the spaceship? (b) What are the dimensions as seen by an observer on the Earth?

Solution. (a) The painting (as well as everything else in the spaceship) looks perfectly normal to everyone on the spaceship, so the captain sees a 1.00 m by 1.50 m painting.

(b) Only the dimension in the direction of motion is shortened, so the height is unchanged at 1.00 m. The length, however, is contracted to

$$L = \frac{L_0}{\gamma} = (1.50 \text{ m})\sqrt{1-(0.90)^2} = 0.65 \text{ m}.$$

So the picture has dimensions 1.00 m × 0.65 m.

Consider two reference frames S and S' which are each characterized by a set of coordinate axes, Fig. 1.1. The axes x and y (z is not shown) refer to S and x' and y' to S'. Now consider an event that occurs at point P represented by the coordinates x', y', z' in reference frame S' at the time t'. Since time is assumed to be absolute in Galilean–Newtonian physics, clocks in the two frames agree with each other and the coordinates of P in S are given by the *Galilean transformation equations*:

$$\begin{aligned} x &= x' + vt' \quad & y &= y' \\ z &= z' & t &= t'. \end{aligned} \tag{1.4}$$

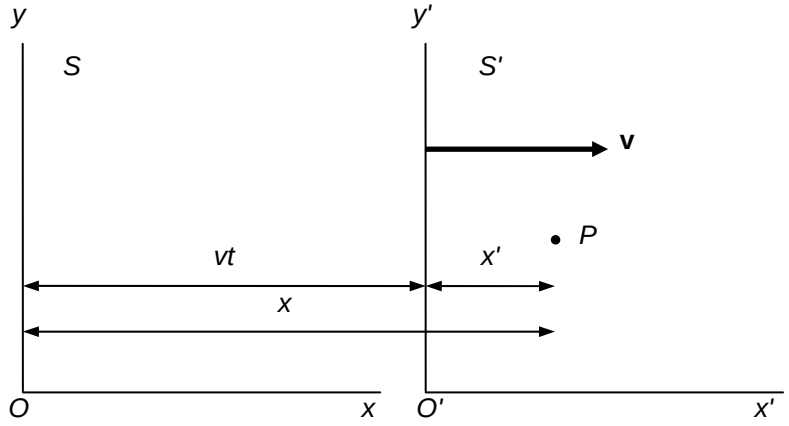


Figure 1.1. Inertial reference frame S' moves to the right at speed v with respect to reference frame S.

Now suppose the point P in Fig. 1.1 is moving. Let the components of its velocity vector in S' be $u'_x = dx'/dt'$, $u'_y = dy'/dt'$, $u'_z = dz'/dt'$. The velocity of P as seen from S will obey the *Galilean velocity transformation equations*:

$$u_x = u'_x + v \quad u_y = u'_y \quad u_z = u'_z. \tag{1.5}$$

We see that the y and z components are unchanged, but the x components differ by v.

In the theory of relativity, the Lorentz transformations relate the positions and times of events in one inertial reference frame to their positions and times in a second inertial reference frame.

$$\begin{aligned} x &= \gamma(x' + vt') \quad & y &= y' \\ z &= z' & t &= \gamma\left(t' - \frac{vx'}{c^2}\right). \end{aligned} \tag{1.6}$$

Velocity addition also must be done in a special way:

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + vu'_x/c^2} \\ u_y &= \frac{u'_y}{\gamma\,(1 + vu'_x/c^2)} \\ u_z &= \frac{u'_z}{\gamma\,(1 + vu'_x/c^2)}. \end{aligned} \tag{1.7}$$

Note that even though the relative velocity $\mathbf{v}$ is in the x direction, the transformation of all the components of a particle's velocity are affected by v and the x component of the particle's velocity; this was not true for the Galilean transformation, Eqs. 1.5.

All these relativistic effects are significant only at high speeds, close to the speed of light, which itself is the ultimate speed in the universe.

Example 1–4. Derive the length contraction formula, Eq. 1.2, from the Lorentz transformation equations.

SOLUTION. Let an object of length L_0 be at rest on the x axis in S. The coordinates of its two end points are x_1 and x_2, so that $x_2 - x_1 = L_0$. At any instant in S', the end points will be at x'_1 and x'_2 as given by the Lorentz transformation equations. The length measured in S' is $L = x'_2 - x'_1$. An observer in S' measures this length by measuring x'_2 and x'_1 at the same time (in the S' reference frame), so $t'_2 = t'_1$. Then, from the first of Eqs. 1.6,

$$L_0 = x_2 - x_1 = \gamma\,(x'_2 + vt'_2 - x'_1 - vt'_1)\,.$$

Since $t'_2 = t'_1$, we have

$$L_0 = \gamma\,(x'_2 - x'_1) = \gamma L,$$

or

$$L = \frac{L_0}{\gamma},$$

which is Eq. 1.2.

Example 1–5. Derive the time-dilation formula, Eq. 1.1, using the Lorentz transformation equations.

SOLUTION. The time Δt_0 between two events that occur at the same place ($x'_2 = x'_1$) in S' is measured to be $\Delta t_0 = t'_2 - t'_1$. Since $x'_2 = x'_1$, then

from the last of Eqs. 1.6, the time Δt between the events as measured in S is

$$\begin{aligned}\Delta t &= t_2 - t_1 = \gamma\left(t_2' + vx_2'/c^2 - t_1' - vx_1'/c^2\right)\\ &= \gamma\,(t_2' - t_1') = \gamma\,\Delta t_0,\end{aligned}$$

which is Eq. 1.1. Notice that we chose S' to be the frame in which the two events occur at the same place, so that $x_1' = x_2'$ and the terms containing x_1' and x_2' cancel out.

Example 1–6. Rocket 1 has speed $v = 0.60\,c$ with respect to Earth. Rocket 2 is fired from rocket 1 with speed $u' = 0.60\,c$. The velocities are along the same straight line. What is the speed of rocket 2 with respect to the Earth?

Solution. We take the direction of motion of the two rockets to be the x (and x') axis. We need use only the first of Eqs. 1.7. Then the speed of rocket 2 with respect to Earth is

$$u = \frac{u' + v}{1 + vu'/c^2} = \frac{0.60\,c + 0.60\,c}{1 + (0.60)^2} = \frac{1.20\,c}{1.36} = 0.88\,c.$$

(The Galilean transformation would have given $u = 1.20\,c$.)

The theory of relativity has changed our notions of space and time, and of momentum, energy, and mass. Space and time are seen to be intimately connected, with time being the fourth dimension in addition to the three dimensions of space.

The *momentum* of an object is given by

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma m\mathbf{v}. \tag{1.8}$$

This formula can be interpreted as a *mass increase*, where the *relativistic mass* is

$$m_{\text{rel}} = \gamma m \tag{1.9}$$

and m is the *rest mass* of the object ($v = 0$).

Mass and energy are interconvertible. The equation

$$E = mc^2 \tag{1.10}$$

tells how much energy E is needed to create a mass m, or vice versa. Said another way, $E = mc^2$ is the amount of energy an object has because of its mass m. The law of conservation of energy must include mass as a form of energy.

The kinetic energy K of an object moving at speed v is given by

$$K = (\gamma - 1)mc^2 = \frac{mc^2}{1 - v^2/c^2} - mc^2, \tag{1.11}$$

where m is the rest mass of the object. The total energy E is

$$E = K + mc^2 = \gamma mc^2. \tag{1.12}$$

The momentum p of an object is related to its total energy E (assuming no potential energy) by

$$E^2 = p^2c^2 + m^2c^4. \tag{1.13}$$

Since the rest mass m of a given particle is the same in any reference frame, the quantity $E^2 - p^2c^2$ must also be the same in any reference frame. We say that the quantity $E^2 - p^2c^2$ is *invariant* under a Lorentz transformation.

Example 1–7. A π^0 meson ($m = 2.4 \times 10^{-28}$ kg) travels at a speed $v = 0.80\,c = 2.4 \times 10^8$ m/s. What is its kinetic energy? Compare to a classical calculation.

Solution. We substitute values into Eq. 1.11

$$K = (\gamma - 1)mc^2$$

where

$$\gamma = \frac{1}{1 - v^2/c^2} = \frac{1}{\sqrt{1 - (0.80)^2}} = 1.67.$$

Then

$$K = (1.67 - 1)(2.4 \times 10^{-28}\ \text{kg})(3.00 \times 10^8\ \text{m/s})^2 = 1.4 \times 10^{-11}\ \text{J}.$$

Notice that the units of mc^2 are $\text{kg} \cdot \text{m}^2/\text{s}^2$, which is the joule. A classical calculation would give

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(2.4 \times 10^{-28}\ \text{kg})(2.4 \times 10^8\ \text{m/s})^2 = 6.9 \times 10^{-12}\ \text{J},$$

about half as much, but this is not a correct result.

Example 1–8. The energy required or released in nuclear reactions and decays comes from a change in mass between the initial and final particles. In one type of radioactive decay, an atom of uranium ($m = 232.03714$ u) decays to an atom of thorium ($m = 228.02873$ u) an atom of helium ($m = 4.00260$ u) where the masses (always rest masses) given are in atomic mass units ($1\text{ u} = 1.6605 \times 10^{-27}$ kg). Calculate the energy released in this decay.

SOLUTION. The initial mass is 232.03714 u, and after the decay it is 228.02873 u + 4.00260 u = 232.03133 u, so there is a decrease in mass of 0.00581 u. This mass decrease, which equals $(0.00581\text{ u})(1.66 \times 10^{-27}\text{ kg}) = 9.64 \times 10^{-30}$ kg, is changed into kinetic energy. Thus

$$E = \Delta mc^2 = (9.64 \times 10^{-30}\text{ kg})(3.0 \times 10^8\text{ m/s})^2 = 8.68 \times 10^{-13}\text{ J}.$$

Since $1\text{ MeV} = 1.60 \times 10^{-13}$ J, the energy released is 5.4 MeV.

Example 1–9. When two moles of hydrogen and one mole of oxygen react to form two moles of water, the energy released is 484 kJ. How much does the mass decrease in this reaction?

SOLUTION. Using Eq. 1.10 we have for the change in mass Δm:

$$\Delta m = \frac{\Delta E}{c^2} = \frac{-484 \times 10^3\text{ J}}{(3.00 \times 10^8\text{ m/s})^2} = -5.38 \times 10^{-12}\text{ kg}.$$

The initial mass of the system is 0.002 kg + 0.016 kg = 0.018 kg. Thus the change in mass is relatively very tiny and can normally be neglected. Conservation of mass is usually a reasonable principle to apply to chemical reactions.

It is known from the classical theory of waves that the frequency and wavelength of sound are altered if the source of the sound and the observer are moving toward or away from each other. There are four different equations for the Doppler shift depending on the direction of the relative motion and whether the source or the observer is moving. The Doppler effect occurs also for light; but the shifted frequency or wavelength is given by slightly different equations, and there are only two of them, because for light—according to special relativity—we can make no distinction between motion of the source and motion of the observer. Recall that sound travels

in a medium such as air, whereas light does not—there is no evidence for an ether!

Suppose a source at rest emits light waves of frequency f_0 and wavelength $\lambda_0 = c/f_0$. If the source and the observer move toward each other with a relative speed v, the observer sees the wavelength λ being somewhat less than λ_0:

$$\lambda = \lambda_0\sqrt{\frac{c-v}{c+v}} = \lambda_0\sqrt{\frac{1-\beta}{1+\beta}} \quad \begin{pmatrix}\text{source and observer moving}\\ \text{toward each other}\end{pmatrix}. \tag{1.14}$$

For relative motion away from each other we have

$$\lambda = \lambda_0\sqrt{\frac{c+v}{c-v}} = \lambda_0\sqrt{\frac{1+\beta}{1-\beta}} \quad \begin{pmatrix}\text{source and observer moving}\\ \text{away from each other}\end{pmatrix}. \tag{1.15}$$

In the latter case, visible light will have its wavelength lengthened toward the red end of the visible spectrum, an effect called a *redshift*.

1.2 Simultaneity. Time dilation. Length contraction

1–1. Lengths and time interval depend on the factor $\sqrt{1-\beta^2}$ according to the theory of relativity (Eqs. 1.1 and 1.2). Evaluate this factor for speeds of: (a) 20 000 m/s (typical speed of a satellite); (b) $0.0100\,c$; (c) $0.100\,c$; (d) $0.900\,c$; $0.990\,c$; $0.999\,c$.

Solution.

(a) $\sqrt{1-\beta^2} = \sqrt{1-((20\,000\text{ m/s})/(3.00\times10^8\text{ m/s}))^2} = 1.00$;
(b) $\sqrt{1-\beta^2} = \sqrt{1-(0.0100)^2} = 0.99995$;
(c) $\sqrt{1-\beta^2} = \sqrt{1-(0.100)^2} = 0.995$;
(d) $\sqrt{1-\beta^2} = \sqrt{1-(0.900)^2} = 0.436$;
(e) $\sqrt{1-\beta^2} = \sqrt{1-(0.990)^2} = 0.141$;
(f) $\sqrt{1-\beta^2} = \sqrt{1-(0.999)^2} = 0.0447$.

1–2. A spaceship passes you at a speed of $0.750\,c$. You measure its length to be 28.2 m. How long would it be when at rest?

SOLUTION. You measure the contracted length. We find the rest length from

$$\begin{aligned} L &= \frac{L_0}{\gamma}; \\ 28.2\text{ m} &= L_0\sqrt{1-(0.750)^2}, \end{aligned}$$

which gives $L_0 = 42.6$ m.

1–3. A certain type of elementary particle travels at a speed of 2.70×10^8 m/s. At this speed, the average lifetime is measured to be 4.76×10^{-6} s. What is the particle's lifetime at rest?

SOLUTION. We find the lifetime at rest from

$$\begin{aligned} \Delta t &= \gamma\,\Delta t_0; \\ 4.76 \times 10^{-6}\text{ s} &= \frac{\Delta t_0}{\sqrt{1-((2.70 \times 10^8\text{ m/s})/(3.00 \times 10^8\text{ m/s}))^2}}, \end{aligned}$$

which gives $\Delta t_0 = 2.07 \times 10^{-6}$ s.

1–4. If you were travel to a star 100 light-years from Earth at a speed of 2.50×10^8 m/s, what would you measure this distance to be?

SOLUTION. You measure the contracted length:

$$L = \frac{L_0}{\gamma} = (100\text{ ly})\sqrt{1-\left(\frac{2.50 \times 10^8\text{ m/s}}{3.00 \times 10^8\text{ m/s}}\right)^2} = 55.3\text{ ly}.$$

1–5. What is the speed of a pion if its average lifetime is measured to be 4.10×10^{-8} s? At rest, its mean lifetime is 2.60×10^{-8} s.

SOLUTION. We determine the speed from the time dilation:

$$\begin{aligned} \Delta t &= \gamma\,\Delta t_0; \\ 4.10 \times 10^{-8}\text{ s} &= \frac{(2.60 \times 10^{-8}\text{ s})}{\sqrt{1-\beta^2}}, \end{aligned}$$

which gives $v = 0.773\,c$.

1–6. Suppose you decide to travel to a star 75 light-years away. How fast would you have to travel so the distance would be only 25 light-years?

SOLUTION. We determine the speed from the length contraction:

$$\begin{aligned} L &= \frac{L_0}{\gamma}; \\ 25 \text{ ly} &= (75 \text{ ly})\sqrt{1-\beta^2}, \end{aligned}$$

which gives $v = 0.94\,c$.

1–7. At what speed do the relativistic formulas for length and time intervals differ from classical values by 1.00 percent? (This is a reasonable way to estimate when to do relativistic calculations rather than classical.)

SOLUTION. For a 1.00 per cent change, the factor in the expressions for time dilation and length contraction must equal $1 - 0.0100 = 0.9900$:

$$\sqrt{1-\beta^2} = 0.9900,$$

which gives $v = 0.141\,c$.

1–8. Suppose a news report stated that starship *Enterprise*[1] had just returned from a 5-year voyage while traveling at $0.84\,c$. (a) If the report meant 5.0 years of *Earth time*, how much time elapsed on the ship? (b) If the report meant 5.0 years of *ship time*, how much time passed on Earth?

SOLUTION. In the Earth frame, the clock on the *Enterprise* will run slower.

(a) We find the elapsed time on the ship from

$$\begin{aligned} \Delta t &= \gamma\,\Delta t_0; \\ 5.0 \text{ yr} &= \frac{\Delta t_0}{\sqrt{1-(0.84)^2}}, \end{aligned}$$

which gives $\Delta t_0 = 2.7$ yr.

(b) We find the elapsed time on the Earth from

$$\Delta t = \gamma\,\Delta t_0 = \frac{(5.0 \text{ yr})}{\sqrt{1-(0.84)^2}} = 9.2 \text{ yr}.$$

1–9. A certain star is 95.0 light-years away. How long would it take a spacecraft traveling $0.960\,c$ to reach that star from Earth, as measured

[1] A fictional starship in the *Star Trek*, a famous science fiction television series of the late 1960s, that later achieved iconic status as a worldwide television phenomenon.

by observers: (a) on Earth, (b) on the spacecraft? (c) What is the distance traveled according to observers on the spacecraft? (d) What will the spacecraft occupants compute their speed to be from the results of (b) and (c)?

SOLUTION. (a) To an observer on Earth, 95.0 ly is the rest length, so the time will be

$$t_{\text{Earth}} = \frac{L_0}{v} = \frac{(95.0\text{ ly})}{(0.960\,c)} = 99.0\text{ yr}.$$

(b) We find the dilated time on the spacecraft from

$$\begin{aligned} \Delta t &= \gamma\,\Delta t_0; \\ 99.0\text{ yr} &= \frac{\Delta t_0}{\sqrt{1-(0.960)^2}}, \end{aligned}$$

which gives $\Delta t_0 = 27.7$ yr.

(c) To the spacecraft observer, the distance to the star is contracted:

$$L = \frac{L_0}{\gamma} = (95.0\text{ ly})\sqrt{1-(0.960)^2} = 26.6\text{ ly}.$$

(d) To the spacecraft observer, the speed of the spacecraft is

$$v = \frac{L}{\Delta t} = \frac{(26.6\text{ ly})}{(27.7\text{ y})} = 0.960\,c,$$

as expected.

1–10. A friend of yours travels by you in her fast sports car at a speed of $0.660\,c$. It is measured in your frame to be 4.80 m long and 1.25 m high. (a) What will be its length and height at rest? (b) How many seconds would you say elapsed on your friend's watch when 20.0 s passed on yours? (c) How fast did you appear to be traveling according to your friend? (d) How many seconds would she say elapsed on your watch when she saw 20.0 s pass on hers?

SOLUTION. (a) You measure the contracted length. We find the rest length from

$$\begin{aligned} L &= \frac{L_0}{\gamma}; \\ 4.80\text{ m} &= L_0\sqrt{1-(0.660)^2}, \end{aligned}$$

which gives $L_0 = 6.39$ m.

Distances perpendicular to the motion do not change, so the rest height is 1.25 m.

(b) We find the dilated time in the sports vehicle from

$$\begin{aligned} \Delta t &= \gamma\, \Delta t_0; \\ 20.0 \text{ s} &= \frac{\Delta t_0}{\sqrt{1-(0.660)^2}}, \end{aligned}$$

which gives $\Delta t_0 = 15.0$ s.

(c) To your friend, you moved at the same relative speed: $0.660\,c$.

(d) She would measure the same time dilation: 15.0 s.

1–11. How fast must an average pion be moving to travel 15 m before it decays? The average lifetime, at rest, is 2.6×10^{-8} s.

Solution. In the Earth frame, the average lifetime of the pion will be dilated:

$$\Delta t = \gamma\, \Delta t_0.$$

The speed as a fraction of the speed of light is

$$\begin{aligned} \beta &= \frac{d}{c\,\Delta t} = \frac{d\sqrt{1-\beta^2}}{c\,\Delta t_0}\,; \\ \beta &= \frac{(15 \text{ m})\sqrt{1-\beta^2}}{(3.00 \times 10^8 \text{ m/s})(2.6 \times 10^{-8} \text{ s})}\,, \end{aligned}$$

which gives $v = 0.89\,c = 2.7 \times 10^8$ m/s.

1.3 Galilean and Lorentz transformations

1–12. Suppose in Fig. 1.1 that the origins of S and S' overlap at $t = t' = 0$ and that S' moves at speed $v = 30$ m/s with respect to S. In S', a person is resting at a point whose coordinates are $x' = 25$ m, $y' = 20$ m, and $z' = 0$. Calculate this person's coordinates in S (x, y, z) at (a) $t = 2.5$ s, (b) $t = 10.0$ s. Use the Galilean transformation.

Solution. With the standard orientation of the reference frames, the Galilean transformation is

$$x = x' + vt, \quad y = y', \quad z = z'.$$

(a) For the given data we have

$$x = 25 \text{ m} + (30 \text{ m/s})(2.5 \text{ s}) = 100 \text{ m}; \quad y = 20 \text{ m}; \quad z = 0.$$

Thus the coordinates of the person are (100 m, 20 m, 0).

(b) For the given data we have

$$x = 25 \text{ m} + (30 \text{ m/s})(10.0 \text{ s}) = 325 \text{ m}; \quad y = 20 \text{ m}; \quad z = 0.$$

Thus the coordinates of the person are (325 m, 20 m, 0).

1–13. Repeat Problem 1–12 using the Lorentz transformation and a relative speed $v = 1.80 \times 10^8$ m/s, but choose the time to be (a) 2.5 μs and (b) 10.0 μs.

SOLUTION. With the standard orientation of the reference frames, the Lorentz transformation is

$$x' = \gamma(x - vt), \quad y = y', \quad z = z';$$

with

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \\ &= \frac{1}{\sqrt{1 - \left((1.80 \times 10^8 \text{ m/s})/(3.00 \times 10^8 \text{ m/s})\right)^2}} = 1.25. \end{aligned}$$

(a) For the given data we have

$$25 \text{ m} = (1.25)\left(x - (1.80 \times 10^8 \text{ m/s})(2.5 \times 10^{-6} \text{ s})\right),$$

which gives $x = 470$ m; $y = 20$ m; $z = 0$. Thus the coordinates of the person are (470 m, 20 m, 0).

(b) For the given data we have

$$25 \text{ m} = (1.25)\left(x - (1.80 \times 10^8 \text{ m/s})(10.0 \times 10^{-6} \text{ s})\right),$$

which gives $x = 1\,820$ m; $y = 20$ m; $z = 0$. Thus the coordinates of the person are (1 820 m, 20 m, 0).

1–14. A person on a rocket traveling at $0.50\,c$ (with respect to the Earth) observes a meteor come from behind and pass her at a speed she

measures as $0.50\,c$. How fast is the meteor moving with respect to the Earth?

SOLUTION. We choose the Earth for the S frame and the rocket for the S' frame. The speed of the meteor in the S' frame is $0.50\,c$. We find the speed of the meteor in the Earth frame from the velocity transformation:

$$u = \frac{u' + v}{1 + vu'/c^2} = \frac{0.50\,c + 0.50\,c}{1 + (0.50)^2} = 0.80\,c.$$

1–15. Two spaceships leave Earth in opposite directions, each with a speed of $0.50\,c$ with respect to Earth. (a) What is the velocity of spaceship 1 relative to spaceship 2? (b) What is the velocity of spaceship 2 relative to spaceship 1?

SOLUTION. (a) We choose the Earth for the S frame and spaceship 2 for the S' frame, so $v = -0.50\,c$. The speed of spaceship 1 in the S frame is $0.50\,c$. We find the speed of spaceship 1 in S' from the velocity transformation:

$$u' = \frac{u - v}{1 - vu/c^2} = \frac{0.50\,c - (-0.50\,c)}{1 - (-1)(0.50)^2} = 0.80\,c.$$

(b) We could redefine our reference frames, but we know that the velocity of spaceship 2 relative to spaceship 1 must be $-0.80\,c$.

1–16. A spaceship leaves Earth traveling at $0.71\,c$. A second spaceship leaves the first at a speed of $0.87\,c$ with respect to the first. Calculate the speed of the second ship with respect to Earth if it is fired (a) in the same direction the first spaceship is already moving, (b) directly backward toward Earth.

SOLUTION. We take the positive direction in the direction of the first spaceship.

(a) In the reference frame of the Earth, the first spaceship is moving at $+0.71\,c$, and the second spaceship is moving at $+0.87\,c$ relative to the first. Thus the speed of the second spaceship relative to the Earth is

$$u = \frac{u' + v}{1 + vu'/c^2} = \frac{+0.87\,c + 0.71\,c}{1 + (0.71)(0.87)} = 0.98\,c.$$

(b) In the reference frame of the Earth, the first spaceship is moving at $+0.71\,c$, and the second spaceship is moving at $-0.87\,c$ relative to the

first. Thus the speed of the second spaceship relative to the Earth is

$$u = \frac{u' + v}{1 + vu'/c^2} = \frac{-0.87\,c + 0.71\,c}{1 + (0.71)(-0.87)} = -0.42\,c.$$

1–17. In Problem 1–12, suppose that the person moves with a velocity whose components are $u'_x = u'_y = 25.0$ m/s. What will be her velocity with respect to S? (Give magnitude and direction.)

SOLUTION. With the standard orientation of the reference frames, the Galilean velocity transformation is

$$u_x = u'_x + v, \quad u_y = u'_y, \quad u_z = u'_z.$$

For the given data we have

$$u_x = 25.0 \text{ m/s} + 30 \text{ m/s} = 55 \text{ m/s}; \quad u_y = 25.0 \text{ m/s}; \quad u_z = 0.$$

The magnitude of the velocity is

$$u = \sqrt{u_x^2 + u_y^2} = \sqrt{(55 \text{ m/s})^2 + (25.0 \text{ m/s})^2} = 60 \text{ m/s}.$$

We find the angle the velocity makes with the x-axis from

$$\tan\theta = \frac{u_y}{u_x} = \frac{25 \text{ m/s}}{55 \text{ m/s}} = 0.455,$$

so $\theta = 24°$.

1–18. In Problem 1–13, suppose that the person moves with a velocity (with a rocket) whose components are $u'_x = u'_y = 2.0 \times 10^8$ m/s. What will be her velocity (magnitude and direction) with respect to S?

SOLUTION. With the standard orientation of the reference frames, the Lorentz velocity transformation is

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2}, \quad u_y = \frac{u'_y}{\gamma\,(1 + vu'_x/c^2)}, \quad u_z = \frac{u'_z}{\gamma\,(1 + vu'_x/c^2)}.$$

For the given data we have

$$\begin{aligned} u_x &= (2.0 \times 10^8 \text{ m/s} + 1.80 \times 10^8 \text{ m/s}) \\ &\quad \times \left(1 + \frac{(2.0 \times 10^8 \text{ m/s})(1.80 \times 10^8 \text{ m/s})}{(3.00 \times 10^8 \text{ m/s})^2}\right)^{-1} \\ &= 2.71 \times 10^8 \text{ m/s}; \end{aligned}$$

$$\begin{aligned} u_y &= (2.0\times10^8\ \text{m/s})\sqrt{1-\left(\frac{1.80\times10^8\ \text{m/s}}{3.00\times10^8\ \text{m/s}}\right)^2} \\ &\quad\times\left(1+\frac{(2.0\times10^8\ \text{m/s})(1.80\times10^8\ \text{m/s})}{(3.00\times10^8\ \text{m/s})^2}\right)^{-1} \\ &= 1.14\times10^8\ \text{m/s}; \end{aligned}$$

$$u_z = 0.$$

The magnitude of the velocity is

$$\begin{aligned} u &= \sqrt{u_x^2+u_y^2} = \sqrt{(2.71\times10^8\ \text{m/s})^2+(1.14\times10^8\ \text{m/s})^2} \\ &= 2.9\times10^8\ \text{m/s}. \end{aligned}$$

We find the angle the velocity makes with the x-axis from

$$\tan\theta = \frac{u_y}{u_x} = \frac{1.14\times10^8\ \text{m/s}}{2.71\times10^8\ \text{m/s}} = 0.421,$$

so $\theta = 23°$.

1–19. A spaceship traveling at $0.66\,c$ away from Earth fires a module with a speed of $0.82\,c$ at right angles to its own direction of travel (as seen by the spaceship). What is the speed of the module, and its direction of travel (relative to the spaceship's direction), as seen by an observer on Earth?

Solution. We choose the Earth for the S frame and the spaceship for the S' frame. The velocity components of the module in the S' frame are $u'_x = 0$, $u'_y = 0.82\,c$. We find the velocity components of the module in the Earth frame from the velocity transformation:

$$\begin{aligned} u_x &= \frac{u'_x+v}{1+vu'_x/c^2} = \frac{0+0.66\,c}{1+0} = 0.66\,c; \\ u_y &= \frac{u'_y}{\gamma\,(1+vu'_x/c^2)} = \frac{(0.82\,c)\sqrt{1-(0.66)^2}}{1+0} = 0.616\,c. \end{aligned}$$

The magnitude of the velocity is

$$u = \sqrt{u_x^2+u_y^2} = \sqrt{(0.66\,c)^2+(0.616\,c)^2} = 0.90\,c = 2.7\times10^8\ \text{m/s}.$$

We find the angle the velocity makes with the x-axis from

$$\tan\theta = \frac{u_y}{u_x} = \frac{0.616\,c}{0.66\,c} = 0.933,$$

so $\theta = 43°$.

1–20. If a particle moves in the xy plane of system S (Fig. 1.1) with speed u in a direction that makes an angle θ with the x axis, show that it makes an angle θ' in S' given by

$$\tan\theta' = \frac{\sin\theta}{\gamma\,(\cos\theta - v/u)}.$$

SOLUTION. The velocity components of the particle in the S frame are $u_x = u\cos\theta$, $u_y = u\sin\theta$. We find the velocity components of the particle in the S' frame from the velocity transformation:

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}; \qquad u'_y = \frac{u_y}{\gamma\,(1 - vu_x/c^2)}.$$

We find the angle the velocity makes with the x'-axis from

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{u_y}{\gamma(u_x - v)} = \frac{u\sin\theta}{\gamma(u\cos\theta - v)} = \frac{\sin\theta}{\gamma(\cos\theta - v/u)}.$$

1–21. A stick of length L_0, at rest in reference frame S, makes an angle θ with the x axis. In reference frame S', which moves to the right with velocity $\mathbf{v} = v\mathbf{i}$ with respect to S, determine (a) the length L of the stick, and (b) the angle θ' it makes with the x' axis.

SOLUTION. (a) In the frame S', the x-component of the stick will be contracted:

$$L_x = \frac{L_0\cos\theta}{\gamma}.$$

The y-component is unchanged:

$$L_y = L_0\sin\theta.$$

The magnitude of the length in S' is

$$\begin{aligned} L &= \sqrt{L_x^2 + L_y^2} = \sqrt{(L_0\cos\theta)^2\,(1-\beta^2) + (L_0\sin\theta)^2} \\ &= L_0\sqrt{(\cos\theta)^2\,(1-\beta^2) + (\sin\theta)^2} = L_0\sqrt{1 - \beta^2(\cos\theta)^2}. \end{aligned}$$

(b) We find the angle that the stick makes with the x'-axis from

$$\tan\theta' = \frac{L_y}{L_x} = \frac{\gamma L_0 \sin\theta}{L_0 \cos\theta} = \gamma \tan\theta.$$

Thus we have

$$\theta' = \arctan\left(\gamma \tan\theta\right).$$

1–22. The scene of action is legendary American Wild West. A marshal riding on a train traveling 50 m/s sees a duel between two men standing on the Earth 50 m apart parallel to the train. The marshal's instruments indicate that in his reference frame the two men fired simultaneously. (a) Which of the two men, the first one the train passes (A) or the second one (B) should be arrested for firing the first shot? That is, in the gunfighter's frame of reference, who fired first? (b) How much earlier did he fire? (c) Who was struck first?

SOLUTION. (a) We choose the train as frame S' and the Earth as frame S. Thus for the two firings we have

$$\Delta x = x_B - x_A = 50 \text{ m}, \quad \text{and} \quad \Delta t' = t'_B - t'_A = 0.$$

We find the separation of the two firings in the S' frame from

$$\Delta x = \gamma(\Delta x' + v\,\Delta t') = \gamma(\Delta x' + 0),$$

or

$$\Delta x' = \frac{\Delta x}{\gamma},$$

which is the contraction of the length.

We find the time interval between firings in the S frame from

$$\Delta t = \gamma\left(\Delta t' + \frac{v\,\Delta x'}{c^2}\right) = \gamma\left(0 + \frac{v\,\Delta x'}{c^2}\right) = \frac{\gamma v\,\Delta x'}{c^2} = \frac{v\,\Delta x}{c^2}.$$

Because $\Delta x > 0$, we see that $\Delta t > 0$, so B fires after A in the S frame; A fires first.

(b) The time difference between firings in the Earth frame is

$$\Delta t = \frac{v\,\Delta x}{c^2} = \frac{(50 \text{ m/s})(50 \text{ m})}{(3.00 \times 10^8 \text{ m/s})^2} = 2.8 \times 10^{-14} \text{ s}.$$

(c) If u is the speed of a bullet in the Earth frame and T represents the time at which a gunfighter is struck, we have

$$T_A = \frac{\Delta x}{u} + t_B, \quad \text{and} \quad T_B = \frac{\Delta x}{u} + t_A,$$

or

$$T_B - T_A = t_A - t_B = -\Delta t.$$

Thus we see that $\Delta T < 0$, or B is struck first in the Earth frame, as expected, because A fired first.

In frame S' we have

$$\Delta T' = \gamma \left(\Delta T - \frac{v\,\Delta x}{c^2} \right) = \gamma \left(-\Delta t - \frac{v\,\Delta x}{c^2} \right).$$

Because $\Delta t > 0$, and $\Delta x > 0$, we see that $\Delta T' < 0$, so B is hit first in the S' frame also.

1–23. A farm boy studying physics believes that he can fit a 13.0-m long pole into a 10.0-m long barn if he runs fast enough, carrying the pole. Can he do it? Explain in detail. How does this fit with the idea that when he is running the barn looks even shorter than 10.0 m?

SOLUTION. To an observer in the barn reference frame, if the boy runs fast enough, the measured contracted length of the pole will be less than 13.0 m, so the observer can say that the two ends of the pole were inside the barn simultaneously. We find the necessary speed for the contracted pole to fit inside the barn from

$$\begin{aligned} L_{\text{pole}} &= \frac{L_{0\,\text{pole}}}{\gamma}\,; \\ 10.0\text{ m} &= (13.0\text{ m})\sqrt{1-\beta^2}\,, \end{aligned}$$

which gives $v = 0.64\,c$.

To the boy, the barn is moving and thus the length of the barn, as he would measure it, is less than the length of the pole:

$$L_{\text{barn}} = L_{0\,\text{barn}}\sqrt{1-\beta^2} = (10.0\text{ m})\sqrt{1-(0.64)^2} = 7.7\text{ m}.$$

However, simultaneity is relative. Thus when the two ends are simultaneously inside the barn to the barn observer, those two events are not

simultaneous to the boy. Thus he would claim that the observer in the barn determined that the ends of the pole were inside the barn at different times, which is also what the boy would say. It is not possible in the boy's frame to have both ends of the pole inside the barn simultaneously.

1.4 Relativistic momentum and mass

1–24. What is the momentum of a proton traveling at $v = 0.85\,c$?

SOLUTION. The momentum of the proton is

$$\begin{aligned} p &= \gamma m v = \frac{(1.67 \times 10^{-27}\ \text{kg})(0.85)(3.00 \times 10^{8}\ \text{m/s})}{\sqrt{1-(0.85)^2}} \\ &= 8.1 \times 10^{-19}\ \text{kg} \cdot \text{m/s}. \end{aligned}$$

1–25. At what speed will an object's relativistic mass be twice its rest mass?

SOLUTION. We find the speed from

$$\begin{aligned} m_{\text{rel}} &= \gamma m; \\ 2m &= \frac{m}{\sqrt{1-\beta^2}}, \end{aligned}$$

which gives $v = 0.866\,c$.

1–26. A particle of rest mass m travels at a speed $v = 0.20\,c$. At what speed will its momentum be doubled?

SOLUTION. For the momentum to be doubled we have

$$\begin{aligned} p_2 &= 2p_1; \\ \gamma_2 m v_2 &= \gamma_1 (2m) v_1; \end{aligned}$$

or

$$\frac{\beta_2^2}{1-\beta_2^2} = \frac{4\beta_1^2}{1-\beta_1^2} = \frac{4(0.20)^2}{1-(0.20)^2},$$

which gives $v_2 = 0.38\,c$.

1–27. (a) A particle travels at $v = 0.10\,c$. By what percentage will a calculation of its momentum be wrong if you use the classical formula? (b) Repeat for $v = 0.50\,c$.

SOLUTION. The two expressions for the momentum are

$$p_{\rm rel} = \gamma m v, \quad p_{\rm cl} = mv.$$

Thus the error is

$$\frac{p_{\rm rel} - p_{\rm cl}}{p_{\rm rel}} = \frac{mv\left(1-\beta^2\right)^{-1/2} - mv}{mv\left(1-\beta^2\right)^{-1/2}} = 1 - \sqrt{1-\beta^2}.$$

(a) For the given speed we have

$$\frac{p_{\rm rel} - p_{\rm cl}}{p_{\rm rel}} = 1 - \sqrt{1-\beta^2} = 1 - \sqrt{1-(0.10)^2} = 0.005 = 0.5\%.$$

(b) For the given speed we have

$$\frac{p_{\rm rel} - p_{\rm cl}}{p_{\rm rel}} = 1 - \sqrt{1-\beta^2} = 1 - \sqrt{1-(0.50)^2} = 0.13 = 13\%.$$

1–28. What is the percent change in momentum of a proton that accelerates (a) from $0.45\,c$ to $0.90\,c$, (b) from $0.90\,c$ to $0.98\,c$?

SOLUTION. The fractional change in momentum is

$$\frac{p_2 - p_1}{p_1} = \frac{mv_2\left(1-\beta_2^2\right)^{-1/2} - mv_1\left(1-\beta_1^2\right)^{-1/2}}{mv_1\left(1-\beta_1^2\right)^{-1/2}} = \frac{v_2\sqrt{1-\beta_1^2}}{v_1\sqrt{1-\beta_2^2}} - 1.$$

(a) For the given speeds we have

$$\frac{p_2 - p_1}{p_1} = \frac{0.90\,c\sqrt{1-(0.45)^2}}{0.45\,c\sqrt{1-(0.90)^2}} - 1 = 3.1 = 310\%.$$

Note that this is about $3\times$ what use of the classical expression would give.

(b) For the given speeds we have

$$\frac{p_2 - p_1}{p_1} = \frac{0.98\,c\sqrt{1-(0.90)^2}}{0.90\,c\sqrt{1-(0.98)^2}} - 1 = 1.4 = 140\%.$$

Note that this is about $1.5\times$ what use of the classical expression would give.

1.5 Energy and mass

1–29. A certain chemical reaction requires 4.82×10^4 J of energy input for it to go. What is the increase in mass of the products over the reactants?

SOLUTION. We find the increase in mass from

$$\Delta m = \frac{\Delta E}{c^2} = \frac{4.82 \times 10^4 \text{ J}}{(3.00 \times 10^8 \text{ m/s})^2} = 5.36 \times 10^{-13} \text{ kg}.$$

Note that this is so small, most chemical reactions are considered to have mass conserved.

1–30. When a uranium nucleus at rest breaks apart in the process known as fission in a nuclear reactor, the resulting fragments have a total kinetic energy of about 200 MeV. How much mass was lost in the process?

SOLUTION. We find the loss in mass from

$$\Delta m = \frac{\Delta E}{c^2} = \frac{(200 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV})}{(3.00 \times 10^8 \text{ m/s})^2} = 3.56 \times 10^{-28} \text{ kg}.$$

1–31. Calculate the rest energy of an electron in joules and in MeV ($1 \text{ MeV} = 1.60 \times 10^{-13}$ J).

SOLUTION. The rest energy of the electron is

$$\begin{aligned} E &= mc^2 = (9.109 \times 10^{-31} \text{ kg})(2.998 \times 10^8 \text{ m/s})^2 \\ &= 8.187 \times 10^{-14} \text{ J} = \frac{8.187 \times 10^{-14} \text{ J}}{1.602 \times 10^{-13} \text{ J/MeV}} = 0.511 \text{ MeV}. \end{aligned}$$

1–32. Calculate the rest mass of a proton in MeV/c^2.

SOLUTION. The mass of the proton is

$$\begin{aligned} m &= \frac{\Delta E}{c^2} = \frac{mc^2}{c^2} = \frac{(1.67 \times 10^{-27} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2}{(1.60 \times 10^{-13} \text{ J/MeV})c^2} \\ &= 939 \text{ MeV}/c^2. \end{aligned}$$

1–33. In 2008, total worldwide energy consumption was about 4.74×10^{20} J. How much mass would have to be converted to energy to fuel this need?

SOLUTION. We find the necessary mass conversion from

$$\Delta m = \frac{\Delta E}{c^2} = \frac{4.74 \times 10^{20} \text{ J}}{(3.00 \times 10^8 \text{ m/s})^2} = 5.27 \times 10^3 \text{ kg}.$$

1–34. How much energy can be obtained from conversion a 1.0 gram of mass? How much mass could this energy raise to a height of 100 m?

SOLUTION. We find the energy equivalent of the mass from

$$E = mc^2 = (1.0 \times 10^{-3} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 = 9.0 \times 10^{13} \text{ J}.$$

If this energy increases the gravitational energy, we have

$$\begin{aligned} E &= Mgh; \\ 9.0 \times 10^{13} \text{ J} &= M(9.80 \text{ m/s}^2)(100 \text{ m}), \end{aligned}$$

which gives $M = 9.2 \times 10^{10}$ kg.

1–35. Show that when the kinetic energy of a particle equals its rest energy, the speed of the particle is about $0.866\,c$.

SOLUTION. If the kinetic energy is equal to the rest energy, we have

$$K = \gamma mc^2 - mc^2 = mc^2,$$

or

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 2,$$

which gives $v = 0.866\,c$.

1–36. At what speed will an object's kinetic energy be 25 percent of its rest energy?

SOLUTION. If the kinetic energy is 25% of the rest energy, we have

$$K = \gamma mc^2 - mc^2 = 0.25\,mc^2,$$

or

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.25,$$

which gives $v = 0.60\,c$.

1–37. (a) How much work is required to accelerate a proton from rest up to a speed of $0.997\,c$? (b) What would be the momentum of this proton?

SOLUTION. (a) We find the work required from

$$\begin{aligned} W &= \Delta K = mc^2\,(\gamma - 1) = (939\text{ MeV})\left(\frac{1}{\sqrt{1-(0.997)^2}} - 1\right) \\ &= 11.2 \times 10^3\text{ MeV} = 11.2\text{ GeV} = 1.79 \times 10^{-9}\text{ J}. \end{aligned}$$

(b) The momentum of the proton is

$$\begin{aligned} p &= \frac{mv}{\sqrt{1-\beta^2}} = \frac{(1.67 \times 10^{-27}\text{ kg})(0.997)(3.00 \times 10^8\text{ m/s})}{\sqrt{1-(0.997)^2}} \\ &= 6.45 \times 10^{-18}\text{ kg}\cdot\text{m/s}. \end{aligned}$$

1–38. Calculate the kinetic energy and momentum of a proton traveling 2.60×10^8 m/s.

SOLUTION. The speed of the proton is

$$v = \frac{2.60 \times 10^8\text{ m/s}}{3.00 \times 10^8\text{ m/s}} = 0.867\,c.$$

The kinetic energy is

$$\begin{aligned} K &= mc^2\,(\gamma - 1) = (939\text{ MeV})\left(\frac{1}{\sqrt{1-(0.867)^2}} - 1\right) \\ &= 943\text{ MeV} = 1.51 \times 10^{-10}\text{ J}. \end{aligned}$$

The momentum of the proton is

$$\begin{aligned} p &= \gamma mv = \frac{(1.67 \times 10^{-27}\text{ kg})(2.60 \times 10^8\text{ m/s})}{\sqrt{1-(0.867)^2}} \\ &= 8.70 \times 10^{-19}\text{ kg}\cdot\text{m/s}. \end{aligned}$$

1–39. What is the momentum of a 750-MeV proton (that is, its kinetic energy is 750 MeV)?

SOLUTION. The total energy of the proton is

$$E = K + mc^2 = 750 \text{ MeV} + 939 \text{ MeV} = 1\,689 \text{ MeV}.$$

The relation between the momentum and energy is

$$\begin{aligned} p^2c^2 &= E^2 - (mc^2)^2; \\ p^2(3.00 \times 10^8 \text{ m/s})^2 &= \left((1\,689 \text{ MeV})^2 - (939 \text{ MeV})^2\right) \\ &\quad \times (1.60 \times 10^{-13} \text{ J/MeV})^2, \end{aligned}$$

which gives $p = 7.49 \times 10^{-19}$ kg $\cdot$ m/s.

1–40. What is the speed of a proton accelerated by a potential difference of 95 MV?

SOLUTION. The kinetic energy acquired by the proton is

$$K = qV = (1 \text{ e})(95 \text{ MV}) = 95 \text{ MeV}.$$

We find the speed from

$$\begin{aligned} K &= mc^2\,(\gamma - 1)\,; \\ 95 \text{ MeV} &= (939 \text{ MeV}) \left(\frac{1}{\sqrt{1 - \beta^2}} - 1 \right), \end{aligned}$$

which gives $v = 0.42\,c$.

1–41. What is the speed of an electron whose kinetic energy is 1.00 MeV?

SOLUTION. We find the speed from

$$\begin{aligned} K &= mc^2\,(\gamma - 1)\,; \\ 1.00 \text{ MeV} &= (0.511 \text{ MeV}) \left(\frac{1}{\sqrt{1 - \beta^2}} - 1 \right), \end{aligned}$$

which gives $v = 0.941\,c$.

1–42. What is the speed of an electron when it hits a television screen after being accelerated by the 25 000 V of the picture tube?

SOLUTION. The kinetic energy acquired by the electron is

$$K = qV = (1 \text{ e})(0.025 \text{ MV}) = 0.025 \text{ MeV}.$$

We find the speed from

$$\begin{aligned} K &= mc^2\,(\gamma - 1)\,; \\ 0.025\ \text{MeV} &= (0.511\ \text{MeV})\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right), \end{aligned}$$

which gives $v = 0.302\,c$.

1–43. Two identical particles of rest mass m approach each other at equal and opposite speeds, v. The collision is completely inelastic and results in a single particle at rest. What is the rest mass of the new particle? How much energy was lost in the collision? How much kinetic energy is lost in this collision?

SOLUTION. If M is the mass of the new particle, for conservation of energy we have

$$\begin{aligned} 2(K + mc^2) &= Mc^2; \\ 2\gamma mc^2 &= Mc^2, \end{aligned}$$

which gives

$$M = 2\gamma m.$$

Because energy is conserved, there was no loss.

The final particle is at rest, so the kinetic energy loss is the initial kinetic energy of the two colliding particles:

$$K_{\text{loss}} = 2K = (M - 2m)c^2 = 2mc^2\,(\gamma - 1)\,.$$

1–44. Calculate the speed of a proton ($m = 1.67 \times 10^{-27}$ kg) whose kinetic energy is exactly half its total energy.

SOLUTION. The total energy of the proton is

$$\begin{aligned} E &= K + mc^2; \\ 2K &= K + mc^2, \end{aligned}$$

which gives $K = mc^2$.

We find the speed from

$$K = \gamma mc^2 - mc^2 = mc^2,$$

or

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 2,$$

which gives $v = 0.866\,c$.

1–45. What is the speed and momentum of an electron ($m = 9.11 \times 10^{-31}$ kg) whose kinetic energy equals its rest energy?

SOLUTION. We find the speed from

$$K = \gamma mc^2 - mc^2 = mc^2,$$

or

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 2,$$

which gives $v = 0.866\,c$.

The momentum of the electron is

$$\begin{aligned} p &= \gamma mv = (2)(9.11 \times 10^{-31}\ \text{kg})(0.866)(3.00 \times 10^{8}\ \text{m/s}) \\ &= 4.73 \times 10^{-22}\ \text{kg} \cdot \text{m/s}. \end{aligned}$$

1–46. Suppose a spacecraft of mass 27 000 kg is accelerated to $0.21\,c$. (a) How much kinetic energy would it have? (b) If you used the classical formula for kinetic energy, by what percentage would you be in error?

SOLUTION. (a) The kinetic energy is

$$\begin{aligned} K &= mc^2\,(\gamma - 1) \\ &= (27\,000\ \text{kg})(3.00 \times 10^{8}\ \text{m/s})^2 \left(\frac{1}{\sqrt{1-(0.21)^2}} - 1 \right) \\ &= 5.54 \times 10^{19}\ \text{J} = 5.5 \times 10^{19}\ \text{J}. \end{aligned}$$

(b) When we use the classical expression, we get

$$K_{\text{cl}} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(27\,000\ \text{kg})\left((0.21)(3.00 \times 10^{8}\ \text{m/s})\right)^2 = 5.35 \times 10^{19}\ \text{J}.$$

The error is

$$\frac{5.35 - 5.54}{5.54} = -0.03 = -3\%.$$

1–47. Calculate the kinetic energy and momentum of a proton ($m = 1.67 \times 10^{-27}$ kg) traveling 8.4×10^7 m/s. By what percentages would your calculations have been in error if you had used classical formulae?

SOLUTION. The speed of the proton is

$$v = \frac{8.4 \times 10^7 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}}\, c = 0.280\, c.$$

The kinetic energy is

$$\begin{aligned} K &= mc^2(\gamma - 1) = (939 \text{ MeV})\left(\frac{1}{1-(0.280)^2} - 1\right) \\ &= 39 \text{ MeV} = 6.3 \times 10^{-12} \text{ J}. \end{aligned}$$

The momentum of the proton is

$$\begin{aligned} p &= \gamma m v = \frac{(1.67 \times 10^{-27} \text{ kg})(8.4 \times 10^7 \text{ m/s})}{\sqrt{1-(0.280)^2}} \\ &= 1.46 \times 10^{-19} \text{ kg} \cdot \text{m/s} = 1.5 \times 10^{-19} \text{ kg} \cdot \text{m/s}. \end{aligned}$$

From the classical expressions, we get

$$K_{\text{cl}} = \tfrac{1}{2} m v^2 = \tfrac{1}{2}(1.67 \times 10^{-27} \text{ kg})(8.4 \times 10^7 \text{ m/s})^2 = 5.9 \times 10^{-12} \text{ J},$$

with an error of

$$\frac{5.9 - 6.3}{6.3} = -0.06 = -6\%.$$

$$p = mv = (1.67 \times 10^{-27} \text{ kg})(8.4 \times 10^7 \text{ m/s}) = 1.40 \times 10^{-19} \text{ kg} \cdot \text{m/s},$$

with an error of

$$\frac{1.40 - 1.46}{1.46} = -0.04 = -4\%.$$

1–48. The americium nucleus, ${}^{241}_{95}\text{Am}$, decays to a neptunium nucleus, ${}^{237}_{93}\text{Np}$, by emitting an α particle of mass 4.00260 u and kinetic energy 5.5 MeV. Estimate the mass of the neptunium nucleus, ignoring its recoil, given that the americium mass is 241.05682 u.

SOLUTION. If we ignore the recoil of the neptunium nucleus, the increase in kinetic energy is the kinetic energy of the α particle, which equals the loss in mass:

$$\begin{aligned} K_\alpha &= (m_{\mathrm{Am}} - (m_{\mathrm{Np}} + m_\alpha))\, c^2; \\ 5.5\ \mathrm{MeV} &= (241.05682\ \mathrm{u} - (m_{\mathrm{Np}} + 4.00260\ \mathrm{u}))\, c^2 (931.5\ \mathrm{MeV/uc^2}), \end{aligned}$$

which gives $m_{\mathrm{Np}} = 237.04832$ u.

1–49. An electron ($m = 9.11 \times 10^{-31}$ kg) is accelerated from rest to speed v by a conservative force. In this process, its potential energy decreases by 5.60×10^{-14} J. Determine the electron's speed, v.

SOLUTION. The increase in kinetic energy comes from the decrease in potential energy:

$$\begin{aligned} K &= -\Delta U = mc^2\,(\gamma - 1)\,; \\ -(-5.60 \times 10^{-14}\ \mathrm{J}) &= (9.11 \times 10^{-31}\ \mathrm{kg})(3.00 \times 10^8\ \mathrm{m/s})^2 \\ &\quad \times \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right), \end{aligned}$$

which gives $v = 0.804\, c$.

1–50. Make a graph of the kinetic energy versus momentum for (a) a particle of nonzero rest mass, and (b) a particle with zero rest mass.

SOLUTION. On Fig. 1.2 the required graphs are drawn in dimensionless coordinates ($\xi = p/mc,\ \eta = K/mc^2$).

(a) For any $m > 0$ the kinetic energy obeys the general equation

$$K = \sqrt{p^2c^2 + (mc^2)^2} - mc^2.$$

In particular, when $p \ll mc$ we get the classical-limit relationship

$$K \approx \frac{p^2}{2m}.$$

(b) For zero rest mass ($m \ll p/c$) the general equation becomes

$$K = pc.$$

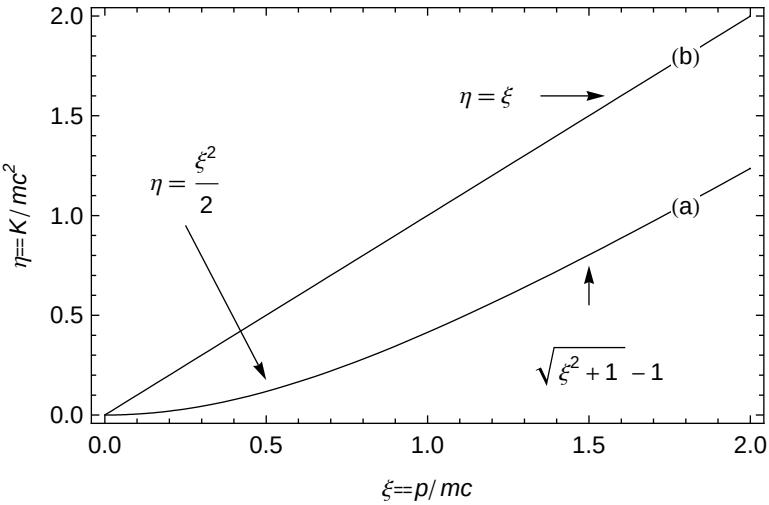


Figure 1.2. Problem 1–50.

1–51. What magnetic field intensity is needed to keep 900-GeV protons revolving in a circle of radius 1.0 km (at, say, the Fermilab synchrotron)? Use the relativistic mass. The proton's rest mass is 0.938 GeV/c^2. (1 GeV = 10^9 eV.)

Solution. The total energy of the proton is

$$E = m_{\rm rel}c^2 = K + mc^2 = 900\ \text{GeV} + 0.938\ \text{GeV} = 901\ \text{GeV},$$

so the relativistic mass is 901 GeV/c^2. We find the speed from

$$\begin{aligned} m_{\rm rel} &= \gamma m; \\ 901\ \text{GeV}/c^2 &= \gamma\left(0.938\ \text{GeV}/c^2\right), \end{aligned}$$

which gives

$$\gamma = 960.6,$$

so $v \approx 1.00\,c$.

The speed is constant so the relativistic mass is constant. The magnetic force provides the radial acceleration:

$$qvB = \frac{m_{\rm rel}v^2}{r},$$

or

$$\begin{aligned} B &= \frac{m_{\rm rel}v}{qr} = \frac{\gamma mv}{qr} = \frac{(960.6)(1.67\times10^{-27}\ \text{kg})(3.00\times10^{8}\ \text{m/s})}{(1.6\times10^{-19}\ \text{C})(1.0\times10^{3}\ \text{m})} \\ &= 3.0\ \text{T}. \end{aligned}$$

1–52. A negative muon traveling at 33 percent the speed of light collides head on with a positive muon traveling at 50 percent the speed of light. The two muons (each of mass $105.7\ \text{MeV}/c^2$) annihilate, and produce how much electromagnetic energy?

SOLUTION. Because the total energy of the muons becomes electromagnetic energy, we have

$$\begin{aligned} E &= K_1 + m_1c^2 + K_2 + m_2c^2 = \gamma_1 mc^2 + \gamma_2 mc^2 \\ &= \frac{105.7\ \text{MeV}}{\sqrt{1-(0.33)^2}} + \frac{105.7\ \text{MeV}}{\sqrt{1-(0.50)^2}} = 234\ \text{MeV}. \end{aligned}$$

1–53. Show that the energy of a particle of charge e revolving in a circle of radius r in a magnetic field B is given by E (in eV) $= Brc$ in the relativistic limit ($v \approx c$).

SOLUTION. The magnetic force provides the radial acceleration:

$$qvB = \frac{m_{\rm rel}v^2}{r},$$

or

$$m_{\rm rel} = \frac{qBr}{v} = \frac{E}{c^2}.$$

With $v \approx c$, and $q = 1$ e, we get

$$E(\text{eV}) = \frac{(1)Brc^2}{c} = Brc.$$

Note that the relativistic mass is constant during the revolution.

1–54. Show that the kinetic energy K of a particle of rest mass m is related to its momentum p by the equation

$$p = \frac{\sqrt{K^2 + 2Kmc^2}}{c}.$$

SOLUTION. The total energy is

$$E = K + mc^2$$

and is related to the momentum by

$$E^2 = p^2c^2 + m^2c^4 = (K + mc^2)^2 = K^2 + 2Kmc^2 + m^2c^4,$$

which gives

$$p^2c^2 = K^2 + 2Kmc^2,$$

or

$$p = \frac{\sqrt{K^2 + 2Kmc^2}}{c}.$$

1–55. (a) In reference frame S, a particle has momentum $\mathbf{p} = p_x\mathbf{i}$ along the positive x axis. Show that in frame S', which moves with speed v as in Fig. 1.1, the momentum has components

$$p'_x = \gamma\left(p_x - \frac{vE}{c^2}\right), \quad p'_y = p_y,$$
$$p'_z = p_z, \qquad\qquad E' = \gamma\left(E - p_x v\right).$$

(These transformation equations hold, actually, for any direction of $\mathbf{p}$.) (b) Show that p_x, p_y, p_z, E/c transform according to the Lorentz transformation in the same way as x, y, z, ct.

SOLUTION. (a) We let m be the mass of the particle. Its velocity has an x-component only, u_x, so its momentum components in frame S are

$$p_x = \frac{mu_x}{\sqrt{1 - u_x^2/c^2}}, \; p_y = 0, \; p_z = 0.$$

The energy of the particle in frame S is

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1 - u_x^2/c^2}}.$$

The velocity in frame S' is

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2},\ u'_y = 0,\ u'_z = 0;$$

so the particle's momentum is

$$p'_x = \frac{m u'_x}{\sqrt{1 - u'^2_x/c^2}},\ p'_y = 0 = p_y,\ p'_z = 0 = p_z.$$

If we consider the denominator, we have

$$\begin{aligned}
\sqrt{1 - \frac{u'^2_x}{c^2}} &= \sqrt{1 - \frac{(u_x - v)^2}{(1 - u_x v/c^2)^2 c^2}} \\
&= \frac{\sqrt{(1 - u_x v/c^2)^2 - (u_x^2 - 2u_x v + v^2)/c^2}}{1 - u_x v/c^2} \\
&= \sqrt{1 - \frac{2u_x v}{c^2} + \left(\frac{u_x v}{c^2}\right)^2 - \left(\frac{u_x}{c}\right)^2 + \frac{2u_x v}{c^2} - \left(\frac{v}{c}\right)^2} \\
&\quad \times \left(1 - \frac{u_x v}{c^2}\right)^{-1} \\
&= \frac{\sqrt{(1 - u_x^2/c^2)}}{\gamma\,(1 - u_x v/c^2)}\,.
\end{aligned}$$

When we use this and the velocity transformation in the expression for the momentum, we have

$$\begin{aligned}
p'_x &= \frac{m u'_x}{\sqrt{1 - u'^2_x/c^2}} = \frac{\gamma m (u_x - v)(1 - u_x v/c^2)}{(1 - u_x v/c^2)\sqrt{(1 - u_x^2/c^2)}} = \gamma \frac{m u_x - m v}{\sqrt{1 - u_x^2/c^2}} \\
&= \gamma \left(p_x - \frac{v E}{c^2}\right).
\end{aligned}$$

For the transformation of the energy we have

$$\begin{aligned}
E' &= \frac{m c^2}{\sqrt{1 - u'^2_x/c^2}} = \frac{\gamma m c^2 (1 - u_x v/c^2)}{\sqrt{1 - u_x^2/c^2}} = \gamma \frac{m c^2 - m u_x v}{\sqrt{1 - u_x^2/c^2}} \\
&= \gamma\,(E - p_x v)\,.
\end{aligned}$$

(b) The transformations are

$$x' = \gamma\left(x - \frac{v}{c}\,ct\right), \qquad y' = y, \qquad z' = z, \qquad ct' = \gamma\left(ct - \frac{vx}{c}\right);$$

$$p'_x = \gamma\left(p_x - \frac{v}{c}\frac{E}{c}\right), \quad p'_y = p_y, \quad p'_z = p_z, \quad \frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{vp_x}{c}\right).$$

Thus we see that p_x, p_y, p_z, E/c transform in the same way as x, y, z, ct.

1.6 Doppler shift for light

1–56. A certain galaxy has a Doppler shift given by $f_0 - f = 0.797 f_0$. How fast is it moving away from us?

SOLUTION. For a source moving away from us the Doppler shift is

$$f = f_0\sqrt{\frac{c-v}{c+v}}.$$

Thus we have

$$\begin{aligned} \frac{f_0 - f}{f_0} &= 1 - \sqrt{\frac{1-\beta}{1+\beta}}; \\ 0.797 &= 1 - \sqrt{\frac{1-\beta}{1+\beta}}, \end{aligned}$$

which gives $v = 0.921\,c$.

1–57. A quasar emits familiar hydrogen lines whose wavelengths are 3.0 times longer than what we measure in the laboratory. (a) What is the speed of this quasar? (b) What result would you obtain if you used the "classical" (non-relativistic) Doppler shift?

SOLUTION. From $c = f\lambda$, we see that $f = f_0/3.0$, so the quasar is moving away from us.

(a) For the Doppler shift we have

$$\begin{aligned} f &= f_0\sqrt{\frac{1-\beta}{1+\beta}}; \\ \frac{f_0}{3.0} &= f_0\sqrt{\frac{1-\beta}{1+\beta}}, \end{aligned}$$

which gives $v = 0.80\,c$.

(b) For the "classical" Doppler shift, the wavelength from the quasar is

$$\lambda = \frac{v + c}{f_0},$$

and the received frequency is

$$\begin{aligned} f &= \frac{c}{\lambda} = \frac{c}{v + c} f_0; \\ \frac{f_0}{3.0} &= \frac{f_0}{1 + \beta}, \end{aligned}$$

which gives $v = 2.0\,c$.

1–58. A spaceship moving toward Earth at $0.80\,c$ transmits radio signals at 95.0 MHz. At what frequency should Earth receivers be tuned?

SOLUTION. For a source moving toward the Earth the Doppler shift is

$$f = f_0 \sqrt{\frac{c + v}{c - v}} = f_0 \sqrt{\frac{1 + \beta}{1 - \beta}} = (95.0 \text{ MHz}) \sqrt{\frac{1 + 0.80}{1 - 0.80}} = 285 \text{ MHz}.$$

1–59. Starting from Eq. 1.15, show that the Doppler shift in wavelength is $\Delta\lambda/\lambda = v/c$, if $v \ll c$.

SOLUTION. For a source moving away from us the Doppler shift is

$$\lambda = \lambda_0 \sqrt{\frac{1 + \beta}{1 - \beta}}\,.$$

If $v \ll c$, we can use the approximation $1/(1 - x) \approx 1 + x$:

$$\lambda \approx \lambda_0 \sqrt{(1 + \beta)^2} = \lambda_0 (1 + \beta)\,.$$

Thus the fractional change is

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\lambda - \lambda_0}{\lambda_0} = 1 + \beta - 1 = \beta.$$

1.7 General problems

1–60. As a rule of thumb, anything traveling faster than about $0.1\,c$ is called *relativistic*—i. e., for which the correction using special relativity is a significant effect. Is the electron in a hydrogen atom (radius 0.5×10^{-10} m) relativistic? Treat the electron as though it were in a circular orbit around the proton.)

SOLUTION. The electrostatic force provides the radial acceleration:

$$k_{\mathrm{e}}\frac{e^2}{r^2}=\frac{mv^2}{r}.$$

Thus we find the speed from

$$v^2=\frac{(9.0\times10^9\ \mathrm{N\cdot m^2/C^2})(1.6\times10^{-19}\ \mathrm{C})^2}{(9.11\times10^{-31}\ \mathrm{kg})(0.5\times10^{-10}\ \mathrm{m})},$$

which gives $v=2\times10^6$ m/s. Because this is less than $0.1\,c$, the electron is not relativistic.

1–61. An atomic clock is taken to the North Pole, while another stays at the Equator. How far will they be out of synchronization after a year has elapsed?

SOLUTION. Because the North Pole has no tangential velocity, the clock there will measure a year (3.16×10^7 s). The clock at the equator has the tangential velocity of the equator:

$$v=r_{\mathrm{E}}\omega=\frac{(6.38\times10^6\ \mathrm{m})(2\pi\ \mathrm{rad})}{(24\ \mathrm{h})(3\,600\ \mathrm{s/h})}=464\ \mathrm{m/s}.$$

The clock at the equator will run slow:

$$t_{\mathrm{equator}}=t_{\mathrm{North}}\sqrt{1-\beta^2}\approx t_{\mathrm{North}}\left(1-\tfrac{1}{2}\beta^2\right).$$

Thus the difference in times is

$$\begin{aligned}t_{\mathrm{North}}-t_{\mathrm{equator}}&=t_{\mathrm{North}}\,\tfrac{1}{2}\beta^2=(3.16\times10^7\ \mathrm{s})\frac{1}{2}\left(\frac{464\ \mathrm{m/s}}{3.00\times10^8\ \mathrm{m/s}}\right)^2\\&=3.8\times10^{-5}\ \mathrm{s}.\end{aligned}$$

1–62. The nearest star to Earth is Proxima Centauri, 4.3 light-years away. (a) At what constant velocity must a spacecraft travel from Earth

if it is to reach the star in 4.0 years, as measured by travelers on the spacecraft? (b) How long does the trip take according to Earth observers?

SOLUTION. (a) To travelers on the spacecraft, the distance to the star is contracted:

$$L = L_0\sqrt{1-\beta^2} = (4.3 \text{ ly})\sqrt{1-\beta^2}.$$

Because the star is moving toward the spacecraft, to cover this distance in 4.0 yr, the speed of the star must be

$$v = \frac{L}{t} = \frac{4.3 \text{ ly}}{4.0 \text{ yr}}\sqrt{1-\beta^2} = (1.075\,c)\sqrt{1-\beta^2},$$

which gives $v = 0.73\,c$. Thus relative to the Earth-star system, the speed of the spacecraft is $0.73\,c$.

(b) According to observers on Earth, clocks on the spacecraft run slow:

$$t_{\text{Earth}} = \frac{t}{\sqrt{1-\beta^2}} = \frac{4.0 \text{ yr}}{\sqrt{1-(0.73)^2}} = 5.9 \text{ yr}.$$

Note that this agrees with the time found from distance and speed:

$$t_{\text{Earth}} = \frac{L_0}{v} = \frac{4.3 \text{ ly}}{0.73\,c} = 5.9 \text{ yr}.$$

1–63. Derive a formula showing how the apparent density of an object changes with speed v relative to an observer. [*Hint*: Use relativistic mass.]

SOLUTION. The dependence of the relativistic mass on the speed is

$$m_{\text{rel}} = \gamma m.$$

If we consider a box with sides x_0, y_0, and z_0, dimensions perpendicular to the motion, which we take to be the x-axis, do not change, but the length in the direction of motion will contract:

$$x = \frac{x_0}{\gamma}\,.$$

Thus the density is

$$\rho = \frac{m_{\text{rel}}}{xyz} = \frac{\gamma^2 m}{x_0 y_0 z_0} = \frac{\rho_0}{1-\beta^2}.$$

1–64. An airplane travels 1 500 km/h around the world, returning to the same place in a circle of radius essentially equal to that of the Earth. Estimate the difference in time to make the trip as seen by Earth and airplane observers. [*Hint*: Use the binomial expansion.]

SOLUTION. We convert the speed: (1 500 km/h)/(3 600 s/h) = 417 m/s. The flight time as observed on Earth is

$$t_{\text{Earth}} = \frac{2\pi r_{\text{E}}}{v} = \frac{2\pi(6.38 \times 10^6 \text{ m})}{417 \text{ m/s}} = 9.62 \times 10^4 \text{ s}.$$

The clock on the plane will run slow:

$$t_{\text{plane}} = t_{\text{Earth}}\sqrt{1-\beta^2} \approx t_{\text{Earth}}\left(1 - \tfrac{1}{2}\beta^2\right).$$

Thus the difference in times is

$$\begin{aligned} t_{\text{Earth}} - t_{\text{plane}} &= t_{\text{Earth}}\tfrac{1}{2}\beta^2 = (9.62 \times 10^4 \text{ s})\frac{1}{2}\left(\frac{417 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}}\right)^2 \\ &= 9.3 \times 10^{-8} \text{ s}. \end{aligned}$$

1–65. (a) What is the speed of an electron whose kinetic energy is 10 000 times its rest energy? Such speeds are reached in the Stanford Linear Accelerator, SLAC. (b) If the electrons travel in the lab through a tube 3.0 km long (as at SLAC), how long is this tube in the electrons' reference frame?

SOLUTION. (a) We find the speed from

$$\begin{aligned} K &= mc^2(\gamma - 1); \\ 10\,000\, mc^2 &= mc^2\left(\frac{1}{1-\beta^2} - 1\right), \end{aligned}$$

which gives

$$\sqrt{1-\beta^2} = 1.00 \times 10^{-4},$$

or

$$\beta^2 = 1 - 1.00 \times 10^{-8}.$$

When we take the square root, we get

$$\beta = \sqrt{1 - 1.00 \times 10^{-8}} \approx 1 - \tfrac{1}{2}(1.00 \times 10^{-8}) = 1 - 0.50 \times 10^{-8}.$$

Thus the speed is 1. 5 m/s less than c.

(b) The contracted length of the tube is

$$L = L_0\sqrt{1-\beta^2} = (3.0 \text{ km})(1.00 \times 10^{-4}) = 3.0 \times 10^{-4} \text{ km} = 30 \text{ cm}.$$

1–66. How many grams of matter would have to be totally destroyed to run a 100-W lightbulb for 1 year?

SOLUTION. We find the mass change from the required energy:

$$\begin{aligned} E &= Pt = mc^2; \\ (100 \text{ W})(3.16 \times 10^7 \text{ s}) &= m\left(3.00 \times 10^8 \text{ m/s}\right)^2, \end{aligned}$$

which gives $m = 3.5 \times 10^{-8}$ kg.

1–67. What minimum amount of electromagnetic energy is needed to produce an electron and a positron together? A positron is a particle with the same mass as an electron, but has the opposite charge. (Note that electric charge is conserved in this process. See Example 2–8.)

SOLUTION. The minimum energy is required to produce the pair at rest:

$$E_{\text{min}} = 2mc^2 = 2(0.511 \text{ MeV}) = 1.02 \text{ MeV} = 1.64 \times 10^{-13} \text{ J}.$$

1–68. A 1. 68-kg mass oscillates on the end of a spring whose spring constant is $k = 48.7$ N/m. If this system is in a spaceship moving past Earth at $0.900\,c$, what is its period of oscillation according to (a) observers on the ship, and (b) observers on Earth?

SOLUTION. (a) Because the spring is at rest on the spaceship, its period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1.68 \text{ kg}}{48.7 \text{ N/m}}} = 1.17 \text{ s}.$$

(b) The oscillating mass is a clock. According to observers on Earth, clocks on the spacecraft run slow:

$$T_{\text{Earth}} = \gamma T = \frac{1.17 \text{ s}}{\sqrt{1-(0.900)^2}} = 2.68 \text{ s}.$$

1–69. An electron ($m = 9.11 \times 10^{-31}$ kg) enters a uniform magnetic field $B = 1.8$ T, and moves perpendicular to the field lines with a speed $v = 0.92\,c$. What is the radius of curvature of its path?

SOLUTION. The speed is constant so the relativistic mass is constant. The magnetic force provides the radial acceleration:

$$qvB = \frac{m_{\text{rel}}v^2}{r},$$

or

$$\begin{aligned} r &= \frac{m_{\text{rel}}v}{qB} = \frac{mv}{qB\sqrt{1-\beta^2}} = \frac{(9.11\times10^{-31}\text{ kg})(0.92)(3.00\times10^{8}\text{ m/s})}{(1.6\times10^{-19}\text{ C})(1.8\text{ T})\sqrt{1-(0.92)^2}} \\ &= 2.2\times10^{-3}\text{ m} = 2.2\text{ mm}. \end{aligned}$$

1–70. An observer on Earth sees an alien vessel approach at a speed of $0.60\,c$. The *Enterprise* comes to the rescue (Fig. 1.3), overtaking the aliens while moving directly toward Earth at a speed of $0.90\,c$ relative to Earth. What is the relative speed of one vessel as seen by the other?

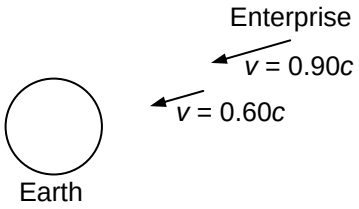


Figure 1.3. Problem 1–70.

SOLUTION. We take the positive direction in the direction of the *Enterprise*. In the reference frame of the alien vessel, the Earth is moving at $-0.60\,c$, and the *Enterprise* is moving at $+0.90\,c$ relative to the Earth. Thus the speed of the *Enterprise* relative to the alien vessel is

$$u = \frac{v+u'}{1+vu'/c^2} = \frac{-0.60\,c+0.90\,c}{1+(-0.60)(+0.90)} = 0.65\,c.$$

Note that the relative speed of the two vessels as seen on Earth is $0.90\,c - 0.60\,c = 0.30\,c$.

1–71. A free neutron can decay into a proton, an electron, and a neutrino. Assume the neutrino's rest mass is zero, and the other masses

can be found in the Appendices A and B. Determine the total kinetic energy shared among the three particles when a neutron decays at rest.

SOLUTION. The kinetic energy comes from the decrease in mass:

$$\begin{aligned} K &= (m_{\mathrm{n}} - (m_{\mathrm{p}} + m_{\mathrm{e}} + m_{\nu}))\, c^2 \\ &= (1.008665\ \mathrm{u} - (1.00728\ \mathrm{u} + 0.000549\ \mathrm{u} + 0)) \\ &\quad \times c^2 \left(931.5\ \mathrm{MeV}/(\mathrm{u}c^2)\right) \\ &= 0.78\ \mathrm{MeV}. \end{aligned}$$

1–72. The Sun radiates energy at a rate of about 4×10^{26} W. (a) At what rate is the Sun's mass decreasing? (b) How long does it take for the Sun to lose a mass equal to that of Earth? (c) Estimate how long the Sun could last if it radiated constantly at this rate.

SOLUTION. (a) We find the rate of mass loss from

$$\frac{\Delta m}{\Delta t} = \frac{\Delta E/\Delta t}{c^2} = \frac{4 \times 10^{26}\ \mathrm{W}}{(3 \times 10^8\ \mathrm{m/s})^2} = 4 \times 10^9\ \mathrm{kg/s}.$$

(b) We find the time from

$$\Delta t = \frac{\Delta m}{\text{rate}} = \frac{5.98 \times 10^{24}\ \mathrm{kg}}{(4.4 \times 10^9\ \mathrm{kg/s})(3.16 \times 10^7\ \mathrm{s/yr})} = 4 \times 10^7\ \mathrm{yr}.$$

(c) We find the time for the Sun to lose all of its mass at this rate from

$$\Delta t = \frac{\Delta m}{\text{rate}} = \frac{2.0 \times 10^{30}\ \mathrm{kg}}{(4.4 \times 10^9\ \mathrm{kg/s})(3.16 \times 10^7\ \mathrm{s/yr})} = 1 \times 10^{13}\ \mathrm{yr}.$$

1–73. An unknown particle is measured to have a negative charge and a speed of 2.24×10^8 m/s. Its momentum is determined to be 3.07×10^{-22} kg · m/s. Identify the particle by finding its mass.

SOLUTION. The speed of the particle is

$$v = \frac{2.24 \times 10^8\ \mathrm{m/s}}{3.00 \times 10^8\ \mathrm{m/s}} = 0.747\, c.$$

We use the momentum to find the rest mass:

$$\begin{aligned} p &= \gamma m v; \\ 3.07 \times 10^{-22}\ \mathrm{kg \cdot m/s} &= \frac{m(0.747)(3.00 \times 10^8\ \mathrm{m/s})}{\sqrt{1 - (0.747)^2}}, \end{aligned}$$

which gives $m = 9.11 \times 10^{-31}$ kg. Because the particle has a negative charge, it is an electron.

1–74. How much energy would be required to break a helium nucleus into its constituents, two protons and two neutrons? The rest masses of a proton (including an electron), a neutron, and helium are, respectively, 1.00783 u, 1.00867 u, and 4.00260 u. (This is called the *total binding energy* of the $^4_2$He nucleus.)

SOLUTION. The binding energy is the energy required to provide the increase in mass:

$$\begin{aligned} E &= ((2m_\mathrm{p} + 2m_\mathrm{n}) - m_\mathrm{He})\, c^2 \\ &= (2(1.00783\ \mathrm{u}) + 2(1.00867\ \mathrm{u}) - 4.00260\ \mathrm{u}) \\ &\quad \times c^2 \left(931.5\ \mathrm{MeV{\cdot}u^{-1}{\cdot}}c^{-2}\right)) \\ &= 28.3\ \mathrm{MeV}. \end{aligned}$$

Note that the two electron masses included in the hydrogen atoms and the helium atom cancel.

1–75. What is the percentage increase in the (relativistic) mass of a car traveling 110 km/h as compared to at rest?

SOLUTION. We convert the speed: $(110\ \mathrm{km/h})/(3\,600\ \mathrm{s/h}) = 30.6\ \mathrm{m/s}$. Because this is much smaller than c, the relativistic mass of the car is

$$m_\mathrm{rel} = \gamma m \approx m\left(1 + \tfrac{1}{2}\beta^2\right).$$

The fractional change in mass is

$$\begin{aligned} \frac{m_\mathrm{rel} - m}{m} &= 1 + \tfrac{1}{2}\beta^2 - 1 = \tfrac{1}{2}\beta^2 = \frac{1}{2}\left(\frac{30.6\ \mathrm{m/s}}{3.00 \times 10^8\ \mathrm{m/s}}\right)^2 \\ &= 5.19 \times 10^{-15} = (5.19 \times 10^{-13})\%. \end{aligned}$$

1–76. Two protons, each having a speed of $0.935\,c$ in the laboratory, are moving toward each other. Determine (a) the momentum of each proton in the laboratory, (b) the total momentum of the two protons in the laboratory, and (c) the momentum of one proton as seen by the other proton.

SOLUTION. (a) The magnitudes of the momenta are equal:

$$\begin{aligned} p &= \gamma mv = \frac{(1.67 \times 10^{-27}\ \text{kg})(0.935)(3.00 \times 10^{8}\ \text{m/s})}{\sqrt{1-(0.935)^2}} \\ &= 1.32 \times 10^{-18}\ \text{kg} \cdot \text{m/s}. \end{aligned}$$

(b) Because the protons are moving in opposite directions, the sum of the momenta is 0.

(c) In the reference frame of one proton, the laboratory is moving at $0.935\,c$. The other proton is moving at $+0.935\,c$ relative to the laboratory. Thus the speed of the other proton relative to the first is

$$u = \frac{v + u'}{1 + vu'/c^2} = \frac{+0.935\,c + (+0.935\,c)}{1 + (+0.935)(+0.935)} = 0.998\,c.$$

The magnitude of the momentum of the other proton is

$$\begin{aligned} p &= \frac{mu}{\sqrt{1-u^2/c^2}} = \frac{(1.67 \times 10^{-27}\ \text{kg})(0.998)(3.00 \times 10^{8}\ \text{m/s})}{\sqrt{1-(0.998)^2}} \\ &= 7.45 \times 10^{-18}\ \text{kg} \cdot \text{m/s}. \end{aligned}$$

1–77. Show analytically that a particle with momentum p and energy E has a speed given by

$$v = \frac{pc^2}{E} = \frac{pc}{\sqrt{m^2c^2 + p^2}}.$$

SOLUTION. The relation between energy and momentum is

$$E = \sqrt{m^2c^4 + p^2c^2} = c\sqrt{m^2c^2 + p^2}.$$

For the momentum, we have

$$p = \gamma mv = \frac{Ev}{c^2},$$

or

$$v = \frac{pc^2}{E} = \frac{pc}{\sqrt{m^2c^2 + p^2}}.$$

1–78. A slab of glass moves to the right with speed v. A flash light is emitted at point A (Fig. 1.4) and passes through the glass arriving at point B a distance L away. The glass has thickness d in the reference frame where it is at rest, and the speed of light in the glass is c/n. How long does it take the light to go from point A to point B according to an observer at rest with respect to points A and B? Check your answer for the cases $v = 0$, $n = 1$, and for $v = c$.

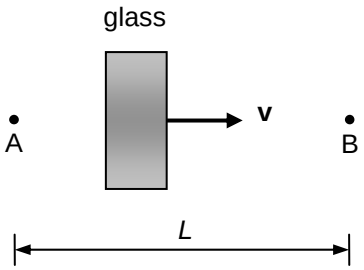


Figure 1.4. Problem 1–78.

SOLUTION. The speed of light in the medium at rest is c/n. We find the speed in the medium according to the observer from the addition of velocities:

$$v' = \frac{(c/n) + v}{1 + (c/n)v/c^2} = \frac{c/n + v}{1 + v/(cn)} = \frac{(c/n)(1 + vn/c)}{1 + v/(cn)}.$$

The thickness of the glass to the observer is contracted:

$$d' = d\sqrt{1 - \beta^2}.$$

Thus the time for light to go from A to B is

$$\begin{aligned} t' &= \frac{L - d'}{c} + \frac{d'}{v'} = \frac{L}{c} - d'\left(\frac{1}{c} - \frac{1}{v'}\right) \\ &= \frac{L}{c} - d\sqrt{1 - \beta^2}\left(\frac{1}{c} - \frac{(n/c)\,(1 + v/(cn))}{1 + vn/c}\right) \\ &= \frac{L}{c} + \frac{d}{c}\,\frac{(n-1)(1-\beta)\sqrt{1-\beta^2}}{1 + vn/c}\,. \end{aligned}$$

If $v = 0$, we have

$$t' = \frac{L}{c} + \frac{d(n-1)}{c} = \frac{L-d}{c} + \frac{d}{(c/n)},$$

which is the time for light to travel through the air plus the time to travel through the glass.

If $n = 1$, we have

$$t' = \frac{L}{c},$$

which is the same as the time in the case where the glass is not present.

If $v = c$, we have

$$t' = \frac{L}{c},$$

which agrees with the fact that the velocity c transforms into c.

1–79. Show that the space-time "distance" $(c\,\Delta t)^2 - (\Delta x)^2$ is invariant, meaning that all observers in all inertial reference frames calculate the same number for this quantity for any pair of events.

SOLUTION. From the Lorentz transformation we have

$$x' = \gamma\left(x - \frac{v}{c}\,ct\right), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right),$$

or

$$\Delta x' = \gamma\left(\Delta x - \beta c\,\Delta t\right), \quad c\,\Delta t' = \gamma\left(c\,\Delta t - \beta\,\Delta x\right).$$

Thus we have

$$\begin{aligned}
(c\,\Delta t')^2 - (\Delta x')^2 &= \gamma^2\left((c\,\Delta t)^2 - 2\beta\,\Delta x c\,\Delta t + \beta^2(\Delta x)^2\right) \\
&\quad - \gamma^2\left((\Delta x)^2 - 2\beta\,\Delta x c\,\Delta t + \beta^2(c\,\Delta t)^2\right) \\
&= \gamma^2\left((c\,\Delta t)^2\left(1-\beta^2\right) - (\Delta x)^2\left(1-\beta^2\right)\right) \\
&= \gamma^2\left(1-\beta^2\right)\left((c\,\Delta t)^2 - (\Delta x)^2\right) \\
&= (c\,\Delta t)^2 - (\Delta x)^2.
\end{aligned}$$

1–80. The fictional starship *Enterprise* obtains its power by combining matter and antimatter, achieving complete conversion of mass into energy. If the mass of the *Enterprise* is approximately 5×10^9 kg, how much mass

must be converted into kinetic energy to accelerate it from rest to one-tenth the speed of light?

SOLUTION. Because the speed achieved is $0.1\,c$, we can assume the mass converted is a small fraction of the mass of the *Enterprise*. We find the mass converted from

$$\begin{aligned} K = mc^2\,(\gamma - 1) &= \Delta m c^2; \\ m\left(\frac{1}{\sqrt{1-(0.1)^2}} - 1\right) &= \Delta m, \end{aligned}$$

which gives

$$\Delta m = 0.005\,m = (0.005)(5 \times 10^9 \text{ kg}) = 3 \times 10^7 \text{ kg}.$$

1–81. A spacecraft (reference frame S') moves past Earth (reference frame S) at velocity $\mathbf{v}$, which points along the x and x' axes. The spacecraft emits light along its y' axis as shown in Fig. 1.5. (a) What angle does this light make with the x axis in the Earth's reference frame? (b) Show that light moves with speed c also in the Earth's reference frame (i. e., given c in frame S'). (c) Compare these relativistic results to what you would have obtained classically (Galilean transformations).

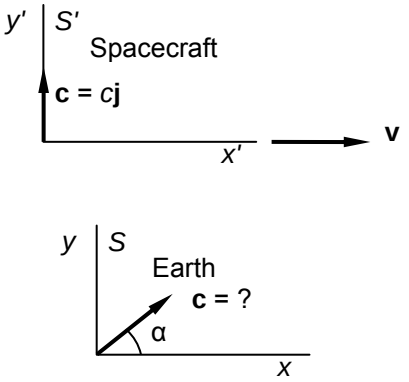


Figure 1.5. Problem 1–81.

SOLUTION. (a) The velocity components of the light in the S' frame are

$$u'_x = 0, \quad u'_y = c.$$

We find the velocity components in the Earth frame from the velocity transformation:

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + u'_x v/c^2} = \frac{0 + v}{1 + 0} = v; \\ u_y &= \frac{u'_y}{\gamma\left(1 + u'_x v/c^2\right)} = \frac{c\sqrt{1-\beta^2}}{1+0} = c\sqrt{1-\beta^2}. \end{aligned}$$

We find the angle the velocity makes with the x-axis from

$$\tan\alpha = \frac{u_y}{u_x} = \frac{c\sqrt{1-\beta^2}}{v} = \sqrt{\beta^{-2}-1},$$

so

$$\alpha = \arctan\left(\sqrt{\beta^{-2}-1}\right).$$

(b) The magnitude of the velocity is

$$u = \sqrt{u_x^2 + u_y^2} = \sqrt{v^2 + c^2\left(1 - \frac{v^2}{c^2}\right)} = c.$$

(c) Classically we have

$$\begin{aligned} u_x &= u'_x + v = 0 + v = v; \\ u_y &= u'_y = c. \end{aligned}$$

The angle would be

$$\theta = \arctan\left(\beta^{-1}\right).$$

The magnitude would be

$$u = \sqrt{u_x^2 + u_y^2} = c\sqrt{1+\beta^2}.$$

2 Early Quantum Theory and Models of the Atom

2.1 Review

Hot matter in condensed states nearly always emits radiation with a continuous distribution of wavelengths rather than a line spectrum. An ideal surface that absorbs all wavelengths of electromagnetic radiation incident upon it is also the best possible emitter of electromagnetic radiation at any wavelength. Such an ideal surface is called a *blackbody*. The wavelength at the peak of the spectrum of a blackbody radiation, λ_{p}, is related to the Kelvin temperature T by

$$\lambda_{\mathrm{p}} T \;=\; 2.90 \times 10^{-3}\ \mathrm{m \cdot K}. \tag{2.1}$$

This is known as *Wien's law*.

Example 2–1. Estimate the temperature of the surface of our Sun, given that the Sun emits light whose peak intensity occurs in the visible spectrum at around 500 nm.

Solution. Wien's law gives

$$T = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{\lambda_{\mathrm{p}}} = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{500 \times 10^{-9}\ \mathrm{m}} \approx 6\,000\ \mathrm{K}.$$

Example 2–2. Suppose a star has a surface temperature of 32 500 K. What color would this star appear?

Solution. From Wien's law we have,

$$\lambda_{\mathrm{p}} = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{T} = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{3.25 \times 10^{4}\ \mathrm{K}} = 89.2\ \mathrm{nm}.$$

The peak is in the UV range of the spectrum. In the visible region, the curve will be descending, so the shortest visible wavelengths will be strongest. Hence the star will appear bluish (or blue-white).

Quantum theory has its origins in *Planck's quantum hypothesis* that molecular oscillations are quantized: their energy E can only be integer (n) multiples of $2\pi\hbar f$, where $\hbar$ is reduced Planck's constant and f is the natural frequency of oscillation:

$$E = n 2\pi\hbar f. \tag{2.2}$$

This hypothesis explained the spectrum of radiation emitted by (black) bodies at high temperature. Einstein proposed that for some experiments, light could be pictured as being emitted and absorbed as quanta (particles), which we now call *photons*, each with energy

$$E = 2\pi\hbar f. \tag{2.3}$$

Example 2–3. Calculate the energy of a photon of blue light, $\lambda = 450$ nm.

SOLUTION. Since $f = c/\lambda$, we have

$$E = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(3.0 \times 10^{8}\ \text{m/s})}{4.5 \times 10^{-7}\ \text{m}} = 4.4 \times 10^{-19}\ \text{J},$$

or $(4.4 \times 10^{-19}\ \text{J})/(1.6 \times 10^{-19}\ \text{J/eV}) = 2.7$ eV.

Example 2–4. Estimate how many visible light photons a 100-W lightbulb emits per second.

SOLUTION. Let's assume an average wavelength in the middle of the visible spectrum, $\lambda \approx 500$ nm. The energy emitted in one second ($= 100$ J) is $E = n2\pi\hbar f$ where n is the number of photons emitted per second and $f = c/\lambda$. Hence

$$n = \frac{E}{2\pi\hbar f} = \frac{E\lambda}{2\pi\hbar c} = \frac{(100\ \text{J})(500 \times 10^{-9}\ \text{m})}{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(3.0 \times 10^{8}\ \text{m/s})} = 2.5 \times 10^{20}.$$

This is an overestimate since much of the 100 J of electric energy input is transformed into heat rather than light. If the efficiency is between 1 percent and 10 percent, then the number of photons emitted is on the order of 10^{19}, still an enormous number.

Einstein proposed the photoelectric effect as a test for the photon theory of light. In the *photoelectric effect*, the photon theory says that each incident photon can strike an electron in a material and eject it if it has sufficient energy. The maximum energy of ejected electrons is then linearly related to the frequency of the incident light:

$$2\pi\hbar f = K_{\max} + W_0, \tag{2.4}$$

where f is the frequency of the incoming light, $K_{\max}$ is the maximum outgoing kinetic energy of the electrons, and W_0 is the *work function*.

Example 2–5. What is the maximum kinetic energy and speed of an electron ejected from a sodium surface whose work function is $W_0 = 2.28$ eV when illuminated by light of wavelength: (a) 410 nm; (b) 550 nm?

SOLUTION. (a) For $\lambda = 410$ nm,

$$2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = 4.85 \times 10^{-19} \text{ J} = 3.03 \text{ eV}.$$

From Eq. 2.4, $K_{\max} = 3.03 \text{ eV} - 2.28 \text{ eV} = 0.75 \text{ eV}$, or 1.2×10^{-19} J.

Since $K = \frac{1}{2}mv^2$, where $m = 9.1 \times 10^{-31}$ kg,

$$v = \sqrt{\frac{2K}{m}} = 5.1 \times 10^5 \text{ m/s}.$$

Notice that we used the nonrelativistic equation for kinetic energy. If v had turned out to be more than about $0.1\,c$, our calculation would have been inaccurate by more than a percent or so, and we would probably prefer to redo it using the relativistic form.

(b) For $\lambda = 550$ nm, $2\pi\hbar f = 3.61 \times 10^{-19}$ J $= 2.26$ eV. Since this photon energy is less than the work function, no electrons are ejected.

Example 2–6. In *photosynthesis*, the process by which pigments such as chlorophyll in plants capture the energy of sunlight to change CO_2 to useful carbohydrate, about nine photons are needed to transform one molecule of CO_2 to carbohydrate and O_2. Assuming light of wavelength $\lambda = 670$ nm (chlorophyll absorbs most strongly in the range 650 nm to 700 nm), how efficient is the photosynthetic process? The reverse chemical reaction releases an energy of 4.9 eV/molecule of CO_2.

SOLUTION. The energy of nine photons, each of energy $2\pi\hbar f = 2\pi\hbar c/\lambda$ is

$$\frac{(9)(6.6 \times 10^{-34} \text{ J} \cdot \text{s})(3.0 \times 10^8 \text{ m/s})}{6.7 \times 10^{-7} \text{ m}} = 2.7 \times 10^{-18} \text{ J} = 17 \text{ eV}.$$

Thus the process is (4.9 eV)/(17 eV) = 29 percent efficient.

The photon theory is also supported by the *Compton effect* (Fig. 2.1). A single photon of wavelength λ strikes an electron in some material, knocking it out its atom. The scattered photon has less energy (since some is given to the electron) and hence has a longer wavelength λ'. The

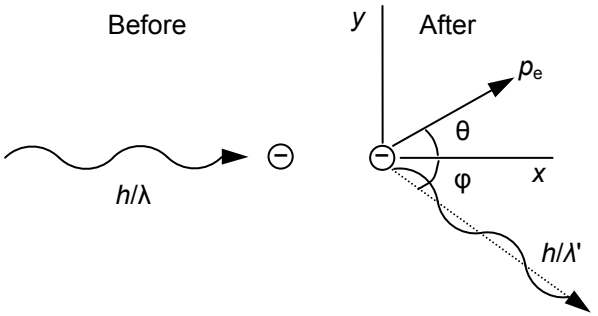


Figure 2.1. The Compton effect.

difference in wavelength in terms of the scattering angle φ

$$\Delta\lambda = \lambda' - \lambda = \lambda_C(1 - \cos\varphi), \tag{2.5}$$

is called the *Compton shift*. The quantity

$$\lambda_C = \frac{2\pi\hbar}{m_e c} = 0.00243 \text{ nm}$$

is the *Compton wavelength*.

Example 2–7. X-rays of wavelength 0.140 nm are scattered from a block of carbon. What will be the wavelengths of X-rays scattered at (a) 0°, (b) 90°, (c) 180°?

SOLUTION. (a) For $\varphi = 0°$, $\cos\varphi = 1$, and Eq. 2.5 gives $\lambda' = \lambda = 0.140$ nm. This makes sense since for $\varphi = 0°$, there really isn't any collision as the photon goes straight through without interacting.

(b) For $\varphi = 90°$, $\cos\varphi = 0$, so

$$\lambda' = \lambda + \lambda_C = 0.140 \text{ nm} + 2.4 \times 10^{-12} \text{ m} = 0.142 \text{ nm};$$

that is, the wavelength is longer by one Compton wavelength.

(c) For $\varphi = 180°$, which means the photon is scattered backward, returning in the direction from which it came (a direct "head-on" collision), $\cos\varphi = -1$, so

$$\lambda' = \lambda + 2\lambda_C = 0.140 \text{ nm} + 2(0.0024 \text{ nm}) = 0.145 \text{ nm}.$$

The photon theory is also supported by the observation of electron-positron *pair production*: a photon disappears and produces an electron and a positron.

Example 2–8. What is (a) the minimum energy of a photon that can produce an electron-positron pair? (b) What is this photon's wavelength?

SOLUTION. (a) Because $E = mc^2$, and the mass created is equal to two electron masses, the photon must have energy

$$\begin{aligned} E &= 2(9.11 \times 10^{-31}\ \text{kg})(3.0 \times 10^{8}\ \text{m/s})^2 \\ &= 1.64 \times 10^{-13}\ \text{J} = 1.02\ \text{MeV}. \end{aligned}$$

A photon with less energy cannot undergo pair production.

(b) Since $E = 2\pi\hbar f = 2\pi\hbar c/\lambda$, the wavelength of a 1.02-MeV photon is

$$\lambda = \frac{2\pi\hbar c}{E} = \frac{(6.6 \times 10^{-34}\ \text{J}\cdot\text{s})(3.0 \times 10^{8}\ \text{m/s})}{1.64 \times 10^{-13}\ \text{J}} = 1.2 \times 10^{-12}\ \text{m},$$

which is 0.0012 nm. Such photons are in the gamma-ray (or very short X-ray) region of the electromagnetic spectrum.

The wave-particle duality refers to the idea that light and matter (such as electrons) have both wave and particle properties. The wavelength of a material object is given by

$$\lambda = \frac{2\pi\hbar}{p}, \tag{2.6}$$

where p is the momentum of the object. This is sometimes called the *de Broglie wavelength*. The principle of complementarity states that we must be aware of both the particle and wave properties of light and of matter for a complete understanding of them.

Example 2–9. Calculate the de Broglie wavelength of a 0.20-kg ball moving with a speed of 15 m/s. Discuss.

SOLUTION.

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{mv} = \frac{6.6 \times 10^{-34}\ \text{J}\cdot\text{s}}{(0.20\ \text{kg})(15\ \text{m/s})} = 2.2 \times 10^{-34}\ \text{m}.$$

This is an incredibly small wavelength. Even if the speed were extremely small, say 10^{-4} m/s, the wavelength would be about 10^{-29} m. Indeed, the

wavelength of any ordinary object is much too small to be measured and detected. The problem is that the properties of waves, such as interference and diffraction, are significant only when the size of objects or slits is not much larger than the wavelength. And there are no known objects or slits to diffract waves only 10^{-30} m long, so the wave properties of ordinary objects go undetected.

But tiny elementary particles, such as electrons, are another matter. Since the the mass m appears in the denominator in Eq. 2.6, a very small mass should have a much larger wavelength.

Example 2–10. Determine the wavelength of an electron that has been accelerated through a potential difference of 100 V.

SOLUTION. We assume that the speed of the electron will be much less than c, so we use nonrelativistic mechanics. (If this assumption were to come out wrong, we would have to recalculate using relativistic formulae.) The gain in kinetic energy will equal the loss in potential energy, so $\frac{1}{2}mv^2 = eV$ and

$$v = \sqrt{\frac{2eV}{m}} = \sqrt{\frac{(2)(1.6\times10^{-19}\text{ C})(100\text{ V})}{9.1\times10^{-31}\text{ kg}}} = 5.9\times10^{6}\text{ m/s}.$$

Then

$$\begin{aligned}\lambda &= \frac{2\pi\hbar}{mv} = \frac{6.6\times10^{-34}\text{ J}\cdot\text{s}}{(9.1\times10^{-31}\text{ kg})(5.9\times10^{6}\text{ m/s})}\\ &= 1.2\times10^{-10}\text{ m} = 0.12\text{ nm}.\end{aligned}$$

Example 2–11. The wave nature of electrons is manifested in experiments where an electron beam interacts with the atoms on the surface of a solid. By studying the angular distribution of the diffracted electrons, one can indirectly measure the geometrical arrangement of atoms. Assume that the electrons strike perpendicular to the surface of a solid (see Fig. 2.2) and that their energy is low, $K = 100$ eV, so that they interact only with the surface layer of atoms. If the smallest angle at which a diffraction maximum occurs is 24°, what is the separation d between the atoms on the surface?

SOLUTION. Treating the electrons as waves, we need to determine the condition where the difference in path traveled by the wave diffracted from

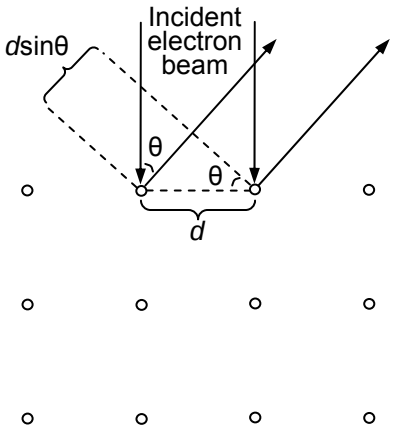


Figure 2.2. Example 2–11.

adjacent atoms is an integer multiple of the de Broglie wavelength, so that constructive interference occurs. The path length difference is $d \sin\theta$; so for the smallest value of θ we must have

$$d \sin\theta = \lambda.$$

However, λ is related to the (non-relativistic) kinetic energy K by

$$K = \frac{p^2}{2m_\mathrm{e}} = \frac{(2\pi\hbar)^2}{2m_\mathrm{e}\lambda^2}.$$

Thus

$$\begin{aligned} \lambda &= \frac{2\pi\hbar}{\sqrt{2m_\mathrm{e}K}} = \frac{6.63 \times 10^{-34}\ \mathrm{J \cdot s}}{\sqrt{2(9.11 \times 10^{-31}\ \mathrm{kg})(100\ \mathrm{eV})(1.6 \times 10^{-19}\ \mathrm{J/eV})}} \\ &= 0.123\ \mathrm{nm}. \end{aligned}$$

The surface inter-atomic spacing is

$$d = \frac{\lambda}{\sin\theta} = \frac{0.123\ \mathrm{nm}}{\sin 24^\circ} = 0.30\ \mathrm{nm}.$$

Early models of the atom include the plum-pudding model and Rutherford's planetary (or nuclear) model. Rutherford's model, which was created to explain the backscattering of α particles from thin metal foils, assumes that an atom consists of a tiny but massive positively charged nucleus surrounded (at a relatively great distance) by electrons. The two main difficulties of the Rutherford model are these: (1) it predicts that light of a continuous range of frequencies will be emitted; (2) it predicts that atoms in general are unstable—electrons should quickly spiral into the nucleus.

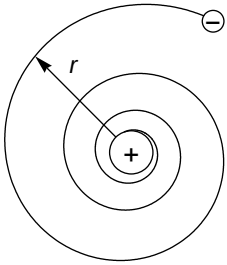


Figure 2.3. Example 2–12.

Example 2–12. According to the classical physics, a charge e moving with an acceleration a radiates at a rate

$$\frac{dE}{dt} = -\frac{2k_\mathrm{e}}{3}\frac{e^2a^2}{c^3}.$$

Find the time interval over which the electron in a classical hydrogen atom will spiral into the nucleus (Fig. 2.3), starting from the Bohr radius r_B.

Solution. For a classical atom, the centripetal acceleration is

$$a = \frac{v^2}{r} = \frac{F_{\text{Coulomb attraction}}}{m_\mathrm{e}} = k_\mathrm{e}\frac{e^2}{r^2m_\mathrm{e}},$$

$$\text{so} \quad \frac{dE}{dt} = -\frac{2k_\mathrm{e}^3}{3}\frac{e^6}{c^3r^4m_\mathrm{e}^2} \quad \text{and} \quad v^2 = k_\mathrm{e}\frac{e^2}{rm_\mathrm{e}}.$$

On the other hand,

$$\begin{aligned} E &= -k_\mathrm{e}\frac{e^2}{r} + \frac{m_\mathrm{e}v^2}{2} = -k_\mathrm{e}\frac{e^2}{2r}, \\ \text{so} \quad \frac{dE}{dt} &= k_\mathrm{e}\frac{e^2}{2r^2}\frac{dr}{dt}. \end{aligned}$$

Thus we have a differential equation

$$\frac{dr}{dt} = -\frac{4k_\mathrm{e}}{3}\frac{e^4}{c^3 m_\mathrm{e}^2 r^2}, \qquad r(0) = r_\mathrm{B}.$$

Separating variables we find

$$\begin{aligned} \int_0^\tau dt &= -\frac{3}{4k_\mathrm{e}^2}\frac{c^3 m_\mathrm{e}^2}{e^4}\int_{r_\mathrm{B}}^0 dr, \\ \tau &= \frac{1}{4}\left(\frac{m_\mathrm{e}c^2}{k_\mathrm{e}e^2}\right)^2 \frac{r_\mathrm{B}^3}{c} \\ &= \frac{1}{4}\left(\frac{0.511\ \mathrm{MeV}}{1.44\ \mathrm{MeV}\cdot 10^{-15}\ \mathrm{m}}\right)^2 \frac{(0.529\times 10^{-10}\ \mathrm{m})^3}{3.00\times 10^8\ \mathrm{m/s}} = 15.5\ \mathrm{ps}. \end{aligned}$$

To explain the line spectra emitted by atoms, as well as the stability of atoms, *Bohr theory* postulated (1) that electrons bound in an atom can only occupy orbits for which the angular momentum is quantized, which results in discrete values for the radius and energy; (2) that an electron in such a *stationary state* emits no radiation; (3) that, if an electron jumps to a lower state, it emits a photon whose energy equals the difference between the two states; (4) that the angular momentum L of atomic electrons is quantized by the rule

$$L = n\hbar, \tag{2.7}$$

where n is an integer called a *quantum number* of the orbit. The radii of all possible orbits in an atom with Z protons and a single electron of mass m and charge e is given by the Bohr formula

$$r_n = \frac{\hbar^2 n^2}{Z k_\mathrm{e} e^2 m_\mathrm{e}} = \frac{n^2}{Z} r_\mathrm{B}, \tag{2.8}$$

where

$$r_{\mathrm{B}} = r_1 = \frac{\hbar^2}{(k_{\mathrm{e}}e^2)m_{\mathrm{e}}} = \frac{(6.58\times10^{-22}\ \mathrm{MeV\cdot s})^2(3\times10^8\ \mathrm{m/s})^2}{(1.44\ \mathrm{MeV}\cdot10^{-15}\ \mathrm{m})(0.511\ \mathrm{MeV})} = 0.529\times10^{-10}\ \mathrm{m} \tag{2.9}$$

is the *Bohr radius*.

The $n = 1$ state in hydrogen is the *ground state*, which has an energy

$$E_1 = -\frac{(k_{\mathrm{e}}e^2)^2m_{\mathrm{e}}}{2\hbar^2} = -13.6\ \mathrm{eV};$$

higher values of n correspond to *excited states* and their energies are

$$E_n = \frac{Z^2}{n^2}E_1. \tag{2.10}$$

Atoms are excited to these higher states by collisions with other atoms or electrons or by absorption of a photon of just the right frequency.

Wavelengths of the spectral lines emitted by single-electron ("hydrogenlike") atoms are given by the equation

$$\frac{1}{\lambda} = \frac{Z^2(k_{\mathrm{e}}e^2)^2m_{\mathrm{e}}}{4\pi\hbar^3c}\left(\frac{1}{n'^2} - \frac{1}{n^2}\right), \tag{2.11}$$

where n refers to the upper state and n' to the lower state. In particular, $n' = 1$ corresponds to the *Lyman series*, $n' = 2$ to the *Balmer series* (Fig. 2.4), and $n' = 3$ to the *Paschen series*. When the constant $Z^2(k_{\mathrm{e}}e^2)^2m_{\mathrm{e}}/4\pi\hbar^3c$ in Eq. 2.11 is evaluated with $Z = 1$, it is found to have the measured value of the *Rydberg constant*, $R_\infty = 1.0974\times10^7\ \mathrm{m}^{-1}$.

Example 2–13. Determine the wavelength of the first Lyman line, the transition from $n = 2$ to $n = 1$. In what region of the electromagnetic spectrum does this lie?

SOLUTION. In this case, $2\pi\hbar f = E_2 - E_1 = 13.6\ \mathrm{eV} - 3.4\ \mathrm{eV} = 10.2\ \mathrm{eV} = 1.63\times10^{-18}$ J. Since $\lambda = c/f$, we have

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{E_2 - E_1} = \frac{(6.63\times10^{-34}\ \mathrm{J\cdot s})(3.00\times10^8\ \mathrm{m/s})}{1.63\times10^{-18}\ \mathrm{J}} = 1.22\times10^{-7}\ \mathrm{m} = 122\ \mathrm{nm},$$

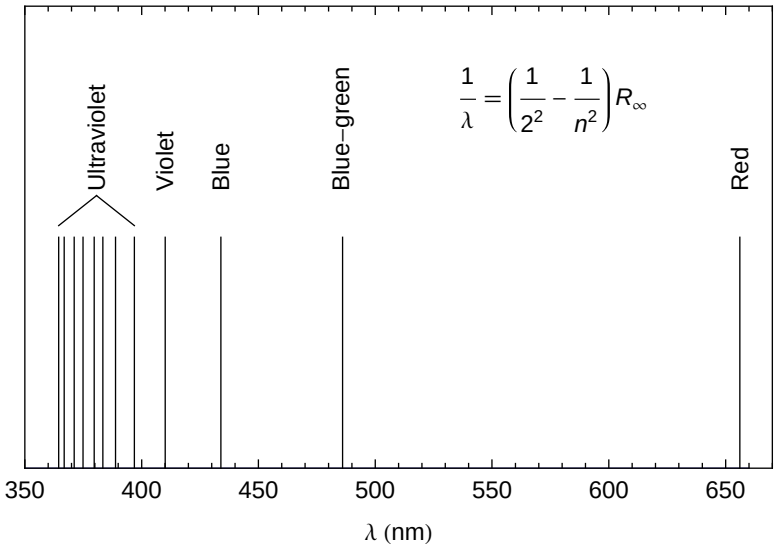


Figure 2.4. Balmer series of lines for hydrogen.

which is in the UV region.

Example 2–14. Determine the wavelength of light emitted when a hydrogen atom makes a transition from the $n = 6$ to the $n = 2$ energy level according to the Bohr model.

Solution. We can use Eq. 2.11. Thus

$$\frac{1}{\lambda} = (1.097 \times 10^7 \text{ m}^{-1}) \left(\frac{1}{4} - \frac{1}{36}\right) = 2.44 \times 10^6 \text{ m}^{-1}.$$

So $\lambda = (2.44 \times 10^6 \text{ m}^{-1})^{-1} = 4.10 \times 10^{-7}$ m $= 410$ nm. This is the fourth line in the Balmer series, and violet in color.

Example 2–15. Determine the maximum wavelength that hydrogen in its ground state can absorb. What would be the next smaller wavelength that would work?

Solution. Maximum λ corresponds to minimal energy, and thus the jump from the ground state to the first excited state, $1 \rightarrow 2$, for which the energy is $(-13.6 \text{ eV})(2^{-2} - 1^{-2}) = 10.2$ eV; the required wavelength,

as we saw in Example 12, is 122 nm. The next possibility is to jump from the ground state to the second excited state, $1 \rightarrow 3$, which requires $(-13.6\text{ eV})(3^{-2} - 1^{-2}) = 12.1\text{ eV}$ and corresponds to a wavelength

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{E_3 - E_1} = \frac{(6.63 \times 10^{-34}\text{ J}\cdot\text{s})(3.00 \times 10^{8}\text{ m/s})}{(12.1\text{ eV})(1.60 \times 10^{-19}\text{ J/eV})} = 103\text{ nm}.$$

Example 2–16. Use the Bohr model to determine the ionization energy of the He^+ ion, which has a single electron. Also calculate the minimum wavelength a photon must have to cause ionization.

SOLUTION. We want to determine the minimum energy required to lift the electron from its ground state and to barely reach the free state at $E = 0$. The ground state energy of He^+ is given by Eq. 2.10 with $n = 1$ and $Z = 2$. Since all the symbols in Eq. 2.10 are the same as for the calculation for hydrogen, except that Z is 2 instead of 1, we see that E_1 will be $Z^2 = 2^2 = 4$ times the E_1 for hydrogen. That is,

$$E_1 = 4(-13.6\text{ eV}) = -54.4\text{ eV}.$$

Thus, to ionize the He^+ ion should require 54.4 eV, and this value agrees with experiment. The minimum wavelength photon that can cause ionization will have energy $2\pi\hbar f = 54.4$ eV and wavelength

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} = \frac{(6.63 \times 10^{-34}\text{ J}\cdot\text{s})(3.00 \times 10^{8}\text{ m/s})}{(54.4\text{ eV})(1.60 \times 10^{-19}\text{ J/eV})} = 22.8\text{ nm}.$$

If the atom absorbed a photon of greater energy (wavelength shorter than 22.8 nm), the atom could still be ionized and the freed electron would have kinetic energy of its own.

Example 2–17. Estimate the average kinetic energy of hydrogen atoms (or molecules) at room temperature, and use the result to explain why nearly all H atoms are in the ground state at room temperature, and hence emit no light.

SOLUTION. According to kinetic theory, the average energy of atoms or molecules in a gas is given by

$$\overline{K} = \tfrac{3}{2}k_{\text{B}}T,$$

where $k_B = 1.38 \times 10^{-23}$ J/K is Boltzmann's constant, and T is the kelvin (absolute) temperature. Room temperature is about $T = 300$ K, so

$$\overline{K} = \tfrac{3}{2}(1.38 \times 10^{-23} \text{ J/K})(300 \text{ K}) = 6.2 \times 10^{-21} \text{ J},$$

or, in electron volts:

$$\overline{K} = \frac{6.2 \times 10^{-21} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = 0.04 \text{ eV}.$$

The average kinetic energy is thus very small compared to the energy between the ground state and the next higher energy state (13.6 eV − 3.4 eV = 10.2 eV). Any atoms in excited states emit light and eventually fall to the ground state. Once in the ground state, collisions with other atoms can transfer energy of only 0.04 eV on the average. A small fraction of atoms can have much more energy, but even kinetic energy that is 10 times the average is not nearly enough to excite atoms above the ground state. Thus, at room temperature, nearly all atoms are in the ground state. Atoms can be excited to upper states by very high temperatures, or by passing a current of high energy electrons through the gas, as in a discharge tube.

De Broglie's hypothesis that electrons (and other matter) have a wavelength $\lambda = 2\pi\hbar/p$ gave an explanation for Bohr's quantized orbits by bringing in the wave-particle duality: the orbits correspond to circular standing waves in which the circumference of the orbit equals a whole number of wavelengths.

2.2 Planck's quantum hypothesis

2–1. How hot is a metal being welded if it radiates most strongly at 440 nm?

SOLUTION. We find the temperature for a peak wavelength of 440 nm:

$$T = \frac{2.90 \times 10^{-3} \text{ m} \cdot \text{K}}{\lambda_p} = \frac{2.90 \times 10^{-3} \text{ m} \cdot \text{K}}{440 \times 10^{-9} \text{ m}} = 6.59 \times 10^3 \text{ K}.$$

2–2. (a) What is the temperature if the peak of a black body spectrum is at 25.0 nm? (b) What is the wavelength at the peak of a blackbody spectrum if the body is at a temperature of 2 800 K?

SOLUTION. (a) The temperature for a peak wavelength of 25.0 nm is

$$T = \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{\lambda_\text{p}} = \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{25.0 \times 10^{-9}\ \text{m}} = 1.16 \times 10^{5}\ \text{K}.$$

(b) We find the peak wavelength from

$$\begin{aligned} \lambda_\text{p} &= \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{T} = \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{2\,800\ \text{K}} = 1.04 \times 10^{-6}\ \text{m} \\ &= 1.04\ \mu\text{m}. \end{aligned}$$

Note that this is not in the visible range.

2–3. An HCl molecule vibrates with a natural frequency of 8.1×10^{13} Hz. What is the difference in energy (in joules and electron volts) between possible values of the oscillation energy?

SOLUTION. Because the energy is quantized, $E = n2\pi\hbar f$, the difference in energy between adjacent levels is

$$\begin{aligned} \Delta E &= 2\pi\hbar f = (6.63 \times 10^{-34}\ \text{J} \cdot \text{s})(8.1 \times 10^{13}\ \text{Hz}) = 5.4 \times 10^{-20}\ \text{J} \\ &= 0.34\ \text{eV}. \end{aligned}$$

2–4. Estimate the peak wavelength for radiation from (a) ice at 0°C, (b) a floodlamp at 3 300 K, (c) helium at 4 K, assuming blackbody emission. In what region of the EM spectrum is each?

SOLUTION. (a) We find the peak wavelength from

$$\begin{aligned} \lambda_\text{p} &= \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{T} = \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{273\ \text{K}} = 1.06 \times 10^{-5}\ \text{m} \\ &= 10.6\ \mu\text{m}. \end{aligned}$$

This wavelength is in the infrared.

(b) We find the peak wavelength from

$$\begin{aligned} \lambda_\text{p} &= \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{T} = \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{3\,300\ \text{K}} = 8.79 \times 10^{-7}\ \text{m} \\ &= 879\ \text{nm}. \end{aligned}$$

This wavelength is in the near infrared.

(c) We find the peak wavelength from

$$\begin{aligned}\lambda_{\rm p} &= \frac{2.90\times10^{-3}\ {\rm m\cdot K}}{T} = \frac{2.90\times10^{-3}\ {\rm m\cdot K}}{4\ {\rm K}} \\ &= 7.25\times10^{-4}\ {\rm m} = 0.73\ {\rm mm}.\end{aligned}$$

This wavelength is in the far infrared.

2–5. Estimate the peak wavelength of light issuing from the pupil of the human eye (which approximates a blackbody) assuming normal body temperature.

SOLUTION. We use a body temperature of $37^\circ{\rm C} = 310$ K. We find the peak wavelength from

$$\begin{aligned}\lambda_{\rm p} &= \frac{2.90\times10^{-3}\ {\rm m\cdot K}}{T} = \frac{2.90\times10^{-3}\ {\rm m\cdot K}}{310\ {\rm K}} = 9.4\times10^{-6}\ {\rm m} \\ &= 9.4\ \mu{\rm m}.\end{aligned}$$

2–6. Planck's radiation law is given by:

$$I(\lambda,T) = \frac{4\pi\hbar c^2\lambda^{-5}}{e^{2\pi\hbar c/\lambda k_{\rm B}T}-1}$$

where $I(\lambda,T)$ is the rate at which energy is radiated per unit area of emitting surface in the normal direction, per unit solid angle, per unit wavelength interval at wavelength λ and Kelvin temperature T. (a) Show that Wien's displacement law follows from this relationship. (b) Determine the value of $\hbar$ from the experimental value of $\lambda_{\rm p}T = 2.90\times10^{-3}\ {\rm m\cdot K}$. (c) Derive the Stefan–Boltzmann law (the T^4 dependence of the rate at which energy is radiated), by integrating Planck's formula over all wavelengths; that is, show that

$$\int I(\lambda,T)\,d\lambda \propto T^4.$$

SOLUTION. (a) To find the wavelength when $I(\lambda,T)$ is maximal at constant temperature, we set $dI/d\lambda = 0$:

$$\begin{aligned}\frac{dI}{d\lambda} &= \frac{d}{d\lambda}\left(\frac{4\pi\hbar c^2\lambda^{-5}}{e^{2\pi\hbar c/\lambda k_{\rm B}T}-1}\right) \\ &= \frac{4\pi c^2\hbar\left(e^{2\pi\hbar c/k_{\rm B}T\lambda}\left(2\pi\hbar c/k_{\rm B}T\lambda-5\right)+5\right)}{k_{\rm B}T\lambda^6\left(e^{2\pi\hbar c/k_{\rm B}T\lambda}-1\right)^2} = 0,\end{aligned}$$

which gives

$$\left(5 - \frac{2\pi\hbar c}{\lambda k_{\mathrm{B}} T}\right) e^{2\pi\hbar c/\lambda k_{\mathrm{B}} T} - 5 = 0,$$

or

$$5 - \frac{2\pi\hbar c}{\lambda k_{\mathrm{B}} T} = 5e^{-2\pi\hbar c/\lambda k_{\mathrm{B}} T}.$$

This equation will have a solution $\lambda_{\mathrm{p}} T = \text{constant}$, which is the Wien displacement law.

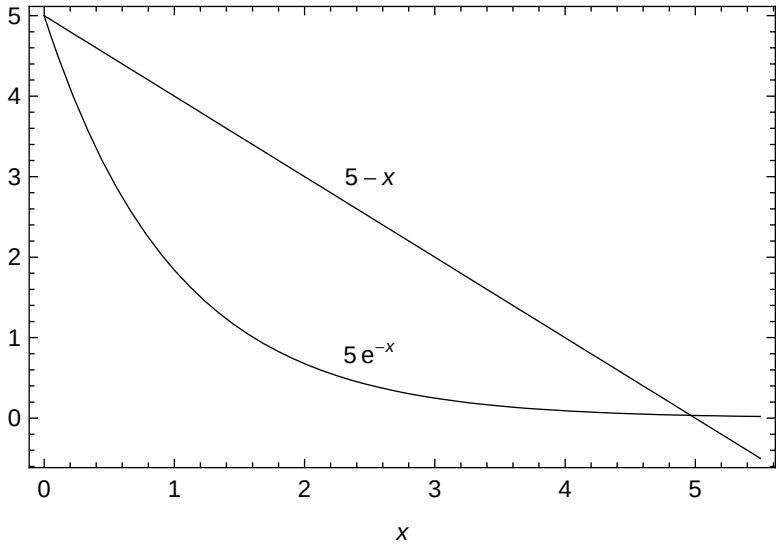


Figure 2.5. Problem 2–6.

(b) To find the value of the constant, we let $x = 2\pi\hbar c/(\lambda_{\mathrm{p}} k_{\mathrm{B}} T)$, so the transcendental equation is

$$5 - x = 5e^{-x}.$$

One way to solve this equation is to plot each side against x (Fig. 2.5). We see from the plot that the solution is very close to $x = 5$. If we let $\delta = 5 - x$, we get

$$\delta = 5e^{(\delta - 5)} \approx 5e^{-5} = 0.034,$$

so $x = 4.966$. Thus we have

$$\begin{aligned} \lambda_{\mathrm{p}} T &= \frac{2\pi\hbar c}{x k_{\mathrm{B}}}; \\ 2.90 \times 10^{-3}\ \mathrm{m \cdot K} &= \frac{2\pi\hbar(3.00 \times 10^{8}\ \mathrm{m/s})}{(4.966)(1.38 \times 10^{-23}\ \mathrm{J/K})}, \end{aligned}$$

which gives $\hbar = 1.05 \times 10^{-34}\ \mathrm{J \cdot s}$.

(c) For the rate at which energy is radiated per unit surface area for all wavelengths we have

$$\int_0^\infty I(\lambda, T)\, d\lambda = \int_0^\infty \frac{4\pi\hbar c^2 \lambda^{-5}}{e^{2\pi\hbar c/\lambda k_{\mathrm{B}} T} - 1}\, d\lambda.$$

If we change variable to $x = 2\pi\hbar c/\lambda k_{\mathrm{B}} T$, then $d\lambda = -\left(2\pi\hbar c/k_{\mathrm{B}} T x^2\right)\, dx$, so we have

$$\int_0^\infty I(\lambda, T)\, d\lambda = \frac{2(k_{\mathrm{B}} T)^4}{(2\pi\hbar)^3 c^2} \int_0^\infty \frac{x^3\, dx}{e^x - 1} \propto T^4.$$

2.3 Photon theory of light and the photoelectric effect

2–7. What is the energy range in eV, of photons in the visible spectrum, of wavelength 400 nm to 750 nm?

Solution. The energy of the photons with wavelengths at the ends of the visible spectrum are

$$\begin{aligned} E_1 &= 2\pi\hbar f_1 = \frac{2\pi\hbar c}{\lambda_1} = \frac{(1.24 \times 10^{3}\ \mathrm{eV \cdot nm})}{(400\ \mathrm{nm})} = 3.10\ \mathrm{eV}; \\ E_2 &= 2\pi\hbar f_2 = \frac{2\pi\hbar c}{\lambda_2} = \frac{(1.24 \times 10^{3}\ \mathrm{eV \cdot nm})}{(750\ \mathrm{nm})} = 1.65\ \mathrm{eV}. \end{aligned}$$

Thus the range of energies is $1.65\ \mathrm{eV} < E < 3.10\ \mathrm{eV}$.

2–8. What is the energy of photons (in eV) emitted by an 88.5-MHz FM radio station?

Solution. The energy of the photon is

$$\begin{aligned} E &= 2\pi\hbar f = (6.63 \times 10^{-34}\ \mathrm{J \cdot s})(88.5 \times 10^{6}\ \mathrm{Hz}) = 5.87 \times 10^{-26}\ \mathrm{J} \\ &= 3.67 \times 10^{-7}\ \mathrm{eV}. \end{aligned}$$

2–9. A typical gamma ray emitted from a nucleus during radioactive decay may have an energy of 300 keV. What is its wavelength? Would we expect significant diffraction of this type of light when it passes through an everyday opening, like a door?

SOLUTION. We find the wavelength from

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{E} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{300 \times 10^3 \text{ eV}} = 4.1 \times 10^{-3} \text{ nm}.$$

Significant diffraction occurs when the opening is on the order of the wavelength. Thus there would be insignificant diffraction through the doorway.

2–10. About 0.1 eV is required to break a "hydrogen bond" in a protein molecule. What are the minimum frequency and maximum wavelength of a photon that can accomplish this?

SOLUTION. The photon energy must be at least 0.1 eV. We find the minimum frequency from

$$\begin{aligned} E_{\min} &= 2\pi\hbar f_{\min}; \\ (0.1 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV}) &= (6.63 \times 10^{-34} \text{ J} \cdot \text{s}) f_{\min}, \end{aligned}$$

which gives $f_{\min} = 2.4 \times 10^{13}$ Hz.
The maximum wavelength is

$$\lambda_{\max} = \frac{c}{f_{\min}} = \frac{3.00 \times 10^8 \text{ m/s}}{2.4 \times 10^{13} \text{ Hz}} = 1.2 \times 10^{-5} \text{ m}.$$

2–11. What minimum frequency of light is needed to eject electrons from a metal whose work function is 4.3×10^{-19} J?

SOLUTION. At the threshold frequency, the kinetic energy of the photoelectrons is zero, so we have

$$\begin{aligned} K_{\max} &= 2\pi\hbar f - W_0 = 0; \\ 2\pi\hbar f_{\min} &= W_0; \\ (6.63 \times 10^{-34} \text{ J} \cdot \text{s}) f_{\min} &= 4.3 \times 10^{-19} \text{ J}, \end{aligned}$$

which gives $f_{\min} = 6.5 \times 10^{14}$ Hz.

2–12. What is the longest wavelength of light that will emit electrons from a metal whose work function is 3.10 eV?

SOLUTION. At the threshold wavelength, the kinetic energy of the photoelectrons is zero, so we have

$$\begin{aligned} K_{\max} &= 2\pi\hbar f - W_0 = 0; \\ \frac{2\pi\hbar c}{\lambda_{\max}} &= W_0, \end{aligned}$$

or

$$\lambda_{\max} = \frac{2\pi\hbar c}{W_0} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{3.10 \text{ eV}} = 400 \text{ nm}.$$

2–13. The work functions for sodium, cesium, copper and iron are 2.3, 2.1, 4.7 and 4.5 eV respectively. Which of these metals will not emit electrons when visible light shines on it?

SOLUTION. The photon of visible light with the maximum energy has the least wavelength:

$$2\pi\hbar f_{\max} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda_{\min}} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{400 \text{ nm}} = 3.10 \text{ eV}.$$

Electrons will not be emitted if this energy is less than the work function. The metals with work functions greater than 3.10 eV are copper and iron.

2–14. In a photoelectric-effect experiment it is observed that no current flows unless the wavelength is less than 570 nm. (a) What is the work function of this material? (b) What is the stopping voltage required if light of wavelength 400 nm is used?

SOLUTION. (a) At the threshold wavelength, the kinetic energy of the photoelectrons is zero, so we have

$$\begin{aligned} K_{\max} &= 2\pi\hbar f - W_0 = 0; \\ W_0 &= \frac{2\pi\hbar c}{\lambda_{\max}} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{570 \text{ nm}} = 2.18 \text{ eV}. \end{aligned}$$

(b) The stopping voltage is the voltage that gives a potential energy change equal to the maximum kinetic energy:

$$\begin{aligned} K_{\max} &= eV_0 = 2\pi\hbar f - W_0; \\ (1 \text{ e})V_0 &= \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{400 \text{ nm}} - 2.18 \text{ eV} = 3.10 \text{ eV} - 2.18 \text{ eV} \\ &= 0.92 \text{ eV}, \end{aligned}$$

so the stopping voltage is 0. 92 V.

2–15. What is the maximum kinetic energy of electrons ejected from barium ($W_0 = 2.48$ eV) when illuminated by white light, $\lambda = 400$ to 750 nm?

SOLUTION. The photon of visible light with the maximum energy has the minimum wavelength:

$$2\pi\hbar f_{\max} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda_{\min}} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{400 \text{ nm}} = 3.10 \text{ eV}.$$

The maximum kinetic energy of the photoelectrons is

$$K_{\max} = 2\pi\hbar f - W_0 = 3.10 \text{ eV} - 2.48 \text{ eV} = 0.62 \text{ eV}.$$

2–16. When UV light of wavelength 255 nm falls on a metal surface, the maximum kinetic energy of emitted electrons is 1. 40 eV. What is the work function of the metal?

SOLUTION. The energy of the photon is

$$2\pi\hbar f = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{255 \text{ nm}} = 4.86 \text{ eV}.$$

We find the work function from

$$\begin{aligned} K_{\max} &= 2\pi\hbar f - W_0; \\ 1.40 \text{ eV} &= 4.86 \text{ eV} - W_0, \end{aligned}$$

which gives $W_0 = 3.46$ eV.

2–17. The threshold wavelength for emission of electrons from a given surface is 350 nm. What will be the maximum kinetic energy of ejected electrons when the wavelength is changed to (a) 280 nm, (b) 380 nm?

SOLUTION. The threshold wavelength determines the work function:

$$W_0 = \frac{2\pi\hbar c}{\lambda_{\max}} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{ nm}}{350 \text{ nm}} = 3.54 \text{ eV}.$$

(a) The energy of the photon is

$$2\pi\hbar f = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{280 \text{ nm}} = 4.43 \text{ eV}.$$

The maximum kinetic energy of the photoelectrons is

$$K_{\max} = 2\pi\hbar f - W_0 = 4.43 \text{ eV} - 3.54 \text{ eV} = 0.89 \text{ eV}.$$

(b) Because the wavelength is greater than the threshold wavelength, the photon energy is less than the work function, so there will be no ejected electrons.

2–18. A certain type of film is sensitive only to light whose wavelength is less than 660 nm. What is the energy (eV and kcal/mol) needed for the chemical reaction to occur which causes the film to change?

SOLUTION. The energy required for the chemical reaction is provided by the photon:

$$E = \frac{2\pi\hbar c}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{ nm}}{660 \text{ nm}} = 1.88 \text{ eV}.$$

Each reaction takes place in a molecule, so we have

$$\begin{aligned} E &= \frac{(1.88 \text{ eV/molecule})(6.02 \times 10^{23} \text{ molecules/mol})}{4\,186 \text{ J/kcal}} \\ &\quad \times (1.60 \times 10^{-19} \text{ J/eV}) \\ &= 43.3 \text{ kcal/mol}. \end{aligned}$$

2–19. When 230-nm light falls on a metal, the current through a photoelectric circuit is brought to zero at a reverse voltage of 1.64 V. What is the work function of the metal?

SOLUTION. The reverse voltage is the voltage that gives a potential energy change equal to the maximum kinetic energy:

$$\begin{aligned} K_{\max} = eV_0 &= 2\pi\hbar f - W_0; \\ (1 \text{ e})(1.64 \text{ V}) &= \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{230 \text{ nm}} - W_0, \end{aligned}$$

which gives $W_0 = 3.75$ eV.

2.4 Photons and the Compton effect

2–20. The quantity $2\pi\hbar/mc$, which has the dimensions of length, is called the *Compton wavelength*. Determine the Compton wavelength for

(a) an electron, (b) a proton. (c) Show that if a photon has wavelength equal to the Compton wavelength of a particle, the photon's energy is equal to the rest energy of the particle.

SOLUTION. (a) For electron:

$$\frac{2\pi\hbar}{m_e c} = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(9.11 \times 10^{-31}\ \text{kg})(3.00 \times 10^{8}\ \text{m/s})} = 2.43 \times 10^{-12}\ \text{m}.$$

(b) For proton:

$$\frac{2\pi\hbar}{m_p c} = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(1.67 \times 10^{-27}\ \text{kg})(3.00 \times 10^{8}\ \text{m/s})} = 1.32 \times 10^{-15}\ \text{m}.$$

(c) For the energy of the photon we have

$$E = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{2\pi\hbar c}{(2\pi\hbar/mc)} = mc^2.$$

2–21. X-rays of wavelength $\lambda = 0.120$ nm are scattered from a carbon block. What is the Compton wavelength shift for photons detected at angles (relative to the incident beam) of (a) 45°, (b) 90°, (c) 180°?

SOLUTION. We find the Compton wavelength shift for a photon scattered from an electron:

$$\lambda' - \lambda = \frac{2\pi\hbar}{m_e c}(1 - \cos\varphi).$$

From the previous problem, $2\pi\hbar/m_e c = 2.43 \times 10^{-12}$ m for an electron.

(a) $\lambda'_a - \lambda = (2.43 \times 10^{-12}\ \text{m})(1 - \cos 45°) = 7.12 \times 10^{-13}$ m.

(b) $\lambda'_b - \lambda = (2.43 \times 10^{-12}\ \text{m})(1 - \cos 90°) = 2.43 \times 10^{-12}$ m.

(c) $\lambda'_c - \lambda = (2.43 \times 10^{-12}\ \text{m})(1 - \cos 180°) = 4.86 \times 10^{-12}$ m.

2–22. For each of the scattering angles in the previous problem, determine (a) the fractional energy loss of the photon and (b) the energy given to the scattered electron.

SOLUTION. (a) The energy of a photon is

$$E = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda}.$$

For the fractional loss, we have

$$\frac{E - E'}{E} = \frac{\lambda^{-1} - \lambda'^{-1}}{\lambda^{-1}} = \frac{\lambda' - \lambda}{\lambda'}.$$

For 45° we get

$$\frac{E - E'_{\mathrm{a}}}{E} = \frac{\lambda'_{\mathrm{a}} - \lambda}{\lambda'_{\mathrm{a}}} = \frac{7.12 \times 10^{-13}\ \mathrm{m}}{0.120 \times 10^{-9}\ \mathrm{m} + 7.12 \times 10^{-13}\ \mathrm{m}} = 5.90 \times 10^{-3}.$$

For 90° we get

$$\frac{E - E'_{\mathrm{b}}}{E} = \frac{\lambda'_{\mathrm{b}} - \lambda}{\lambda'_{\mathrm{b}}} = \frac{2.43 \times 10^{-12}\ \mathrm{m}}{0.120 \times 10^{-9}\ \mathrm{m} + 2.43 \times 10^{-12}\ \mathrm{m}} = 1.98 \times 10^{-2}.$$

For 180° we get

$$\frac{E - E'_{\mathrm{c}}}{E} = \frac{\lambda'_{\mathrm{c}} - \lambda}{\lambda'_{\mathrm{c}}} = \frac{4.86 \times 10^{-12}\ \mathrm{m}}{0.120 \times 10^{-9}\ \mathrm{m} + 4.86 \times 10^{-12}\ \mathrm{m}} = 3.89 \times 10^{-2}.$$

(b) The energy of the incident photon is

$$\begin{aligned} E &= 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{1.24 \times 10^{3}\ \mathrm{eV \cdot nm}}{\lambda} = \frac{1.24 \times 10^{3}\ \mathrm{eV \cdot nm}}{0.120\ \mathrm{nm}} \\ &= 10.3 \times 10^{3}\ \mathrm{eV}. \end{aligned}$$

From conservation of energy, the energy given to the scattered electron is the energy lost by the photon:

$$K = E - E' = \frac{E - E'}{E} E.$$

For 45° we get

$$K_{\mathrm{a}} = \frac{E - E'_{\mathrm{a}}}{E} E = (5.90 \times 10^{-3})(10.3 \times 10^{3}\ \mathrm{eV}) = 60.8\ \mathrm{eV}.$$

For 90° we get

$$K_{\mathrm{b}} = \frac{E - E'_{\mathrm{b}}}{E} E = (1.98 \times 10^{-2})(10.3 \times 10^{3}\ \mathrm{eV}) = 204\ \mathrm{eV}.$$

For 180° we get

$$K_{\mathrm{c}} = \frac{E - E'_{\mathrm{c}}}{E} E = (3.89 \times 10^{-2})(10.3 \times 10^{3}\ \mathrm{eV}) = 401\ \mathrm{eV}.$$

2–23. (a) Show that the fractional energy loss of scattered photons in the Compton effect can be written as $\Delta\lambda/\lambda$. (b) Is this an exact expression? If not, specify what quantity limits the accuracy and what its value can be so this formula is accurate to 0.1 percent.

Solution. (a) The wavelength shift for Compton scattering is

$$\lambda' - \lambda = \frac{2\pi\hbar}{mc}(1 - \cos\varphi).$$

The fractional energy loss of the scattered electron is

$$-\frac{\Delta E}{E} = \frac{2\pi\hbar c\lambda^{-1} - 2\pi\hbar c\lambda'^{-1}}{2\pi\hbar c\lambda^{-1}} = \frac{\lambda' - \lambda}{\lambda'} = \frac{\Delta\lambda}{\lambda'}.$$

If $\lambda' \approx \lambda$, this is

$$-\frac{\Delta E}{E} = \frac{\Delta\lambda}{\lambda}.$$

(b) For the approximation to be accurate to 0.1%, we have

$$\frac{\Delta\lambda\lambda^{-1} - \Delta\lambda\lambda'^{-1}}{\Delta\lambda\lambda'^{-1}} = \frac{\lambda' - \lambda}{\lambda} \leqslant 0.001.$$

This can be written as

$$\frac{2\pi\hbar}{mc}(1 - \cos\varphi) \leqslant 0.001\,\lambda.$$

Thus for a given incident wavelength, this puts a restriction on the scattering angle:

$$\cos\varphi \geqslant 1 - 0.001\,\frac{mc\lambda}{2\pi\hbar}.$$

2–24. In the Compton effect (see Fig. 2.1), use the relativistic equations for conservation of energy and of linear momentum to show that the Compton shift in wavelength is given by Eq. 2.5.

SOLUTION. For the conservation of momentum, we have

$$\begin{aligned} x: \quad \frac{2\pi\hbar}{\lambda} + 0 &= \frac{2\pi\hbar}{\lambda'}\cos\varphi + p\cos\theta, \quad \text{or} \\ p\cos\theta &= \frac{2\pi\hbar}{\lambda} - \frac{2\pi\hbar}{\lambda'}\cos\varphi; \\ y: \quad 0 + 0 &= \frac{2\pi\hbar}{\lambda'}\sin\varphi - p\sin\theta, \quad \text{or} \\ p\sin\theta &= \frac{2\pi\hbar}{\lambda'}\sin\varphi. \end{aligned}$$

If we square and add, to eliminate θ, we get

$$p^2 = \left(\frac{2\pi\hbar}{\lambda}\right)^2 + \left(\frac{2\pi\hbar}{\lambda'}\right)^2 - \frac{2(2\pi\hbar)^2}{\lambda\lambda'}\cos\varphi.$$

For energy conservation, we have

$$\frac{2\pi\hbar c}{\lambda} + mc^2 = \frac{2\pi\hbar c}{\lambda'} + \sqrt{(pc)^2 + (mc^2)^2},$$

or

$$\frac{2\pi\hbar c}{\lambda} + mc^2 - \frac{2\pi\hbar c}{\lambda'} = \sqrt{(pc)^2 + (mc^2)^2}.$$

When we substitute the result from momentum conservation and square, we get

$$\begin{aligned} &\left(\frac{2\pi\hbar c}{\lambda}\right)^2 + (mc^2)^2 + \left(\frac{2\pi\hbar c}{\lambda'}\right)^2 + \frac{8(\pi\hbar c)^2}{\lambda\lambda'} \\ &-\frac{4mc^2\pi\hbar c}{\lambda} - \frac{4mc^2\pi\hbar c}{\lambda'} \\ &\quad = \left(\frac{2\pi\hbar c}{\lambda}\right)^2 + \left(\frac{2\pi\hbar c}{\lambda'}\right)^2 - \frac{8(\pi\hbar c)^2}{\lambda\lambda'}\cos\varphi + (mc^2)^2, \end{aligned}$$

which reduces to

$$4\pi mc^2\hbar c\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{8(\pi\hbar)^2c^2}{\lambda\lambda'}(1 - \cos\varphi),$$

or

$$\lambda' - \lambda = \frac{2\pi\hbar}{mc}(1 - \cos\varphi).$$

2–25. In the Compton effect, a 0.100-nm photon strikes a free electron in a head-on collision and knocks it into the forward direction. The rebounding photon recoils directly backward. Use conservation of (relativistic) momentum to determine (a) the kinetic energy of the electron, and (b) the wavelength of the recoiling photon.

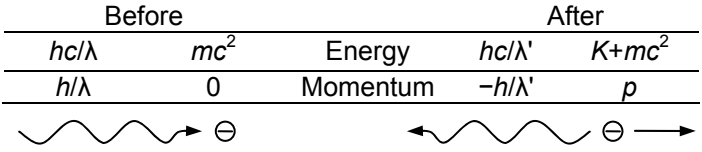


Figure 2.6. Problem 2–25.

Solution. (a) For the conservation of momentum for the head-on collision, we have (Fig. 2.6)

$$\frac{2\pi\hbar}{\lambda} + 0 = -\frac{2\pi\hbar}{\lambda'} + p,$$

or

$$\frac{2\pi\hbar}{\lambda'} = p - \frac{2\pi\hbar}{\lambda}.$$

For energy conservation, we have

$$\frac{2\pi\hbar c}{\lambda} + mc^2 = \frac{2\pi\hbar c}{\lambda'} + K + mc^2.$$

When we use the result for momentum conservation, we get

$$K = \frac{4\pi\hbar c}{\lambda} - pc.$$

The momentum of the electron is related to its kinetic energy:

$$pc = \sqrt{K^2 + 2mc^2K}.$$

Thus we have

$$\frac{4\pi\hbar c}{\lambda} - K = pc = \sqrt{K^2 + 2mc^2K}.$$

After squaring and reducing, we get

$$\begin{aligned}\left(mc^2 + \frac{4\pi\hbar c}{\lambda}\right)K &= \left(\frac{4\pi\hbar c}{\lambda}\right)^2;\\ \left(0.511\times10^6\text{ eV}\right.&\\ \left.+\,2\frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{0.100\text{ nm}}\right)K &= 2\left(\frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{0.100\text{ nm}}\right)^2,\end{aligned}$$

which gives $K = 574$ eV.

(b) We find the wavelength of the recoiling photon from

$$\begin{aligned}\frac{2\pi\hbar c}{\lambda'} &= \frac{2\pi\hbar c}{\lambda} - K;\\ \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{\lambda'} &= \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{0.100\text{ nm}} - 584\text{ eV},\end{aligned}$$

which gives $\lambda' = 0.105$ nm.

2.5 Photon interactions. Pair production

2–26. How much total kinetic energy will an electron-positron pair have if produced by a 2.84-MeV photon?

SOLUTION. The kinetic energy of the pair is

$$K = 2\pi\hbar f - 2mc^2 = 2.84\text{ MeV} - 2(0.511\text{ MeV}) = 1.82\text{ MeV}.$$

2–27. What is the longest wavelength photon that could produce a proton-antiproton pair? (Each has a mass of 1.67×10^{-27} kg.)

SOLUTION. The photon with the longest wavelength has the minimum energy to create the masses:

$$\begin{aligned}2\pi\hbar f_{\min} = \frac{2\pi\hbar c}{\lambda_{\max}} &= 2mc^2;\\ \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})}{\lambda_{\max}} &= 2(1.67\times10^{-27}\text{ kg})(3.00\times10^8\text{ m/s}),\end{aligned}$$

which gives $\lambda_{\max} = 6.62 \times 10^{-16}$ m.

2–28. What is the minimum photon energy needed to produce a μ^+–μ^- pair? The mass of each μ (muon) is 207 times the mass of the electron. What is the wavelength of such a photon?

SOLUTION. The photon with minimum energy to create the masses is

$$2\pi\hbar f_{\min} = 2mc^2 = 2(207)(0.511 \text{ MeV}) = 212 \text{ MeV}.$$

The wavelength is

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{212 \times 10^6 \text{ eV}} = 5.85 \times 10^{-6} \text{ nm} = 5.85 \times 10^{-15} \text{ m}.$$

2–29. A gamma-ray photon produces an electron-positron pair, each with a kinetic energy of 345 keV. What was the energy and wavelength of the photon?

SOLUTION. The energy of the photon is

$$2\pi\hbar f = 2(K + mc^2) = 2(0.345 \text{ MeV} + 0.511 \text{ MeV}) = 1.71 \text{ MeV}.$$

The wavelength is

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{1.71 \times 10^6 \text{ eV}} = 7.24 \times 10^{-4} \text{ nm} = 7.24 \times 10^{-13} \text{ m}.$$

2.6 Wave nature of matter

2–30. Calculate the wavelength of a 0.21-kg ball traveling at 0.10 m/s.

SOLUTION. We find the wavelength from

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(0.21 \text{ kg})(0.10 \text{ m/s})} = 3.2 \times 10^{-32} \text{ m}.$$

2–31. What is the wavelength of a neutron ($m = 1.67 \times 10^{-27}$ kg) traveling at 5.5×10^4 m/s?

SOLUTION. We find the wavelength from

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(1.67 \times 10^{-27} \text{ kg})(5.5 \times 10^4 \text{ m/s})} = 7.2 \times 10^{-12} \text{ m}.$$

Note that $v \ll c$.

2–32. Through how many volts of potential difference must an electron be accelerated to achieve a wavelength of 0. 28 nm?

SOLUTION. We find the speed from

$$\begin{aligned} \lambda &= \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{mv}; \\ 0.28 \times 10^{-9}\ \text{m} &= \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(9.11 \times 10^{-31}\ \text{kg})v}, \end{aligned}$$

which gives $v = 2.60 \times 10^6$ m/s. Because this is much less than c, we can use the classical expression for the kinetic energy. The kinetic energy is equal to the potential energy change:

$$\begin{aligned} eV = K = \tfrac{1}{2}mv^2 &= \tfrac{1}{2}(9.11 \times 10^{-31}\ \text{kg})(2.60 \times 10^6\ \text{m/s})^2 \\ &= 3.08 \times 10^{-18}\ \text{J} = 19.2\ \text{eV}. \end{aligned}$$

Thus the required potential difference is 19 V.

2–33. Calculate the de Broglie wavelength of an electron in your TV picture tube if it is accelerated by 30 000 V. Is it relativistic? How does its wavelength compare to the size of the "neck" of the tube, typically 5 cm? Do we have to worry about diffraction problems blurring our picture on the screen?

SOLUTION. The kinetic energy is equal to the potential energy change:

$$K = eV = (1\ \text{e})(30.0 \times 10^3\ \text{V}) = 30.0 \times 10^3\ \text{eV} = 0.0300\ \text{MeV}.$$

Because this is 6% of mc^2, the electron is relativistic. We find the momentum from

$$E^2 = (K + mc^2)^2 = p^2c^2 + m^2c^4,$$

or

$$\begin{aligned} p^2c^2 &= K^2 + 2Kmc^2; \\ p^2c^2 &= (0.0300\ \text{MeV})^2 + 2(0.0300\ \text{MeV})(0.511\ \text{MeV}), \end{aligned}$$

which gives $pc = 0.178$ MeV, or

$$p = \frac{(0.178\ \text{MeV})(1.60 \times 10^{-13}\ \text{J/MeV})}{3.00 \times 10^8\ \text{m/s}} = 9.47 \times 10^{-23}\ \text{kg}\cdot\text{m/s}.$$

The wavelength is

$$\lambda = \frac{2\pi\hbar}{p} = \frac{6.63 \times 10^{-34}\ \text{J} \cdot \text{s}}{9.47 \times 10^{-23}\ \text{kg} \cdot\ \text{m/s}} = 7.0 \times 10^{-12}\ \text{m}.$$

Because $\lambda \ll 5$ cm, diffraction effects are negligible.

2–34. What is the wavelength of an electron of energy (a) 10 eV, (b) 100 eV, (c) 1.0 keV?

SOLUTION. Because all the energies are much less than mc^2, we can use $K = p^2/(2m)$, so

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mK}} = \frac{2\pi\hbar c}{\sqrt{2mc^2K}}.$$

$$\begin{aligned}
\text{(a)}\ \lambda &= \frac{2\pi\hbar c}{\sqrt{2mc^2K}} = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{\sqrt{2(0.511 \times 10^6\ \text{eV})(10\ \text{eV})}} = 0.39\ \text{nm}. \\
\text{(b)}\ \lambda &= \frac{2\pi\hbar c}{\sqrt{2mc^2K}} = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{\sqrt{2(0.511 \times 10^6\ \text{eV})(100\ \text{eV})}} = 0.12\ \text{nm}. \\
\text{(c)}\ \lambda &= \frac{2\pi\hbar c}{\sqrt{2mc^2K}} = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{\sqrt{2(0.511 \times 10^6\ \text{eV})(1.0 \times 10^3\ \text{eV})}} \\
&= 0.039\ \text{nm}.
\end{aligned}$$

2–35. Show that if an electron and a proton have the same nonrelativistic kinetic energy, the proton has the shorter wavelength.

SOLUTION. With $K = p^2/(2m)$, we have

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mK}}.$$

If we form the ratio for the two particles with equal kinetic energies, we get

$$\frac{\lambda_\text{p}}{\lambda_\text{e}} = \sqrt{\frac{m_\text{e}}{m_\text{p}}}.$$

Because $m_\text{p} > m_\text{e}$, $\lambda_\text{p} < \lambda_\text{e}$.

2–36. Calculate the ratio of the kinetic energy of an electron to that of a proton if their wavelengths are equal. Assume that the speeds are non-relativistic.

SOLUTION. With $K = p^2/(2m)$, we have

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mK}}.$$

If we form the ratio for the two particles with equal wavelengths, we get

$$1 = \sqrt{\frac{m_\mathrm{e}K_\mathrm{e}}{m_\mathrm{p}K_\mathrm{p}}},$$

or

$$\frac{K_\mathrm{e}}{K_\mathrm{p}} = \frac{m_\mathrm{p}}{m_\mathrm{e}} = \frac{1.67\times10^{-27}\ \mathrm{kg}}{9.11\times10^{-31}\ \mathrm{kg}} = 1.84\times10^3.$$

2–37. What is the wavelength of an O_2 molecule in the air at room temperature?

SOLUTION. We find the speed from

$$\begin{aligned} \tfrac{1}{2}mv^2 &= \tfrac{3}{2}k_\mathrm{B}T; \\ \tfrac{1}{2}(32\ \mathrm{u})(1.66\times10^{-27}\ \mathrm{kg/u})v^2 &= \tfrac{3}{2}(1.38\times10^{-23}\ \mathrm{J/K})(300\ \mathrm{K}), \end{aligned}$$

which gives $v = 484$ m/s. The wavelength is

$$\lambda = \frac{2\pi\hbar}{mv} = \frac{6.63\times10^{-34}\ \mathrm{J\cdot s}}{(32\ \mathrm{u})(1.66\times10^{-27}\ \mathrm{kg/u})(484\ \mathrm{m/s})} = 2.6\times10^{-11}\ \mathrm{m}.$$

2–38. A car with a mass of 2 000 kg approaches a freeway underpass that is 10 m across. At what speed must the car be moving, in order for it to have a wavelength such that it might somehow "diffract" after passing through this single "slit"? How do these conditions compare to normal freeway speeds of 30 m/s?

SOLUTION. For diffraction, the wavelength must be of the order of the opening. We find the speed from

$$\begin{aligned} \lambda &= \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{mv}; \\ 10\ \mathrm{m} &= \frac{6.63\times10^{-34}\ \mathrm{J\cdot s}}{(2\,000\ \mathrm{kg})v}, \end{aligned}$$

which gives $v = 3.3\times10^{-38}$ m/s. Not a good speed if you want to get somewhere! At a speed of 30 m/s, $\lambda \ll 10$ m, so there will be no diffraction.

2.7 Electron microscopes

2–39. What voltage is needed to produce electron wavelengths of 0.10 nm? (Assume that the electrons are nonrelativistic.)

SOLUTION. The kinetic energy of the electron is

$$\begin{aligned} K &= \frac{p^2}{2m} = \frac{(2\pi\hbar)^2}{2m\lambda^2} = \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(9.11\times10^{-31}\ \text{kg})(0.10\times10^{-9}\ \text{m})^2} \\ &= 2.41\times10^{-17}\ \text{J} = 150\ \text{eV}. \end{aligned}$$

Because this must equal the potential energy change, the required voltage is 150 V.

2–40. Electrons are accelerated by 2 250 V in an electron microscope. What is the maximum possible resolution?

SOLUTION. The wavelength of the electron is

$$\begin{aligned} \lambda &= \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mK}} = \frac{2\pi\hbar c}{\sqrt{2mc^2K}} = \frac{1.24\times10^3\ \text{eV}\cdot\text{nm}}{\sqrt{2(0.511\times10^6\ \text{eV})(2\,250\ \text{eV})}} \\ &= 2.59\times10^{-2}\ \text{nm}. \end{aligned}$$

If we neglect aberrations, the maximum possible resolution is on the order of the wavelength: 0.026 nm.

2.8 Atomic spectra. The Bohr model

2–41. For the three hydrogen transitions indicated below, with n being the initial state and n' being the final state, is the transition an absorption or an emission? Which is higher, the initial state energy or the final state energy of the atom? Finally, which of these transitions involves the largest energy photon? (a) $n = 1$, $n' = 3$; (b) $n = 6$, $n' = 2$; (c) $n = 4$, $n' = 5$.

SOLUTION. The energy of a level is

$$E_n = -\frac{13.6\ \text{eV}}{n^2}.$$

(a) The transition from $n = 1$ to $n' = 3$ is an absorption, because the final state, $n' = 3$, has a higher energy. The energy of the photon is

$$2\pi\hbar f = E_{n'} - E_n = -(13.6\ \text{eV})\left(\frac{1}{3^2} - \frac{1}{1^2}\right) = 12.1\ \text{eV}.$$

(b) The transition from $n = 6$ to $n' = 2$ is an emission, because the initial state, $n = 6$, has a higher energy. The energy of the photon is

$$2\pi\hbar f = -(E_{n'} - E_n) = (13.6\text{ eV})\left(\frac{1}{2^2} - \frac{1}{6^2}\right) = 3.0\text{ eV}.$$

(c) The transition from $n = 4$ to $n' = 5$ is an absorption, because the final state, $n' = 5$, has a higher energy. The energy of the photon is

$$2\pi\hbar f = E_{n'} - E_n = -(13.6\text{ eV})\left(\frac{1}{5^2} - \frac{1}{4^2}\right) = 0.31\text{ eV}.$$

The photon for the transition from $n = 1$ to $n' = 3$ has the largest energy.

2–42. How much energy is needed to ionize a hydrogen atom in the $n = 2$ state?

SOLUTION. To ionize the atom means removing the electron, or raising it to zero energy:

$$E_{\text{ion}} = 0 - E_n = \frac{13.6\text{ eV}}{n^2} = \frac{13.6\text{ eV}}{2^2} = 3.4\text{ eV}.$$

2–43. The third longest wavelength in the Paschen series in hydrogen corresponds to what transition?

SOLUTION. From $\Delta E = 2\pi\hbar c/\lambda$, we see that the third longest wavelength comes from the transition with the third smallest energy: $n = 6$ to $n' = 3$.

2–44. Calculate the ionization energy of doubly ionized lithium, Li^{2+}, which has $Z = 3$.

SOLUTION. Doubly ionized lithium is like hydrogen, except that there are three positive charges ($Z = 3$) in the nucleus. The square of the product of the positive and negative charges appears in the energy term for the energy levels. We can use the results for hydrogen, if we replace e^2 by Ze^2:

$$E_n = -\frac{Z^2(13.6\text{ eV})}{n^2} = -\frac{3^2(13.6\text{ eV})}{n^2} = -\frac{122\text{ eV}}{n^2}.$$

We find the energy needed to remove the remaining electron from

$$E = 0 - E_1 = 0 - \left(-\frac{122\text{ eV}}{1^2}\right) = 122\text{ eV}.$$

2–45. Determine the wavelength of the second Balmer line ($n = 4$ to $n = 2$ transition). Determine likewise (b) the wavelength of the second Lyman line and (c) the wavelength of the third Balmer line.

SOLUTION. (a) For the jump from $n = 4$ to $n = 2$, we have

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{E_4 - E_2} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{-0.85 \text{ eV} - (-3.4 \text{ eV})} = 486 \text{ nm}.$$

(b) For the jump from $n = 3$ to $n = 1$, we have

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{E_3 - E_1} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{-1.5 \text{ eV} - (-13.6 \text{ eV})} = 102 \text{ nm}.$$

(c) The energy of the $n = 5$ level is

$$E_5 = -\frac{13.6 \text{ eV}}{5^2} = -0.54 \text{ eV}.$$

For the jump from $n = 5$ to $n = 2$, we have

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{E_5 - E_2} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{-0.54 \text{ eV} - (-3.4 \text{ eV})} = 434 \text{ nm}.$$

2–46. Evaluate the Rydberg constant R_∞ using Bohr theory and show that its value is $R_\infty = 1.0974 \times 10^7 \text{ m}^{-1}$.

SOLUTION. For the Rydberg constant we have

$$\begin{aligned} R_\infty &= \frac{(k_e e^2)^2 m_e}{4\pi\hbar^3 c} \\ &= \frac{(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)^2 (1.602177 \times 10^{-19} \text{ C})^4}{4\pi (1.0545716 \times 10^{-34} \text{ J} \cdot \text{s})^3} \\ &\quad \times \frac{(9.109390 \times 10^{-31} \text{ kg})}{(2.997925 \times 10^8 \text{ m/s})} \\ &= 1.0974 \times 10^7 \text{ m}^{-1}. \end{aligned}$$

2–47. What is the longest wavelength light capable of ionizing a hydrogen atom in the ground state?

SOLUTION. The longest wavelength corresponds to the minimum energy, which is the ionization energy:

$$\lambda = \frac{(1.24 \times 10^3 \text{ eV} \cdot \text{nm})}{E_{\text{ion}}} = \frac{(1.24 \times 10^3 \text{ eV} \cdot \text{nm})}{(13.6 \text{ eV})} = 91.2 \text{ nm}.$$

Note that shorter wavelengths would give the ejected electron some kinetic energy.

2–48. What wavelength photon would be required to ionize a hydrogen atom in the ground state and give the ejected electron a kinetic energy of 10.0 eV?

SOLUTION. The energy of the photon is

$$2\pi\hbar f = E_{\text{ion}} + K = 13.6 \text{ eV} + 10.0 \text{ eV} = 23.6 \text{ eV}.$$

We find the wavelength from

$$\lambda = \frac{(1.24 \times 10^3 \text{ eV} \cdot \text{nm})}{2\pi\hbar f} = \frac{(1.24 \times 10^3 \text{ eV} \cdot \text{nm})}{(23.6 \text{ eV})} = 52.5 \text{ nm}.$$

2–49. In the Sun, an ionized helium (He^+) atom makes a transition from the $n = 6$ state to the $n = 2$ state, emitting a photon. Can that photon be absorbed by hydrogen atoms present in the Sun? If so, between what energy states will the hydrogen atom jump?

SOLUTION. Singly ionized helium is like hydrogen, except that there are two positive charges ($Z = 2$) in the nucleus. The square of the product of the positive and negative charges appears in the energy term for the energy levels. We can use the results for hydrogen, if we replace e^2 by Ze^2:

$$E_n = -\frac{Z^2(13.6 \text{ eV})}{n^2} = -\frac{2^2(13.6 \text{ eV})}{n^2} = -\frac{(54.4 \text{ eV})}{n^2}.$$

We find the energy of the photon from the $n = 6$ to $n = 2$ transition:

$$E = E_6 - E_2 = -(54.4 \text{ eV})\left(\frac{1}{6^2} - \frac{1}{2^2}\right) = 12.1 \text{ eV}.$$

Because this is the energy difference for the $n = 1$ to $n = 3$ transition in hydrogen, the photon can be absorbed by a hydrogen atom which will jump from $n = 1$ to $n = 3$.

2–50. Construct the energy-level diagram for the He^+ ion.

SOLUTION. Singly ionized helium is like hydrogen, except that there are two positive charges ($Z = 2$) in the nucleus. The square of the product of the positive and negative charges appears in the energy term for the energy levels. We can use the results for hydrogen, if we replace e^2 by Ze^2:

$$E_n = -\frac{Z^2(13.6\text{ eV})}{n^2} = -\frac{2^2(13.6\text{ eV})}{n^2} = -\frac{54.4\text{ eV}}{n^2}.$$

The diagram is shown in Fig. 2.7.

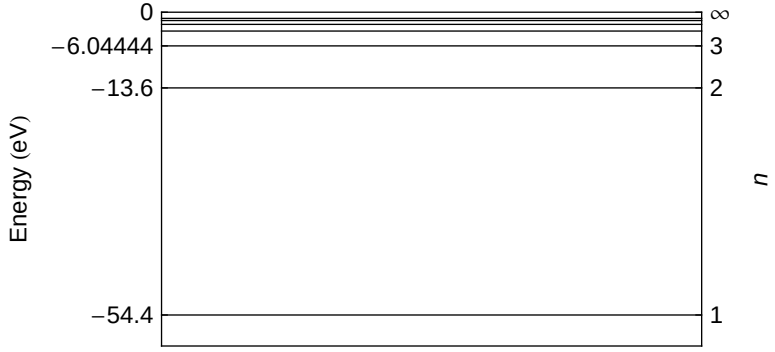


Figure 2.7. Problem 2–50.

2–51. Construct the energy-level diagram for doubly ionized lithium, Li^{2+}.

SOLUTION. Doubly ionized lithium is like hydrogen, except that there are three positive charges ($Z = 3$) in the nucleus. The square of the product of the positive and negative charges appears in the energy term for the energy levels. We can use the results for hydrogen, if we replace e^2 by Ze^2:

$$E_n = -\frac{Z^2(13.6\text{ eV})}{n^2} = -\frac{3^2(13.6\text{ eV})}{n^2} = -\frac{(122.4\text{ eV})}{n^2}.$$

The diagram is shown in Fig. 2.8.

2–52. What is the potential energy and the kinetic energy of an electron in the ground state of the hydrogen atom?

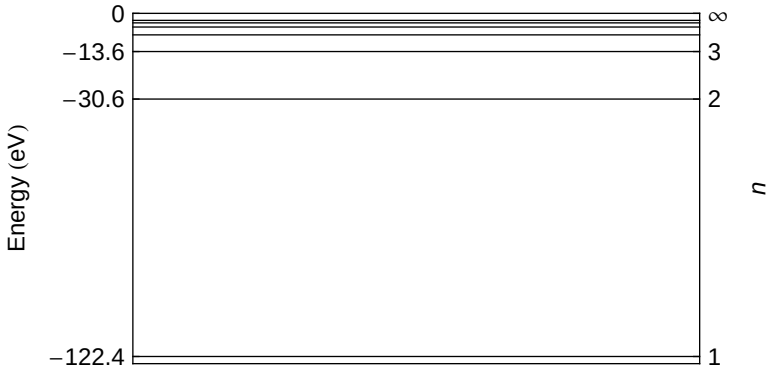


Figure 2.8. Problem 2–51.

SOLUTION. The potential energy for the ground state is

$$\begin{aligned} U &= -k_{\mathrm{e}}\frac{e^2}{r_{\mathrm{B}}} = -\frac{(9.00\times10^{9}\ \mathrm{N\cdot m^2/C^2})(1.60\times10^{-19}\ \mathrm{C})^2}{0.529\times10^{-10}\ \mathrm{m}} \\ &= -4.36\times10^{-18}\ \mathrm{J} = -27.2\ \mathrm{eV}. \end{aligned}$$

The kinetic energy is

$$K = E_1 - U = -13.6\ \mathrm{eV} - (-27.2\ \mathrm{eV}) = +13.6\ \mathrm{eV}.$$

2–53. An excited hydrogen atom could, in principle, have a radius of 1.00 mm. What would be the value of n for a Bohr orbit of this size? What would its energy be?

SOLUTION. We find the value of n from

$$\begin{aligned} r_n &= n^2 r_{\mathrm{B}}; \\ 1.00\times10^{-3}\ \mathrm{m} &= n^2(0.529\times10^{-10}\ \mathrm{m}), \end{aligned}$$

which gives $n = 4.35\times10^{3}$. The energy of this orbit is

$$E = -\frac{13.6\ \mathrm{eV}}{n^2} = -\frac{13.6\ \mathrm{eV}}{(4.35\times10^{3})^2} = -7.2\times10^{-7}\ \mathrm{eV}.$$

Note that energy differences will be very small.

2–54. Is the use of nonrelativistic formulae justified in the Bohr atom? To check, calculate the electron's velocity, v, in terms of c for the ground state of hydrogen, and then calculate $\sqrt{1-\beta^2}$.

SOLUTION. We find the velocity from the quantum condition:

$$\begin{aligned} mvr_1 &= n\hbar; \\ (9.11\times10^{-31}\text{ kg})v(0.529\times10^{-10}\text{ m}) &= (1)(1.055\times10^{-34}\text{ J}\cdot\text{s}), \end{aligned}$$

which gives $v = 2.18\times10^6$ m/s $= 7.3\times10^{-3}c$. The relativistic factor is

$$\sqrt{1-\beta^2} \approx 1 - \tfrac{1}{2}\beta^2 = 1 - 2.7\times10^{-5}.$$

Because this is essentially 1, the use of nonrelativistic formulae is justified.

2–55. Suppose an electron was bound to a proton, as in the hydrogen atom, by the gravitational force rather than by the electric force. What would be the radius, and energy, of the first Bohr orbit?

SOLUTION. If we compare the two forces:

$$F_\text{e} = k_\text{e}\frac{e^2}{r^2},$$

and

$$F_\text{g} = G\frac{m_\text{e}m_\text{p}}{r^2},$$

we see that we can use the hydrogen expressions if we replace $k_\text{e}e^2$ with $Gm_\text{e}m_\text{p}$. For the radius we get

$$\begin{aligned} r_1 &= \frac{\hbar^2}{Gm_\text{e}^2m_\text{p}} \\ &= \frac{(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(9.11\times10^{-31}\text{ kg})^2} \\ &\quad\times(1.67\times10^{-27}\text{ kg})^{-1} \\ &= 1.20\times10^{29}\text{ m}. \end{aligned}$$

Note that this is many times intergalactic distances.

The ground state energy is

$$\begin{aligned} E_1 &= -\frac{G^2 m_{\mathrm{e}}^3 m_{\mathrm{p}}^2}{2\hbar^2} \\ &= -\frac{(6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2})^2(9.11\times10^{-31}\ \mathrm{kg})^3}{2(1.055\times10^{-34}\ \mathrm{J\cdot s})^2} \\ &\quad \times (1.67\times10^{-27}\ \mathrm{kg})^2 \\ &= -4.21\times10^{-97}\ \mathrm{J}. \end{aligned}$$

2–56. Show that the magnitude of the potential energy of an electron in any Bohr orbit of a hydrogen atom is twice the magnitude of its kinetic energy in that orbit.

SOLUTION. The potential energy for the nth state is

$$U = -k_{\mathrm{e}}\frac{e^2}{r_n}.$$

The Coulomb force provides the radial acceleration, so we have

$$k_{\mathrm{e}}\frac{e^2}{r_n^2} = \frac{mv_n^2}{r_n},$$

or

$$mv_n^2 = k_{\mathrm{e}}\frac{e^2}{r_n}.$$

Thus the kinetic energy is

$$K = \frac{mv_n^2}{2} = \frac{k_{\mathrm{e}}e^2}{2r_n} = \frac{1}{2}|U|.$$

2–57. *Correspondence principle*: Show that for large values of n, the difference in radius Δr between two adjacent orbits (with quantum numbers n and $n-1$) is given by

$$\Delta r = r_n - r_{n-1} \approx \frac{2r_n}{n},$$

so $\Delta r/r_n \to 0$ as $n \to \infty$, in accordance with the correspondence principle. (Note that we can check the correspondence principle by either considering large values of n ($n \to \infty$) or by letting $\hbar \to 0$. Are these equivalent?)

SOLUTION. For the difference in radius for adjacent orbits, we have

$$\Delta r = r_n - r_{n-1} = \left(n^2 - (n-1)^2\right) r_1 = (2n-1)r_1.$$

When $n \gg 1$, we get

$$\Delta r \approx 2nr_1 = \frac{2r_n}{n}.$$

In the classical limit, the separation of radii (and energies) should be very small. We see that letting $n \to \infty$ does this. If we substitute the expression for r_n, we see that $\Delta r \propto \hbar^2$, so letting $\hbar \to 0$ also makes the separation of radii small.

2–58. (a) For very large values of n, show that when an electron jumps from the level n to the level $n-1$, the frequency of the light emitted is equal to $f = v/2\pi r_n$. (b) Show that this is also just the frequency predicted by classical theory for an electron revolving in a circular orbit of radius r_n with speed v. (c) Explain why this is consistent with the correspondence principle.

SOLUTION. (a) We find the frequency of the radiation for a jump from level n to level $n-1$ from

$$\begin{aligned} 2\pi\hbar f &= E_n - E_{n-1}; \\ f &= \frac{1}{h}\frac{Z^2 e^4 m}{8\varepsilon_0^2 h^2}\left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) = \frac{Z^2 k_e^2 e^4 m_e}{4\pi\hbar^3}\left(\frac{n^2-(n-1)^2}{n^2(n-1)^2}\right) \\ &\approx \frac{Z^2 k_e^2 e^4 m_e}{4\pi\hbar^3}\frac{2n}{n^4} = \frac{Z^2 k_e^2 e^4 m_e}{2\pi\hbar^3 n^3}. \end{aligned}$$

From the quantum condition we have

$$m_e v r_n = n\hbar,$$

or

$$v = \frac{n\hbar}{m_e r_n}.$$

Thus we get

$$\frac{v}{2\pi r_n} = \frac{n\hbar}{2\pi m_e r_n^2} = \frac{n\hbar}{2\pi m_e \left(n^2\hbar^2/Z k_e e^2 m_e\right)^2} = \frac{Z^2 k_e^2 e^4 m_e}{2\pi\hbar^3 n^3},$$

which is the same as the above frequency.

(b) From the classical theory for an electron revolving in a circular orbit, the time for one revolution is

$$T = \frac{2\pi r_n}{v},$$

so the frequency is

$$f = \frac{1}{T} = \frac{v}{2\pi r_n}.$$

(c) Classically an accelerated charge radiates. For circular motion, the frequency of the radiation is the orbital frequency. This agrees with the Bohr prediction for large values of n, consistent with the correspondence principle.

2.9 General problems

2–59. The Big Bang theory states that the beginning of the universe was accompanied by a huge burst of photons. Those photons are still present today and make up the so called *Cosmic Microwave Background Radiation*. The universe radiates like a blackbody with a temperature of about 2.7 K. Calculate the peak wavelength of this radiation.

SOLUTION. We find the peak wavelength from

$$\begin{aligned} \lambda_{\mathrm{p}} &= \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{T} = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{2.7\ \mathrm{K}} = 1.1 \times 10^{-3}\ \mathrm{m} \\ &= 1.1\ \mathrm{mm}. \end{aligned}$$

2–60. At low temperatures, nearly all the atoms in hydrogen gas will be in the ground state. What minimum frequency photon is needed if the photoelectric effect is to be observed?

SOLUTION. To produce a photoelectron, the hydrogen atom must be ionized, so the minimum energy of the photon is 13.6 eV. We find the minimum frequency of the photon from

$$\begin{aligned} E_{\mathrm{min}} &= 2\pi\hbar f_{\mathrm{min}}; \\ (13.6\ \mathrm{eV})(1.60 \times 10^{-19}\ \mathrm{J/eV}) &= (6.63 \times 10^{-34}\ \mathrm{J \cdot s}) f_{\mathrm{min}}, \end{aligned}$$

which gives $f_{\mathrm{min}} = 3.28 \times 10^{15}$ Hz.

2–61. A beam of 85-eV electrons is scattered from a crystal, as in X-ray diffraction, and a first-order peak is observed at $\theta = 38°$. What is the spacing between planes in the diffracting crystal?

SOLUTION. Because the energy is much less than mc^2, we can use $K = p^2/(2m)$, so the wavelength of the electron is

$$\begin{aligned}\lambda &= \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mK}} = \frac{2\pi\hbar c}{\sqrt{2mc^2K}} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\sqrt{2(0.511 \times 10^6 \text{ eV})(85 \text{ eV})}} \\ &= 0.133 \text{ nm}.\end{aligned}$$

We find the spacing of the planes from

$$\begin{aligned}2d \sin\theta &= n\lambda; \\ 2d \sin 38° &= (1)(0.133 \text{ nm}),\end{aligned}$$

which gives $d = 0.108$ nm.

2–62. Show that the energy E (in electron volts) of a photon whose wavelength is λ (meters) is given by

$$E = \frac{(1.24 \times 10^{-6})}{\lambda}.$$

SOLUTION. For the energy of the photon, we have

$$\begin{aligned}E &= 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})(3.00 \times 10^8 \text{ m/s})}{(1.60 \times 10^{-19} \text{ J/eV})\lambda} \\ &= \frac{1.24 \times 10^{-6} \text{ eV} \cdot \text{m}}{\lambda}.\end{aligned}$$

2–63. A microwave oven produces electromagnetic radiation at $\lambda = 12.2$ cm and produces a power of 760 W. Calculate the number of microwave photons produced by the microwave oven each second.

SOLUTION. The energy of the photon is

$$\begin{aligned}E &= 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})(3.00 \times 10^8 \text{ m/s})}{12.2 \times 10^{-2} \text{ m}} \\ &= 1.63 \times 10^{-24} \text{ J}.\end{aligned}$$

Thus the rate at which photons are produced in the oven is

$$N = \frac{P}{E} = \frac{760 \text{ W}}{1.63 \times 10^{-24} \text{ J}} = 4.66 \times 10^{26} \text{ photons/s}.$$

2–64. Sunlight reaching the Earth's surface has an intensity of about 1 000 W/m^2. Estimate how many photons per square meter per second this represents. Take the average wavelength to be 550 nm.

SOLUTION. The energy of the photon is

$$2\pi\hbar f = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{550 \text{ nm}} = 2.25 \text{ eV}.$$

We find the intensity of photons from

$$\begin{aligned} I_{\text{photons}} &= \frac{I}{2\pi\hbar f} = \frac{1\,000 \text{ W/m}^2}{(2.25 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})} \\ &= 2.78 \times 10^{21} \text{ photons}/(\text{s} \cdot \text{m}^2). \end{aligned}$$

2–65. A beam of red laser light ($\lambda = 633$ nm) hits a black wall and is fully absorbed. If this exerts a total force $F = 5.5$ nN on the wall, how many photons per second are hitting the wall?

SOLUTION. The impulse on the wall is due to the change in momentum of the photons:

$$F\,\Delta t = \Delta p = np = \frac{n2\pi\hbar}{\lambda},$$

or

$$\frac{n}{\Delta t} = \frac{F\lambda}{2\pi\hbar} = \frac{(5.5 \times 10^{-9} \text{ N})(633 \times 10^{-9} \text{ m})}{6.63 \times 10^{-34} \text{ J} \cdot \text{s}} = 5.3 \times 10^{18} \text{ photons/s}.$$

2–66. If a 100-W light bulb emits 3.0 percent of the input energy as visible light (average wavelength 550 nm) uniformly in all directions, estimate how many photons per second of visible light will strike the pupil (4.0 mm diameter) of the eye of an observer 1.0 km away.

SOLUTION. The energy of the photon is

$$2\pi\hbar f = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{ nm}}{550 \text{ nm}} = 2.25 \text{ eV}.$$

Because the light radiates uniformly, the intensity at a distance L is

$$I = \frac{P}{4\pi L^2},$$

so the rate at which energy enters the pupil is

$$\frac{E}{t} = I\pi r^2 = \frac{Pr^2}{4L^2}.$$

Thus the rate at which photons enter the pupil is

$$\begin{aligned} \frac{n}{t} &= \frac{(E/t)}{2\pi\hbar f} = \frac{Pr^2}{4L^2(2\pi\hbar f)} \\ &= \frac{(0.030)(100\text{ W})(2.0\times10^{-3}\text{ m})^2}{4(1.0\times10^{3}\text{ m})^2(2.25\text{ eV})(1.60\times10^{-19}\text{ J/eV})} \\ &= 8.3\times10^{6}\text{ photons/s}. \end{aligned}$$

2–67. An electron and a positron collide head on, annihilate, and create two 0.90 MeV photons traveling in opposite directions. What were the initial kinetic energies of electron and positron?

SOLUTION. Because the energies of the photons are equal, their momenta have equal magnitudes. Thus the total momentum of the system before and after the collision is zero. This means the momentum of the electron has the same magnitude as the momentum of the positron. Because they have the same mass, their energies must be equal. From conservation of energy we have

$$2(2\pi\hbar f) = 2(K + m_{\text{e}}c^2),$$

or

$$\begin{aligned} 2\pi\hbar f &= K + m_{\text{e}}c^2; \\ 0.90\text{ MeV} &= K + 0.511\text{ MeV}, \end{aligned}$$

which gives $K = 0.39$ MeV.

2–68. In Compton's original experiment he saw X-rays scattered where the wavelength shifted from 0.0711 nm to 0.0735 nm. Through what angle did these X-rays scatter?

SOLUTION. We find the scattering angle from

$$\begin{aligned} \lambda' - \lambda &= \lambda_{\rm C}(1 - \cos\varphi); \\ 7.35 \times 10^{-11}\ {\rm m} - 7.11 \times 10^{-11}\ {\rm m} &= (2.426 \times 10^{-12}\ {\rm m})(1 - \cos\varphi), \end{aligned}$$

which gives $\cos\varphi = 0.0107$, and $\varphi = 89.4°$.

2–69. By what potential difference must (a) a proton ($m_{\rm p} = 1.67 \times 10^{-27}$ kg), and (b) an electron ($m_{\rm e} = 9.11 \times 10^{-31}$ kg), be accelerated to have a wavelength $\lambda = 5.0 \times 10^{-12}$ m?

SOLUTION. The required momentum is

$$p = \frac{2\pi\hbar}{\lambda},$$

or

$$pc = \frac{2\pi\hbar c}{\lambda} = \frac{1.24 \times 10^3\ {\rm eV \cdot nm}}{5.0 \times 10^{-3}\ {\rm nm}} = 2.48 \times 10^5\ {\rm eV} = 0.248\ {\rm MeV}.$$

(a) For the proton, $pc \ll m_{\rm p}c^2$, so we can find the required kinetic energy from

$$K = \frac{p^2}{2m_{\rm p}} = \frac{(pc)^2}{2m_{\rm p}c^2} = \frac{(0.248\ {\rm MeV})^2}{2(938\ {\rm MeV})} = 3.3 \times 10^{-5}\ {\rm MeV} = 33\ {\rm eV}.$$

The potential difference to produce this kinetic energy is

$$V = \frac{K}{e} = \frac{(33\ {\rm eV})}{(1\ {\rm e})} = 33\ {\rm V}.$$

(b) For the electron, pc is of the order of $m_{\rm e}c^2$, so we can find the required kinetic energy from

$$\begin{aligned} K &= \sqrt{(pc)^2 + (m_{\rm e}c^2)^2} - m_{\rm e}c^2 \\ &= \sqrt{(0.248\ {\rm MeV})^2 + (0.511\ {\rm MeV})^2} - 0.511\ {\rm MeV} \\ &= 0.057\ {\rm MeV} = 57\ {\rm keV}. \end{aligned}$$

The potential difference to produce this kinetic energy is

$$V = \frac{K}{e} = \frac{57\ {\rm keV}}{1\ {\rm e}} = 57\ {\rm kV}.$$

2–70. In some of Rutherford's experiments (Fig. 2.9) the α particles (mass $= 6.64 \times 10^{-27}$ kg) had a kinetic energy of 4.8 MeV. How close could they get to a gold nucleus (charge $= +79$ e)? Ignore the recoil motion of the nucleus.

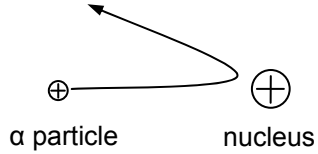


Figure 2.9. Problem 2–70.

SOLUTION. If we ignore the recoil motion of the gold nucleus, at the closest approach the kinetic energy of both particles is zero. The potential energy of the two charges must equal the initial kinetic energy of the α particle:

$$\begin{aligned} K &= k_{\mathrm{e}} \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{r_{\min}}; \\ (4.8 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV}) &= (8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2) \\ &\quad \times \frac{(2)(79)(1.60 \times 10^{-19} \text{ C})^2}{r_{\min}}, \end{aligned}$$

which gives $r_{\min} = 4.7 \times 10^{-14}$ m.

2–71. By what fraction does the mass of an H atom decrease when it makes an $n = 3$ to $n = 1$ transition?

SOLUTION. The decrease in mass occurs because of the loss in energy when a photon is emitted:

$$\frac{\Delta m}{m} = \frac{\Delta E}{mc^2} = \frac{(-13.6 \text{ eV})\left((1)^{-2} - (3)^{-2}\right)}{939 \times 10^6 \text{ eV}} = -1.29 \times 10^{-8}.$$

2–72. For what maximum kinetic energy is a collision between an electron and a hydrogen atom in its ground state definitely elastic?

SOLUTION. The collision must be elastic as long as the electron does not have enough energy to raise the hydrogen atom to the $n = 2$ level, so

no energy is transferred to the atom. Thus we have

$$K < E_2 - E_1 = (-13.6 \text{ eV}) \left(\frac{1}{2^2} - \frac{1}{1^2} \right) = 10.2 \text{ eV}.$$

2–73. Using Bohr theory, derive an equation for the angular velocity ω, and frequency f, of an electron in a hydrogen atom. Determine (a) for the ground state and (b) for the first excited state ($n = 2$).

SOLUTION. The Coulomb force provides the radial acceleration, so we have

$$k_\text{e} \frac{e^2}{r_n^2} = \frac{m_\text{e} v_n^2}{r_n},$$

or

$$v_n = \sqrt{\frac{k_\text{e} e^2}{m_\text{e} r_n}}.$$

For the angular velocity, we get

$$\omega_n = \frac{v_n}{r_n} = \sqrt{\frac{k_\text{e} e^2}{m_\text{e} {r_n}^3}} = \sqrt{\frac{k_\text{e}^4 e^8 m_\text{e}^2}{n^6 \hbar^6}} = \frac{k_\text{e}^2 e^4 m_\text{e}}{n^3 \hbar^3},$$

where we have used the expression

$$r_n = \frac{n^2 \hbar^2}{k_\text{e} e^2 m_\text{e}}$$

for the radius of the orbit.

For the frequency, we get

$$f_n = \frac{\omega_n}{2\pi} = \frac{k_\text{e}^2 e^4 m_\text{e}}{2\pi n^3 \hbar^3}.$$

(a) For the ground state, we get

$$\begin{aligned} \omega_1 &= (8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)^2 \\ &\quad \times \frac{(1.60 \times 10^{-19} \text{ C})^4 (9.11 \times 10^{-31} \text{ kg})}{(1)^3 (1.055 \times 10^{-34} \text{ J} \cdot \text{s})^3} \\ &= 4.11 \times 10^{16} \text{ rad/s}. \end{aligned}$$

The frequency is

$$f_1 = \frac{\omega_1}{2\pi} = \frac{4.11 \times 10^{16} \text{ rad/s}}{2\pi} = 6.54 \times 10^{15} \text{ Hz}.$$

(b) For the first excited state, we get

$$\begin{aligned} \omega_2 &= (8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)^2 \\ &\quad \times \frac{(1.60 \times 10^{-19} \text{ C})^4 (9.11 \times 10^{-31} \text{ kg})}{(2)^3 (1.055 \times 10^{-34} \text{ J} \cdot \text{s})^3} \\ &= 5.14 \times 10^{15} \text{ rad/s}. \end{aligned}$$

The frequency is

$$f_2 = \frac{\omega_2}{2\pi} = \frac{5.14 \times 10^{15} \text{ rad/s}}{2\pi} = 8.17 \times 10^{14} \text{ Hz}.$$

2–74. Calculate the ratio of the gravitational to electric force for the electron in a hydrogen atom. Can the gravitational force be safely ignored?

SOLUTION. The ratio of the forces is

$$\begin{aligned} \frac{F_\text{g}}{F_\text{e}} &= \frac{\left(G m_\text{e} m_\text{p}/r^2\right)}{\left(k_\text{e} e^2/r^2\right)} = \frac{G m_\text{e} m_\text{p}}{k_\text{e} e^2} \\ &= \frac{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)}{(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(1.60 \times 10^{-19} \text{ C})^2} \\ &\quad \times (9.11 \times 10^{-31} \text{ kg})(1.67 \times 10^{-27} \text{ kg}) \\ &= 4.4 \times 10^{-40}. \end{aligned}$$

Yes, the gravitational force may be safely ignored.

2–75. A child's swing has a natural frequency of 0.75 Hz. (a) What is the separation between possible energy values (in joules)? (b) If the swing reaches a vertical height of 45 cm above its lowest point and has a mass of 20 kg (including the child), what is the value of the quantum number n? (c) What is the fractional change in energy between levels whose quantum numbers are n (as just calculated) and $n + 1$? Would quantization be measurable in this case?

SOLUTION. (a) Because the energy is quantized, $E = n2\pi\hbar f$, the difference in energy between adjacent levels is

$$\Delta E = 2\pi\hbar f = (6.63 \times 10^{-34}\ \text{J} \cdot \text{s})(0.75\ \text{Hz}) = 5.0 \times 10^{-34}\ \text{J}.$$

(b) The total energy will be the maximum potential energy, so we have

$$\begin{aligned} E = mgh &= n2\pi\hbar f; \\ (20\ \text{kg})(9.80\ \text{m/s}^2)(0.45\ \text{m}) &= n(5.0 \times 10^{-34}\ \text{J}), \end{aligned}$$

which gives $n = 1.8 \times 10^{35}$.

(c) The fractional change in energy is

$$\frac{\Delta E}{E} = \frac{2\pi\hbar f}{n2\pi\hbar f} = \frac{1}{n} = \frac{1}{1.8 \times 10^{35}} = 5.6 \times 10^{-36},$$

not measurable.

2–76. Electrons accelerated by a potential difference of 12.3 V pass through a gas of hydrogen atoms at room temperature. What wavelengths of light will be emitted?

SOLUTION. The potential difference produces a kinetic energy of 12.3 eV, so it is possible to provide this much energy to the hydrogen atom through collisions. From the ground state, the maximum energy of the atom is

$$-13.6\ \text{eV} - (-12.3\ \text{eV}) = -1.3\ \text{eV}.$$

From the energy level diagram, we see that this means the atom could be excited to the $n = 3$ state, so the possible transitions when the atom returns to the ground state are $n = 3$ to $n = 2$, $n = 3$ to $n = 1$, and $n = 2$ to $n = 1$. For the wavelengths we have

$$\begin{aligned} \lambda_{3\to2} &= \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{E_3 - E_2} = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{-1.5\ \text{eV} - (-3.4\ \text{eV})} = 653\ \text{nm}; \\ \lambda_{3\to1} &= \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{E_3 - E_1} = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{-1.5\ \text{eV} - (-13.6\ \text{eV})} = 102\ \text{nm}; \\ \lambda_{2\to1} &= \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{E_2 - E_1} = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{-3.4\ \text{eV} - (-13.6\ \text{eV})} = 122\ \text{nm}. \end{aligned}$$

2–77. Atoms can be formed in which a muon (mass = 207 times the mass of an electron) replaces one of the electrons in an atom. Calculate, using Bohr theory, the energy of the photon emitted when a muon makes a transition from $n = 2$ to $n = 1$ in a muonic $^{208}_{82}$Pb atom (lead whose nucleus has a mass 208 times the proton mass and charge $+82e$.

SOLUTION. The energy levels are

$$\begin{aligned} E_n &= -\frac{Z^2 k_e^2 e^4 m}{2\hbar^2 n^2} = -\frac{(82)^2 k_e^2 e^4 (207) m_e}{2\hbar^2 n^2} = \frac{(82)^2 (207)(-13.6 \text{ eV})}{n^2} \\ &= -\frac{1.89 \times 10^7 \text{ eV}}{n^2} = -\frac{18.9 \text{ MeV}}{n^2}. \end{aligned}$$

For the $n = 2$ to $n = 1$ transition, the photon energy is

$$2\pi\hbar f = E_2 - E_1 = (-18.9 \text{ MeV}) \left(\frac{1}{2^2} - \frac{1}{1^2} \right) = 14.2 \text{ MeV}.$$

Note that this would be a gamma-ray photon.

2–78. In a particular photoelectric experiment a stopping potential of 2.10 volts is measured when ultraviolet light having a wavelength of 290 nm is incident on the metal. Using the same setup, what will the new stopping potential be if blue a wavelength of 440 nm is used, instead of the ultraviolet light?

SOLUTION. The energy of the ultraviolet photon is

$$2\pi\hbar f_1 = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda_1} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{290 \text{ nm}} = 4.28 \text{ eV}.$$

The stopping potential is the potential difference that gives a potential energy change equal to the maximum kinetic energy:

$$\begin{aligned} K_{\max 1} &= eV_{01} = 2\pi\hbar f_1 - W_0; \\ (1 \text{ e})(2.10 \text{ V}) &= 4.28 \text{ eV} - W_0, \end{aligned}$$

which gives $W_0 = 2.18$ eV.

The energy of the blue photon is

$$2\pi\hbar f_2 = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda_2} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{440 \text{ nm}} = 2.82 \text{ eV}.$$

We find the new stopping potential from

$$\begin{aligned} K_{\max 2} &= eV_{02} = 2\pi\hbar f_2 - W_0; \\ (1\text{ e})V_{02} &= 2.82\text{ eV} - 2.18\text{ eV}, \end{aligned}$$

which gives $V_{02} = 0.64$ V.

2–79. In an X-ray tube, the high voltage between filament and target is V. After being accelerated through this voltage, an electron strikes the target where it is decelerated (by positively charged nuclei) and in the process one or more X-ray photons are emitted. (a) Show that the photon of shortest wavelength will have $\lambda_0 = 2\pi\hbar c/eV$. (b) What is the shortest wavelength of X-ray emitted when accelerated electrons strike the face of a 30-kV television picture tube?

Solution. (a) The electron has a charge e, so the potential difference produces a kinetic energy of eV. The shortest wavelength photon is produced when all the kinetic energy is lost and a photon emitted:

$$2\pi\hbar f_{\max} = \frac{2\pi\hbar c}{\lambda_0} = eV,$$

which gives

$$\lambda_0 = \frac{2\pi\hbar c}{eV}.$$

(b) Numerically,

$$\lambda_0 = \frac{2\pi\hbar c}{eV} = \frac{1.24 \times 10^3\text{ eV}\cdot\text{nm}}{30 \times 10^3\text{ eV}} = 0.041\text{ nm}.$$

2–80. Show that the wavelength of a particle of mass m with kinetic energy K is given by the relativistic formula

$$\lambda = \frac{2\pi\hbar c}{\sqrt{K^2 + 2mc^2K}}.$$

Solution. We find the momentum from

$$E^2 = (K + mc^2)^2 = p^2c^2 + m^2c^4,$$

or

$$p^2c^2 = K^2 + 2mc^2K.$$

The wavelength is

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar c}{pc} = \frac{2\pi\hbar c}{\sqrt{K^2 + 2mc^2K}}.$$

2–81. What is the kinetic energy and wavelength of a "thermal" neutron (one that is in equilibrium at room temperature)?

SOLUTION. The kinetic energy of a thermal neutron is

$$K = \tfrac{3}{2}k_{\rm B}T = \tfrac{3}{2}(1.38 \times 10^{-23}\ {\rm J/K})(300\ {\rm K}) = 6.21 \times 10^{-21}\ {\rm J} = 0.039\ {\rm eV}.$$

We find the speed from

$$\begin{aligned} K &= \tfrac{1}{2}mv^2; \\ 6.21 \times 10^{-21}\ {\rm J} &= \tfrac{1}{2}(1.67 \times 10^{-27}\ {\rm kg})v^2, \end{aligned}$$

which gives $v = 2.73 \times 10^3$ m/s.

The wavelength is

$$\begin{aligned} \lambda &= \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{mv} = \frac{6.63 \times 10^{-34}\ {\rm J \cdot s}}{(1.67 \times 10^{-27}\ {\rm kg})(2.73 \times 10^3\ {\rm m/s})} \\ &= 1.5 \times 10^{-10}\ {\rm m} = 0.15\ {\rm nm}. \end{aligned}$$

2–82. What is the theoretical limit of resolution for an electron microscope whose electrons are accelerated through 60 kV? (Relativistic formulae should be used.)

SOLUTION. We find the momentum from

$$E^2 = (K + mc^2)^2 = p^2c^2 + m^2c^4,$$

or

$$p^2c^2 = K^2 + 2mc^2K.$$

The wavelength is

$$\begin{aligned} \lambda &= \frac{2\pi\hbar}{p} = \frac{2\pi\hbar c}{pc} = \frac{2\pi\hbar c}{\sqrt{K^2 + 2mc^2K}} \\ &= \frac{1.24 \times 10^3\ {\rm eV \cdot nm}}{\sqrt{(60 \times 10^3\ {\rm eV})^2 + 2(0.511 \times 10^6\ {\rm eV})(60 \times 10^3\ {\rm eV})}} \\ &= 4.9 \times 10^{-3}\ {\rm nm}. \end{aligned}$$

The theoretical resolution limit is of the order of the wavelength, or 5×10^{-12} m.

2–83. The intensity of the Sun's light in the vicinity of the Earth is about 1 000 W/m^2. Imagine a spacecraft with a mirrored square sail of dimension 1.0 km. Estimate how much thrust (in newtons) this craft will experience due to collisions with the Sun's photons. [*Hint*: Assume the photons bounce off the sail with no change in magnitude of their momentum.]

SOLUTION. The energy of a photon in terms of the momentum is

$$E = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = pc.$$

If the sail is perpendicular to the sunlight, the rate at which photons are striking the sail is

$$\frac{N}{\Delta t} = \frac{IA}{E} = \frac{IA}{pc}.$$

Because the photons reflect from the sail, the change in momentum of a photon is

$$\Delta p = 2p.$$

The impulse on the sail is due to the change in momentum of the photons:

$$F\,\Delta t = N\,\Delta p,$$

or

$$F = \frac{N}{\Delta t}\,\Delta p = \frac{IA}{pc}\,(2p) = \frac{2IA}{c} = \frac{2(1\,000\ \text{W/m}^2)(1 \times 10^3\ \text{m})^2}{3.00 \times 10^8\ \text{m/s}} = 7\ \text{N}.$$

2–84. The human eye can respond to as little as 10^{-18} J of light energy. If this comes with a wavelength at the peak of visual sensitivity, 550 nm, how many photons lead to an observable flash?

SOLUTION. We find the number of photons from

$$\begin{aligned} N &= \frac{E}{2\pi\hbar f} = \frac{E}{(2\pi\hbar c/\lambda)} \\ &= \frac{10^{-18}\ \text{J}}{((1.24 \times 10^3\ \text{eV} \cdot \text{nm})/(550\ \text{nm}))\,(1.60 \times 10^{-19}\ \text{J/eV})} \approx 3. \end{aligned}$$

2–85. Light with a wavelength of 300 nm strikes a metal whose work function is 2.2 eV. What is the shortest de Broglie wavelength for the electrons that are produced as photoelectrons?

SOLUTION. The energy of the photon is

$$2\pi\hbar f = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{300 \text{ nm}} = 4.13 \text{ eV}.$$

The maximum kinetic energy of the photoelectrons is

$$K_{\max} = 2\pi\hbar f - W_0 = 4.13 \text{ eV} - 2.2 \text{ eV} = 1.93 \text{ eV}.$$

Because this is much less than the rest energy of the electron,

$$K_{\max} = \frac{p_{\max}^2}{2m}.$$

Thus the shortest wavelength of the electron is

$$\begin{aligned} \lambda_{\min} &= \frac{2\pi\hbar}{p_{\max}} = \frac{2\pi\hbar}{\sqrt{2mK_{\max}}} \\ &= \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{\sqrt{2(9.11 \times 10^{-31} \text{ kg})(1.93 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})}} \\ &= 8.8 \times 10^{-10} \text{ m} = 0.88 \text{ nm}. \end{aligned}$$

3 Quantum Mechanics

3.1 Review

In 1925, Schrödinger and Heisenberg separately worked out a new theory, *quantum mechanics*, which is now considered to be the basic theory at the atomic level. It is a statistical theory rather than a deterministic one.

An important aspect of quantum mechanics is the Heisenberg *uncertainty principle*. It results from the wave-particle duality and the unavoidable interaction between the observed object and the observer.

One form of the uncertainty principle states that the position x and momentum p_x of an object cannot both be measured precisely at the same time. The products of the uncertainties, $\Delta x\,\Delta p_x$, can be no less than $\hbar(= h/2\pi)$:

$$\Delta x\,\Delta p_x \geqslant \frac{\hbar}{2}. \tag{3.1}$$

Another form states that the energy can be uncertain, or nonconserved, by an amount ΔE for a time Δt where

$$\Delta E\,\Delta t \geqslant \frac{\hbar}{2}. \tag{3.2}$$

Example 3–1. An electron moves in a straight line with a constant speed $v = 1.10 \times 10^6$ m/s which has been measured to a precision of 0.10 percent. What is the maximum precision with which its position could be simultaneously measured?

Solution. The momentum of the electron is

$$\begin{aligned} p &= mv = (9.11 \times 10^{-31}\ \text{kg})(1.10 \times 10^6\ \text{m/s}) \\ &= 1.00 \times 10^{-24}\ \text{kg} \cdot \text{m/s}. \end{aligned}$$

The uncertainty in the momentum is 0.10 percent of this, or $\Delta p = 1.0 \times 10^{-27}$ kg · m/s. From the uncertainty principle, the best simultaneous position measurement will have an uncertainty of

$$\Delta x = \frac{\hbar}{2\,\Delta p} = \frac{1.06 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(1.0 \times 10^{-27}\ \text{kg} \cdot \text{m/s})} = 0.53 \times 10^{-7}\ \text{m} = 53\ \text{nm}.$$

This is about 1 000 times the diameter of an atom.

Example 3–2. What is the uncertainty in position, imposed by the uncertainty principle, on a 150-g baseball thrown at (42 ± 1) m/s?

SOLUTION. The uncertainty in the momentum is

$$\Delta p = m\,\Delta v = (0.150\ \text{kg})(1\ \text{m/s}) = 0.15\ \text{kg}\cdot\text{m/s}.$$

Hence the uncertainty in a position measurement could be as small as

$$\Delta x = \frac{\hbar}{2\,\Delta p} = \frac{1.06\times 10^{-34}\ \text{J}\cdot\text{s}}{(2)(0.15\ \text{kg}\cdot\text{m/s})} = 3.5\times 10^{-34}\ \text{m},$$

which is a distance incredibly smaller than any we could imagine observing or measuring. Indeed, the uncertainty principle sets no relevant limit on measurement for macroscopic objects.

Example 3–3. The J/ψ meson discovered in 1974, was measured to have an average mass of 3 100 MeV/c^2 (note the use of energy units since $E = mc^2$) and an intrinsic width of 63 keV/c^2. By this we mean that the masses of different J/ψ mesons were actually measured to be slightly different from one another. This mass "width" is related to the very short lifetime of the J/ψ before it decays into other particles. Estimate its lifetime using the uncertainty principle.

SOLUTION. The uncertainty of 63 keV/c^2 in the J/ψ's mass is an uncertainty in its rest energy, which in joules is

$$\Delta E = (63\times 10^3\ \text{eV})(1.60\times 10^{-19}\ \text{J/eV}) = 1.01\times 10^{-14}\ \text{J}.$$

Then we expect its lifetime τ $(= \Delta t$ here) to be

$$\tau \approx \frac{\hbar}{2\,\Delta E} = \frac{1.06\times 10^{-34}\ \text{J}\cdot\text{s}}{(2)(1.01\times 10^{-14}\ \text{J})} \approx 0.52\times 10^{-20}\ \text{s}.$$

Lifetimes this short are difficult to measure directly, and the assignment of very short lifetimes depends on this use of the uncertainty principle.

The uncertainty principle applies also for the angular variables:

$$\Delta L_z\,\Delta\varphi \geqslant \frac{\hbar}{2},$$

where L_z is the component of the angular momentum along a given axis (z) and φ is the angular position in the plane perpendicular to that axis.

A particle such as an electron is represented by a *wave function* ψ. The square of the wave function, $|\psi|^2$, at any point in space represents the *probability* of finding the particle at that point. The wave function must be *normalized*, meaning that

$$\int_{\text{all space}} |\psi|^2 \, dV = 1, \tag{3.3}$$

since the particle must be found at one place or another.

In nonrelativistic wave mechanics, ψ satisfies the *Schrödinger equation*:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x), \tag{3.4}$$

here in its one-dimensional time-independent form, where U is the potential energy as a function of position and E is the total energy of the particle.

The more general form of of the Schrödinger equation, including time dependence, for a particle of mass m moving in one dimension, is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + U(x)\psi(x) = i\hbar\frac{\partial\psi(x)}{\partial t}, \tag{3.5}$$

A *free particle* subject to no forces has a sinusoidal wave function

$$\psi = A\sin kx + B\cos kx \tag{3.6}$$

with the wavenumber

$$k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{p}{\hbar} = \frac{2\pi\hbar}{\lambda\hbar} = \frac{2\pi}{\lambda} \tag{3.7}$$

and p is the particle's momentum. Such a wave of fixed momentum is spread out indefinitely in space as a plane wave.

A *wave packet*, localized in space, is a superposition of sinusoidal waves with a range of momenta.

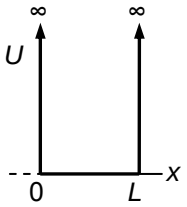


Figure 3.1. Potential energy U vs x for an infinitely deep square well potential.

For a particle confined to an *infinitely deep square well potential*, or *rigid box* (Fig. 3.1), the Schrödinger equation gives the wave functions inside the well

$$\psi = A \sin kx, \quad A = \sqrt{\frac{2}{L}} \quad \text{and} \quad k = \frac{n\pi}{L}. \tag{3.8}$$

The energy is quantized,

$$E = \frac{\hbar^2 k^2}{2m} = n^2 \frac{\pi^2 \hbar^2}{2mL^2}, \tag{3.9}$$

where n is an integer. Fig. 3.2 shows the wave functions (Eq. 3.8) and the probability distribution for $n = 1$, 2, and 3.

Example 3–4. (a) Calculate the three lowest energy levels for an electron trapped in an infinitely deep square well potential of width $L = 1.00 \times 10^{-10}$ m (about the diameter of a hydrogen atom in its ground state), (b) If a photon were emitted when the electron jumps from the $n = 2$ state to the $n = 1$ state, what would its wavelength be?

Solution. (a) The ground state ($n = 1$) has energy

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(9.11 \times 10^{-31} \text{ kg})(1.00 \times 10^{-10} \text{ m})^2} = 6.03 \times 10^{-18} \text{ J}.$$

In electron volts this is

$$E_1 = \frac{6.03 \times 10^{-18} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = 37.7 \text{ eV}.$$

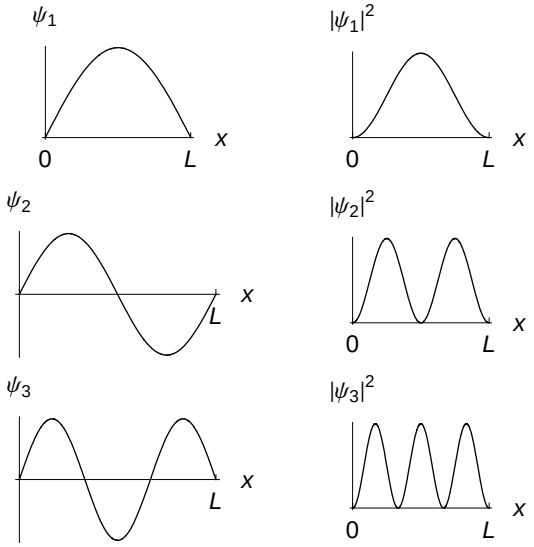


Figure 3.2. Wave functions and the probability distribution for a particle in a rigid box for the states with $n = 1$, 2, and 3.

Then

$$\begin{aligned} E_2 &= (2)^2 E_1 = 151 \text{ eV} \\ E_3 &= (3)^2 E_1 = 339 \text{ eV}. \end{aligned}$$

(b) The energy difference is

$$E_2 - E_1 = 151 \text{ eV} - 38 \text{ eV} = 113 \text{ eV} = 1.81 \times 10^{-17} \text{ J},$$

and this would equal the energy of the emitted photon (energy conservation). Its wavelength would be

$$\begin{aligned} \lambda = \frac{c}{f} = \frac{2\pi\hbar c}{E} &= \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})(3.00 \times 10^{8} \text{ m/s})}{1.81 \times 10^{-17} \text{ J}} \\ &= 1.10 \times 10^{-8} \text{ m} = 11.0 \text{ nm}, \end{aligned}$$

which is in the ultraviolet region of the spectrum.

Example 3–5. Show that the normalization constant A for all wave functions describing a particle in an infinite potential well of width L has a value of $A = \sqrt{2/L}$.

SOLUTION. The wave functions are

$$\psi = A \sin \frac{n\pi x}{L}$$

To normalize ψ, we must have (Eq. 3.3)

$$1 = \int_0^L |\psi|^2 \, dx = \int_0^L A^2 \left(\sin \frac{n\pi x}{L} \right)^2 dx.$$

We need integrate only from 0 to L since $\psi = 0$ for all other values of x. To evaluate this integral we let $\theta = n\pi x/L$ and use the trigonometric identity $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$. Then, with $dx = L\, d\theta/(n\pi)$, we have

$$\begin{aligned} 1 &= A^2 \int_0^{n\pi} \sin^2 \theta \left(\frac{L}{n\pi} \right) d\theta = \frac{A^2 L}{2n\pi} \int_0^{n\pi} (1 - \cos 2\theta)\, d\theta \\ &= \frac{A^2 L}{2n\pi} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Bigg|_0^{n\pi} = \frac{A^2 L}{2}. \end{aligned}$$

Thus $A^2 = 2/L$ and

$$A = \sqrt{\frac{2}{L}}.$$

Example 3–6. A tiny bacterium with a mass of about 10^{-14} kg is confined between two rigid walls 0.1 mm apart. (a) Estimate its minimum speed, (b) If, instead, its speed is about 1 mm in 100 s, estimate the quantum number of its state.

SOLUTION. (a) The minimum speed occurs in the ground state, $n = 1$; since $E = \frac{1}{2}mv^2$, we have

$$\begin{aligned} v &= \sqrt{\frac{2E}{m}} = \sqrt{\frac{\pi^2 \hbar^2}{m^2 L^2}} = \frac{\pi \hbar}{mL} = \frac{\pi (1.055 \times 10^{-34} \text{ J} \cdot \text{s})}{(10^{-14} \text{ kg})(10^{-4} \text{ m})} \\ &\approx 3 \times 10^{-16} \text{ m/s}. \end{aligned}$$

This is a speed so small that we could not measure it and the object would seem at rest, consistent with classical physics.

(b) Given $v = 10^{-3}$ m/100 s $= 10^{-5}$ m/s, the kinetic energy of the bacterium is

$$E = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(10^{-14}\text{ kg})(10^{-5}\text{ m/s})^2 = 0.5 \times 10^{-24}\text{ J}.$$

From Eq. 3.9, the quantum number of this state is

$$\begin{aligned} n &= \sqrt{E\left(\frac{2mL^2}{\pi^2\hbar^2}\right)} = \sqrt{\frac{(0.5 \times 10^{-24}\text{ J})(2)(10^{-14}\text{ kg})(10^{-4})^2}{\pi^2(1.055 \times 10^{-34}\text{ J}\cdot\text{s})^2}} \\ &\approx \sqrt{1 \times 10^{21}} \approx 3 \times 10^{10}. \end{aligned}$$

This number is so large that we could never distinguish between adjacent energy states (between $n = 3 \times 10^{10}$ and $3 \times 10^{10} + 1$). The energy states would appear to form a continuum. Thus, even though the energies involved here are small ($\ll 1$ eV), we are still dealing with a macroscopic object (though visible only under a microscope) and the quantum result is not distinguishable from a classical one. This is in accordance with the correspondence principle.

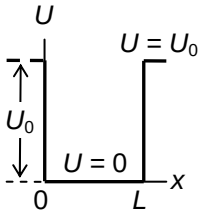


Figure 3.3. Potential energy U vs x for a finite one-dimensional square well.

In a *finite potential well* (Fig. 3.3), the wave function extends into the classically forbidden region where the total energy is less than the potential energy. That this is possible is consistent with the uncertainty principle. The solutions to the Schrödinger equation in these areas are decaying exponentials.

Because quantum-mechanical particles can penetrate such classically forbidden areas, they can *tunnel* through thin barriers even though the potential energy in the barrier, U_0 is greater than the total energy of

the particle, E. Quantitatively, we can describe the tunneling probability with a *transmission coefficient*, T, and a *reflection coefficient*, R. Note that $T + R = 1$, since an incident particle must either reflect or tunnel through. If T is small ($\ll 1$), then

$$T \approx \frac{16E(U_0 - E)}{U_0^2}\, e^{-2GL}, \tag{3.10}$$

where

$$G = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

This approximate expression is valid when $e^{-2GL} \ll 1$. In fact, this exponential dependence on the penetration parameter G and on the width of the barrier L is the most important, and the most useful, feature of the tunneling formula Eq. 3.10.

Example 3–7. A 50-eV electron approaches a quare barrier 70 eV high and (a) 1.0 nm thick, (b) 0.10 nm thick. What is the probability that the electron will tunnel through?

Solution. (a) First we write the energy in SI units.

$$U_0 - E = (70 \text{ eV} - 50 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV}) = 3.2 \times 10^{-18} \text{ J}.$$

Then, using Eq. 3.10, we have

$$2GL = 2\frac{\sqrt{2(9.11 \times 10^{-31} \text{ kg})(3.2 \times 10^{-18} \text{ J})}}{1.06 \times 10^{-34} \text{ J} \cdot \text{s}}\,(1.0 \times 10^{-9} \text{ m}) = 45.56,$$

and

$$T = \frac{16E(U_0 - E)}{U_0^2}\, e^{-2GL} = \frac{16(50 \text{ eV})(20 \text{ eV})}{(70 \text{ eV})^2}\, e^{-45.56} = 5.3 \times 10^{-20},$$

which is extremely small.

(b) For $L = 0.10$ nm, $2GL = 4.56$ and

$$T = \frac{16(50 \text{ eV})(20 \text{ eV})}{(70 \text{ eV})^2}\, e^{-4.56} = 0.034.$$

Thus the electron has a 3 percent chance of penetrating a 0.1-nm-thick barrier, but only 5 chances in 10^{20} to penetrate a 1-nm barrier. By reducing

the barrier thickness by a factor of 10, the probability of tunneling through was increased 10^{18} times! Clearly the transmission coefficient is extremely sensitive to the values of L, $U_0 - E$, and m.

It can be a challenge to solve the Shrödinger equation for the bound-state energy levels of an arbitrary potential well. An alternative approach that can yield good approximate results for the energy levels is the *WKB approximation* named for the physicists Gregor Wentzel, Hendrik Kramers, and Léon Brillouin, who pioneered its application to quantum mechanics.

The WKB approximation begins from three physical statements: (i) according to de Broglie, the magnitude of momentum p of a quantum-mechanical particle is $p = 2\pi\hbar/\lambda$; (ii) the magnitude of momentum is related to the kinetic energy K by the relation $K = p^2/2m$; (iii) if there are no conservative forces, then in Newtonian mechanics the energy E for a particle is constant and equal at each point to the sum of kinetic and potential energies at that point: $E = K + U(x)$, where x is the coordinate. These three relations can be combined to find the wavelength of the particle at a coordinate x: solving the equation

$$E = K + U(x) = \frac{p^2}{2m} + U(x)$$

for p yields $p = \sqrt{2m\,(E - U(x))}$, and the equation $p = 2\pi\hbar/\lambda$, in its turn, gives

$$\lambda(x) = \frac{2\pi\hbar}{\sqrt{2m\,(E - U(x))}}. \tag{3.11}$$

Thus we envision a quantum-mechanical particle in a potential well $U(x)$ as being like a free particle, but with a wavelength $\lambda(x)$ that is a function of position.

As the particle moves into a region of increasing potential energy, $U(x)$ gets larger approaching E from below (since $K \geqslant 0$) and the quantity $E - U(x)$ gets smaller, so the particle's wavelength $\lambda(x)$ gets larger.

At a point where $E = U(x)$, Newtonian mechanics says that the particle has zero kinetic energy and must be instantaneously at rest. Such point is called *classical turning point*, since this is where a Newtonian particle must stop its motion and reverse direction. As an example, an object oscillating in simple harmonic motion with amplitude A moves back and

forth between points $x = -A$ and $x = +A$; each of these is a classical turning point, since there the potential energy $\frac{1}{2}kx^2$ equals the total energy $\frac{1}{2}kA^2$. In the WKB expression for $\lambda(x)$, the wavelength at a classical turning point becomes infinite.

For a particle in a box with length L, the walls of the box are classical turning points (see Fig. 3.1). Furthermore, the number of wavelengths that fit within the box must be a half-integer (see Fig. 3.2), so that $L = (n/2)\lambda$ and hence $L/\lambda = n/2$, where $n = 1, 2, 3, \ldots$. [Note that this is a restatement of Eq. 3.8.] The WKB scheme for finding the allowed bound-state energy levels of an *arbitrary* potential well is an extension of these observations. It demands that for an allowed energy E, there must be a half-integer number of wavelengths between the classical turning points for that energy. Since the wavelength in the WKB approximation is not a constant but depends on x, the number of wavelengths between the classical turning points a and b for a given value of the energy is the integral of $1/\lambda(x)$ between those points:

$$\int_a^b \frac{dx}{\lambda(x)} = \frac{n}{2}, \qquad n = 1,\ 2,\ 3,\ \ldots$$

Using Eq. 3.11 for $\lambda(x)$ we find that the *WKB condition for an allowed bound-state energy* can be written as

$$\int_a^b \sqrt{2m\,(E - U(x))}\,dx = \pi\hbar n, \qquad n = 1,\ 2,\ 3,\ \ldots. \tag{3.12}$$

Example 3–8. As a check on Eq. 3.12, apply it to a particle (a) In an infinitely deep square well with walls at $x = 0$ and $x = L$; (b) In a finite square well shown in Fig. 3.3. Compare and contrast your results.

Solution. (a) We need to evaluate the integral and find the allowed energy levels in terms of the WKB approximation. In so doing we take into account that since the walls of the box are infinitely high, the points $x = 0$ and $x = L$ are classical turning points for *any* energy E. Besides, inside the box, the potential energy is zero. So,

$$\int_a^b \sqrt{2m\,(E - U(x))}\,dx = \int_0^L \sqrt{2mE}\,dx = \sqrt{2mE}\int_0^L dx = \sqrt{2mE}\,L.$$

Thus Eq. 3.12 gives

$$\sqrt{2mE}\,L = n\pi\hbar$$

whence it follows that

$$E = \frac{\pi^2\hbar^2 n^2}{2mL^2}.$$

We see that the allowed energy levels according to the WKB approximation are the same as those given by Eq. 3.9.

(b) Assume $E < U_0$. Then the classical turning points are at $x = 0$ and $x = L$. Since $U(x) = 0$ in the region between the turning points, the result is the *same* as (a). The height U_0 never enters the calculation. For the finite square well, the WKB expression given by Eq. 3.12 predicts the same bound-state energies as for an infinite square well of the same width. The results (a) and (b) show that the WKB approximation does a poor job when the potential energy function changes discontinuously, as for a finite potential well. This technique is best used with smoothly varying potential $U(x)$. Problems 3–38 and 3–39 consider situations in which the potential energy function changes gradually and the WKB approximation is much more useful.

3.2 The wave function

3–1. The neutrons in a parallel beam, each having kinetic energy $\frac{1}{40}$ eV, are directed through two slits 1.0 mm apart. How far apart will the interference peaks be on a screen 1.0 m away?

SOLUTION. We find the wavelength of the neutron from

$$\begin{aligned}\lambda &= \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mK}} \\ &= \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(1.67\times10^{-27}\ \text{kg})(0.025\ \text{eV})(1.6\times10^{-19}\ \text{J/eV})}} \\ &= 1.81\times10^{-10}\ \text{m}.\end{aligned}$$

The peaks of the interference pattern are given by

$$d\sin\theta = m\lambda, \qquad m = 1,\ 2,\ \ldots\ ,$$

and the positions on the screen are

$$y = L\tan\theta.$$

For small angles, $\sin\theta \approx \tan\theta$, so we have

$$y = \frac{mL\lambda}{d}.$$

Thus the separation is

$$\Delta y = \frac{L\lambda}{d} = \frac{(1.0\text{ m})(1.81\times10^{-10}\text{ m})}{1.0\times10^{-3}\text{ m}} = 1.8\times10^{-7}\text{ m}.$$

3–2. Bullets of mass 2.0 g are fired in parallel paths with speeds of 120 m/s through a hole 3.0 mm wide. How far from the hole must you be to detect a 1.0-cm spread in the beam?

SOLUTION. We find the wavelength of the bullet from

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{mv} = \frac{6.63\times10^{-34}\text{ J}\cdot\text{s}}{(2.0\times10^{-3}\text{ kg})(120\text{ m/s})} = 2.8\times10^{-33}\text{ m}.$$

The half-angle for the central circle of the diffraction pattern is given by

$$\sin\theta = \frac{1.22\,\lambda}{D}.$$

For small angles, $\sin\theta \approx \tan\theta$, so we have

$$\begin{aligned} r &= L\tan\theta \approx L\sin\theta = \frac{1.22\,L\lambda}{D}; \\ 0.50\times10^{-2}\text{ m} &= \frac{1.22\,L(2.8\times10^{-33}\text{ m})}{3.0\times10^{-3}\text{ m}}, \end{aligned}$$

which gives $L = 4.5\times10^{27}$ m. Diffraction effects are negligible for macroscopic objects.

3.3 The Heisenberg uncertainty principle

3–3. A proton is traveling with a speed of $(4.825\pm0.012)\times10^{5}$ m/s. With what maximum accuracy can its position be ascertained?

SOLUTION. We find the uncertainty in the momentum:

$$\begin{aligned} \Delta p &= m\,\Delta v = (1.67\times10^{-27}\text{ kg})(0.024\times10^{5}\text{ m/s}) \\ &= 4.00\times10^{-24}\text{ kg}\cdot\text{m/s}. \end{aligned}$$

We find the uncertainty in the proton's position from

$$\Delta x \geqslant \frac{\hbar}{2\,\Delta p} = \frac{1.055 \times 10^{-34}\ \mathrm{J \cdot s}}{(2)(4.00 \times 10^{-24}\ \mathrm{kg \cdot m/s})} = 1.3 \times 10^{-11}\ \mathrm{m}.$$

Thus the accuracy of the position is $\pm 0.66 \times 10^{-11}$ m.

3–4. An electron remains in an excited state of an atom for typically 10^{-8} s. What is the minimum uncertainty in the energy of the state (in eV)?

SOLUTION. We find the minimum uncertainty in the energy of the state from

$$\Delta E \geqslant \frac{\hbar}{2\,\Delta t} = \frac{1.055 \times 10^{-34}\ \mathrm{J \cdot s}}{(2)(10^{-8}\ \mathrm{s})} = 0.53 \times 10^{-26}\ \mathrm{J} = 3.3 \times 10^{-8}\ \mathrm{eV}.$$

3–5. If an electron's position can be measured to an accuracy of 1.6×10^{-8} m, how accurately can its velocity be known?

SOLUTION. We find the uncertainty in the momentum:

$$\Delta p \geqslant \frac{\hbar}{2\,\Delta x} = \frac{1.055 \times 10^{-34}\ \mathrm{J \cdot s}}{(2)(1.6 \times 10^{-8}\ \mathrm{m})} = 3.3 \times 10^{-27}\ \mathrm{kg \cdot m/s}.$$

We find the uncertainty in the velocity from

$$\begin{aligned} \Delta p &= m\,\Delta v; \\ 3.3 \times 10^{-27}\ \mathrm{kg \cdot m/s} &= (9.11 \times 10^{-31}\ \mathrm{kg})\,\Delta v, \end{aligned}$$

which gives $\Delta v = 3.6 \times 10^{3}$ m/s.

3–6. A 12-g bullet leaves a rifle at a speed of 150 m/s. (a) What is the wavelength of this bullet? (b) If the position of the bullet is known to an accuracy of 0.55 cm (radius of the barrel), what is the minimum uncertainty in its vertical momentum? (c) If the accuracy of the bullet were determined only by the uncertainty principle (an unreasonable assumption), by how much might the bullet miss a pinpoint target 300 m away?

SOLUTION. (a) We find the wavelength of the bullet from

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{mv} = \frac{6.63 \times 10^{-34}\ \mathrm{J \cdot s}}{(12 \times 10^{-3}\ \mathrm{kg})(150\ \mathrm{m/s})} = 3.7 \times 10^{-34}\ \mathrm{m}.$$

(b) We find the uncertainty in the momentum component perpendicular to the motion:

$$\Delta p_y \geqslant \frac{\hbar}{2\,\Delta y} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(0.55 \times 10^{-2}\ \text{m})} = 0.96 \times 10^{-32}\ \text{kg} \cdot \text{m/s}.$$

(c) We find the possible uncertainty in the y-position at the target from

$$\begin{aligned} \frac{y}{L} &= \frac{\Delta v_y}{v_x} = \frac{\Delta p_y}{p_x}; \\ \frac{y}{300\ \text{m}} &= \frac{0.96 \times 10^{-32}\ \text{kg} \cdot \text{m/s}}{(12 \times 10^{-3}\ \text{kg})(150\ \text{m/s})}, \end{aligned}$$

which gives $y = 1.6 \times 10^{-30}$ m.

3–7. An electron and a 150-g baseball are each traveling 75 m/s measured to an accuracy of 0.065 percent. Calculate and compare the uncertainty in position of each.

Solution. The uncertainty in the velocity is

$$\Delta v = \left(\frac{0.065}{100}\right)(75\ \text{m/s}) = 0.0488\ \text{m/s}.$$

For the electron, we have

$$\Delta x \geqslant \frac{\hbar}{2m\,\Delta v} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(9.11 \times 10^{-31}\ \text{kg})(0.0488\ \text{m/s})} = 1.2 \times 10^{-3}\ \text{m}.$$

For the baseball, we have

$$\Delta x \geqslant \frac{\hbar}{2m\,\Delta v} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(0.150\ \text{kg})(0.0488\ \text{m/s})} = 0.72 \times 10^{-32}\ \text{m}.$$

The uncertainty for the electron is greater by a factor of 1.7×10^{29}.

3–8. Use the uncertainty principle to show that if an electron were present in the nucleus ($r \approx 10^{-15}$ m), its kinetic energy (use relativity) would be hundreds of MeV. (Since such electron energies are not observed, we conclude that electrons are not present in the nucleus.) [*Hint*: a particle can have energy as large as its uncertainty.]

SOLUTION. We use the radius as the uncertainty in position for the electron. We find the uncertainty in the momentum from

$$\Delta p \geqslant \frac{\hbar}{2\Delta x} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(10^{-15}\ \text{m})} = 0.5275 \times 10^{-19}\ \text{kg} \cdot \text{m/s}.$$

If we assume that the lowest value for the momentum is the least uncertainty, we estimate the lowest possible energy as

$$\begin{aligned} E &= K + mc^2 = \sqrt{p^2c^2 + m^2c^4} = \sqrt{(\Delta p)^2c^2 + m^2c^4} \\ &= \big((0.5275 \times 10^{-19}\ \text{kg} \cdot \text{m/s})^2(3.00 \times 10^8\ \text{m/s})^2 \\ &\quad + (9.11 \times 10^{-31}\ \text{kg})^2(3.00 \times 10^8\ \text{m/s})^4\big)^{1/2} \\ &= 1.58 \times 10^{-11}\ \text{J} \approx 99\ \text{MeV}. \end{aligned}$$

3–9. An electron in the $n = 2$ state of hydrogen remains there on the average about 10^{-8} s before jumping to the $n = 1$ state. (a) Estimate the uncertainty in the energy of the $n = 2$ state, (b) What fraction of the transition energy is this? (c) What is the wavelength, and width (in nm), of this line in the spectrum of hydrogen?

SOLUTION. (a) We find the minimum uncertainty in the energy of the state from

$$\Delta E \geqslant \frac{\hbar}{2\Delta t} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(10^{-8}\ \text{s})} = 0.53 \times 10^{-26}\ \text{J} = 3.3 \times 10^{-8}\ \text{eV}.$$

Note that, because the ground state is stable, we associate the uncertainty with the excited state.

(b) The transition energy is

$$E = -(13.6\ \text{eV})\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.2\ \text{eV},$$

so we have

$$\frac{\Delta E}{E} = \frac{3.3 \times 10^{-8}\ \text{eV}}{10.2\ \text{eV}} = 3.2 \times 10^{-9}.$$

(c) The wavelength of the line is

$$\lambda = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{E} = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{10.2\ \text{eV}} = 122\ \text{nm}.$$

If we treat the width of the line as a differential, we have

$$\Delta\lambda = -(1.24 \times 10^3 \text{ eV} \cdot \text{nm})\frac{\Delta E}{E^2}.$$

If we ignore the sign, we get

$$\Delta\lambda = \lambda\,\frac{\Delta E}{E} = (122 \text{ nm})(3.2 \times 10^{-9}) = 3.9 \times 10^{-7} \text{ nm}.$$

3–10. How accurately can the position of a 2.50-keV electron be measured assuming its energy is known to 1.00 percent?

SOLUTION. The momentum of the electron is

$$\begin{aligned} p &= \sqrt{2mK} \\ &= \sqrt{2(9.11 \times 10^{-31} \text{ kg})(2.50 \text{ keV})(1.60 \times 10^{-16} \text{ J/keV})} \\ &= 2.70 \times 10^{-23} \text{ kg} \cdot \text{m/s}. \end{aligned}$$

Because the changes are small, we can treat them as differentials. Thus the change in momentum is

$$\Delta p = \frac{1}{2}\sqrt{\frac{2m}{K}}\,\Delta K,$$

or

$$\frac{\Delta p}{p} = \frac{1}{2}\,\frac{\Delta K}{K}.$$

We find the uncertainty in the electron's position from

$$\begin{aligned} \Delta x &\geqslant \frac{\hbar}{2\Delta p} = \frac{2\hbar}{2p\,(\Delta K/K)} = \frac{2(1.055 \times 10^{-34} \text{ J} \cdot \text{s})}{(2)(2.70 \times 10^{-23} \text{ kg} \cdot \text{m/s})(0.0100)} \\ &= 3.91 \times 10^{-10} \text{ m}. \end{aligned}$$

3–11. In a double-slit experiment on electrons (or photons) suppose that we use indicators to determine which slit each electron went through. These indicators must tell us the y coordinate to within $d/2$, where d is the distance between slits. Use the uncertainty principle to show that the interference pattern will be destroyed. [*Note*: first show that the angle θ

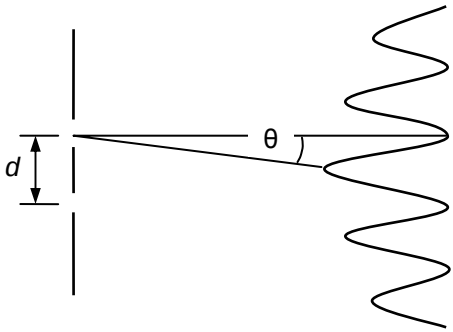


Figure 3.4. Problem 3–11.

between maxima and minima of the interference pattern is given by $\frac{1}{2}\lambda/d$, Fig. 3.4.]

SOLUTION. The electron has an initial momentum p_x and a wavelength $\lambda = h/p_x$. For the maxima of the double-slit interference we have

$$d \sin\theta = m\lambda, \qquad m = 0,\ 1,\ 2,\ \ldots\ .$$

If the angles are small, the separation of maxima is

$$\Delta\theta_{\max} = \frac{\lambda}{d},$$

so the angle between a maximum and a minimum is

$$\Delta\theta = \frac{\lambda}{2d}.$$

The separation on the screen will be

$$H = L\,\Delta\theta = \frac{\lambda L}{2d}.$$

The uncertainty in the y-position at the slits of $d/2$ produces an uncertainty in the y-momentum of

$$\Delta p_y \geqslant \frac{\hbar}{2\,\Delta y} = \frac{\hbar}{2\,(d/2)} = \frac{\hbar}{d}.$$

This produces an uncertainty of the y-position at the screen of

$$\Delta H = \frac{\Delta p_y}{p_x} L = \frac{(\hbar/d)}{(2\pi\hbar/\lambda)} L = \frac{\lambda L}{2\pi d}.$$

This is on the order of the separation of maxima and minima, so the pattern is destroyed.

3.4 Time-dependent Schrödinger equation

3–12. (a) Show that $\Psi(x,t) = Ae^{i(kx-\omega t)}$ is a solution to the time-dependent Schrödinger equation for a free particle [$U(x) = U_0 =$ constant] but that $\Psi(x,t) = A\cos(kx-\omega t)$ and $\Psi(x,t) = A\sin(kx-\omega t)$ are not. (b) Show that the valid solution of part (a) satisfies conservation of energy if the de Broglie relations hold, $\lambda = 2\pi\hbar/p$, $\omega = E/\hbar$. That is, show that direct substitution in Eq. 3.5 gives

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + U_0.$$

SOLUTION. The Schrödinger equation for a free particle is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U_0\Psi = i\hbar\frac{\partial\Psi}{\partial t}.$$

(a) For the proposed solution $\Psi = Ae^{i(kx-\omega t)}$ we have

$$\begin{aligned} \frac{\partial\Psi}{\partial t} &= -i\omega Ae^{i(kx-\omega t)} = -i\omega\Psi; \\ \frac{\partial^2\Psi}{\partial x^2} &= (ik)^2 Ae^{i(kx-\omega t)} = -k^2\Psi. \end{aligned}$$

If we substitute these in the equation, we get

$$-\frac{\hbar^2}{2m}(-k^2\Psi) + U_0\Psi = -i^2\hbar\omega\Psi = \hbar\omega\Psi.$$

Because both sides have the same functional dependence, the solution is valid.

For the proposed solution $\Psi = A\cos(kx-\omega t)$ we have

$$\begin{aligned} \frac{\partial\Psi}{\partial t} &= +A\omega\sin(kx-\omega t); \\ \frac{\partial^2\Psi}{\partial x^2} &= -Ak^2\cos(kx-\omega t). \end{aligned}$$

If we substitute these in the equation, we get

$$-\frac{\hbar^2}{2m}\left(-Ak^2\cos(kx-\omega t)\right)+U_0A\cos(kx-\omega t)=i\hbar A\omega\sin(kx-\omega t);$$
$$\left(\frac{\hbar^2k^2}{2m}+U_0\right)A\cos(kx-\omega t)=i\hbar A\omega\sin(kx-\omega t).$$

Because cosine and sine functions are different functions of x and t, the solution is not valid.

For the proposed solution $\Psi = A\sin(kx-\omega t)$ we have

$$\begin{aligned}\frac{\partial\Psi}{\partial t} &= -A\omega\cos(kx-\omega t);\\ \frac{\partial^2\Psi}{\partial x^2} &= -Ak^2\sin(kx-\omega t).\end{aligned}$$

If we substitute these in the equation, we get

$$\begin{aligned}-\frac{\hbar^2}{2m}\left(-Ak^2\sin(kx-\omega t)\right)+U_0A\sin(kx-\omega t) &= -i\hbar A\omega\cos(kx-\omega t);\\ \left(\frac{\hbar^2k^2}{2m}+U_0\right)A\sin(kx-\omega t) &= -i\hbar A\omega\cos(kx-\omega t).\end{aligned}$$

Because cosine and sine functions are different functions of x and t, the solution is not valid.

(b) For the valid solution we have

$$-\frac{\hbar^2}{2m}(-k^2\Psi)+U_0\Psi=-i^2\hbar\omega\Psi=\hbar\omega\Psi,$$

or

$$\hbar\omega=\frac{\hbar^2k^2}{2m}+U_0.$$

3.5 Free particles; plane waves and wave packets

3–13. Write the wave function for (a) a free electron and (b) a free proton, each having a constant velocity $v = 4.0\times10^5$ m/s.

Solution. The wave function for a free particle is

$$\psi = A\sin kx + B\cos kx.$$

(a) For the free electron we have

$$k = \frac{p}{\hbar} = \frac{mv}{\hbar} = \frac{(9.11 \times 10^{-31}\ \text{kg})(4.0 \times 10^{5}\ \text{m/s})}{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}} = 3.5 \times 10^{9}\ \text{m}^{-1}.$$

Thus we have

$$\psi = A \sin\left(3.5 \times 10^{9}\ \text{m}^{-1}\right) x + B \cos\left(3.5 \times 10^{9}\ \text{m}^{-1}\right) x.$$

(b) For the free proton we have

$$k = \frac{p}{\hbar} = \frac{mv}{\hbar} = \frac{(1.67 \times 10^{-27}\ \text{kg})(4.0 \times 10^{5}\ \text{m/s})}{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}} = 6.3 \times 10^{12}\ \text{m}^{-1}.$$

Thus we have

$$\psi = A \sin\left(6.3 \times 10^{12}\ \text{m}^{-1}\right) x + B \cos\left(6.3 \times 10^{12}\ \text{m}^{-1}\right) x.$$

3–14. A free electron has a wave function $\psi(x) = A \sin(1.0 \times 10^{10} x)$, where x is given in meters. Determine the electron's (a) wavelength, (b) momentum, (c) speed, and (d) kinetic energy.

SOLUTION. The wave function is

$$\psi = A \sin\left(1.0 \times 10^{10}\ \text{m}^{-1}\right) x = A \sin kx.$$

(a) We find the wavelength from

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{1.0 \times 10^{10}\ \text{m}^{-1}} = 6.3 \times 10^{-10}\ \text{m} = 0.63\ \text{nm}.$$

(b) The momentum is

$$p = \hbar k = (1.055 \times 10^{-34}\ \text{J} \cdot \text{s})(1.0 \times 10^{10}\ \text{m}^{-1}) = 1.1 \times 10^{-24}\ \text{kg} \cdot \text{m/s}.$$

(c) The speed is

$$v = \frac{p}{m} = \frac{1.1 \times 10^{-24}\ \text{kg} \cdot \text{m/s}}{9.11 \times 10^{-31}\ \text{kg}} = 1.2 \times 10^{6}\ \text{m/s}.$$

(d) The kinetic energy is

$$K = \frac{p^2}{2m} = \frac{(1.1 \times 10^{-24}\ \text{kg} \cdot \text{m/s})^2}{2(9.11 \times 10^{-31}\ \text{kg})} = 6.1 \times 10^{-19}\ \text{J} = 3.8\ \text{eV}.$$

3–15. Show that the uncertainty principle holds for a "wave packet" that is formed by two waves of similar wavelength λ_1 and λ_2. To do so, use as the two waves $\psi_1 = A \sin k_1 x$ and $\psi_2 = A \sin k_2 x$. Then show that the width of each "wave packet" is $\Delta x = 2\pi/(k_1 - k_2) = 2\pi/\Delta k$. Finally, show that $\Delta x\, \Delta p = 2\pi\hbar$ for this simple situation.

SOLUTION. We form the wave packet from

$$\psi = \psi_1 + \psi_2 = A \sin k_1 x + A \sin k_2 x,$$

where $k_1 = 2\pi/\lambda_1$, and $k_2 = 2\pi/\lambda_2$. If we use a trigonometric identity we get

$$\psi = A(\sin k_1 x + \sin k_2 x) = 2A \sin\left(\frac{(k_1 + k_2)x}{2}\right) \cos\left(\frac{(k_1 - k_2)x}{2}\right).$$

If the wavelengths are almost equal, we have

$$\Delta k = k_1 - k_2,$$

and

$$k \approx \tfrac{1}{2}(k_1 + k_2);$$

thus

$$\psi = 2A \cos\left(\tfrac{1}{2}\, \Delta k\, x\right) \sin kx.$$

The width of the packet corresponds to the distance from one zero to the next zero of the cosine function:

$$\tfrac{1}{2}\, \Delta k\, \Delta x = \pi,$$

which gives

$$\Delta x = \frac{2\pi}{\Delta k}.$$

The momentum is $p = \hbar k$, so $\Delta k = \Delta p/\hbar$. When we use this in the expression for the width, we have

$$\Delta x = \frac{2\pi}{(\Delta p/\hbar)},$$

or

$$\Delta x\, \Delta p = 2\pi\hbar.$$

3.6 Particle in an indefinitely deep square well potential

3–16. Show that for a particle in a perfectly rigid box, the wavelength of the wave function for any state is the de Broglie wavelength.

SOLUTION. The energy levels for the particle in a rigid box are

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{p_n^2}{2m}.$$

Thus the momentum is

$$p_n = \frac{n\pi\hbar}{L}.$$

The wavefunctions for the rigid box are

$$\psi_n = A \sin k_n x,$$

with $k_n = n\pi/L$. The wavelength is

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n} = \frac{2\pi\hbar}{p_n},$$

which is the de Broglie wavelength.

3–17. What is the minimum speed of an electron trapped in a 0.10-nm-wide infinitely deep square well?

SOLUTION. The minimum speed corresponds to the minimum momentum or the maximum wavelength. This is the lowest energy state, so $\lambda_{\max} = 2L$. Thus we have

$$\begin{aligned} p_{\min} &= \frac{2\pi\hbar}{\lambda_{\max}}; \\ mv_{\min} &= \frac{2\pi\hbar}{2L}; \\ (9.11 \times 10^{-31}\ \text{kg})v_{\min} &= \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{2(0.10 \times 10^{-9}\ \text{m})}, \end{aligned}$$

which gives $v_{\min} = 3.6 \times 10^6$ m/s.

3–18. An $n = 3$ to $n = 1$ transition for an electron trapped in a rigid box produces a 240-nm photon. What is the width of the box?

SOLUTION. The energy levels for the electron in a rigid box are

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

For a transition from $n_3 = 3$ to $n_1 = 1$, we have

$$\begin{aligned} 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} &= \Delta E = \frac{(n_3^2 - n_1^2)\pi^2\hbar^2}{2mL^2}; \\ \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{240 \text{ nm}} & \\ \times (1.60 \times 10^{-19} \text{ J/eV}) &= \frac{(3^2 - 1^2)\pi^2(1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(9.11 \times 10^{-31} \text{ kg})L^2}, \end{aligned}$$

which gives $L = 7.6 \times 10^{-10}$ m $= 0.76$ nm.

3–19. An electron trapped in an infinitely deep square well has a ground-state energy $E = 8.0$ eV. (a) What is the longest wave-length photon this system can emit, and (b) what is the width of the well?

SOLUTION. (a) The energy levels for the electron in an infinite square well are

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = n^2 E_1.$$

The longest wavelength photon has the least energy, so the transition must be from $n_2 = 2$ to $n_1 = 1$. Thus we have

$$\begin{aligned} 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} &= \Delta E = (n_2^2 - n_1^2)E_1; \\ \frac{(1.24 \times 10^3 \text{ eV} \cdot \text{nm})}{\lambda} &= (2^2 - 1^2)(8.0 \text{ eV}), \end{aligned}$$

which gives $\lambda = 52$ nm.

(b) We find the width of the well from the ground state energy:

$$\begin{aligned} E_1 &= \frac{\pi^2\hbar^2}{2mL^2}; \\ (8.0 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV}) &= \frac{\pi^2(1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(9.11 \times 10^{-31} \text{ kg})L^2}, \end{aligned}$$

which gives $L = 2.2 \times 10^{-10}$ m $= 0.22$ nm.

3–20. The longest-wavelength line in the spectrum emitted by an electron trapped in an infinitely deep square well is 690 nm. What is the width of the well?

SOLUTION. The longest wavelength photon has the least energy, so the transition must be from $n_2 = 2$ to $n_1 = 1$. Thus we have

$$\begin{aligned} 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} &= \Delta E = \frac{(n_2^2 - n_1^2)\pi^2\hbar^2}{2mL^2}; \\ \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{690 \text{ nm}} & \\ \times (1.60 \times 10^{-19} \text{ J/eV}) &= \frac{(2^2 - 1^2)\pi^2(1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(9.11 \times 10^{-31} \text{ kg})L^2}, \end{aligned}$$

which gives $L = 7.9 \times 10^{-10}$ m $= 0.79$ nm.

3–21. For a particle in a box with rigid walls, determine whether our results for the ground state are consistent with the uncertainty principle by calculating the product $\Delta p\,\Delta x$. Take $\Delta x \approx L$, since the particle is known to be at least within the box. For Δp, note that although p is known ($= \hbar k$), the direction of $\mathbf{p}$ is not known, so the x component could vary from $-p$ to $+p$; hence take $\Delta p \approx 2p$.

SOLUTION. The energy level for the ground state of a particle in a rigid box is

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2} = \frac{p_1^2}{2m}.$$

Thus the momentum is

$$p_1 = \frac{\pi\hbar}{L}.$$

Because the direction is not known, we have $\Delta p \approx 2p_1 = 2\pi\hbar/L$. Because the particle can be anywhere in the box, we have $\Delta x \approx L$.

Thus we get

$$\Delta p\,\Delta x \approx \left(\frac{2\pi\hbar}{L}\right) L = 2\pi\hbar,$$

which is consistent with the uncertainty principle.

3–22. Write a formula for the positions of (a) the maxima and (b) the minima in $|\psi|^2$ for a particle in the nth state in an infinite square well.

SOLUTION. The wavefunctions for the rigid box are

$$\psi_n = A \sin k_n x,$$

with $k_n = n\pi/L$. Thus we have

$$|\psi_n|^2 = A^2 \left(\sin \frac{n\pi x}{L}\right)^2.$$

(a) The maximal value of $|\psi_n|^2$ is A^2, which occurs n times. We find the positions of the maxima from

$$\begin{aligned} \sin \frac{n\pi x_{\max}}{L} &= \pm 1; \\ \frac{n\pi x_{\max}}{L} &= \frac{\pi}{2}, \frac{3\pi}{2}, \ \ldots \end{aligned}$$

or

$$x_{\max} = \frac{L}{n}\left(m - \frac{1}{2}\right), \qquad m = 1,\ 2,\ \ldots\ ,\ n.$$

(b) The minimal value of $|\psi_n|^2$ is 0, which occurs $n+1$ times. We find the positions of the minima from

$$\begin{aligned} \sin \frac{n\pi x_{\min}}{L} &= 0; \\ \frac{n\pi x_{\min}}{L} &= 0,\ \pi,\ 2\pi,\ \ldots\ ; \end{aligned}$$

or

$$x_{\min} = \left(\frac{L}{n}\right) m, \qquad m = 0,\ 1,\ 2,\ \ldots\ ,\ n.$$

3–23. Determine the lowest four energy levels and wave functions for an electron trapped in an infinitely deep potential well of width 2.0 nm.

SOLUTION. The energy levels for the electron in an infinite potential well are

$$\begin{aligned} E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2} \\ &= \frac{n^2\pi^2(1.055 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{2(9.11 \times 10^{-31}\ \text{kg})(2.0 \times 10^{-9}\ \text{m})^2(1.60 \times 10^{-19}\ \text{J/eV})} \\ &= (9.42 \times 10^{-2}\ \text{eV})n^2. \end{aligned}$$

The wave functions are

$$\begin{aligned}\psi_n &= \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} = \sqrt{\frac{2}{2.0\text{ nm}}}\sin\left(\frac{n\pi x}{2.0\text{ nm}}\right)\\ &= \left(1.00\text{ nm}^{-1/2}\right)\sin\left(1.57\text{ nm}^{-1}\,nx\right).\end{aligned}$$

Thus we have

$$\begin{aligned}E_1 &= (9.42\times10^{-2}\text{ eV})(1)^2 = 0.094\text{ eV};\\ \psi_1 &= \left(1.00\text{ nm}^{-1/2}\right)\sin\left(1.57\text{ nm}^{-1}\,x\right);\\ E_2 &= (9.42\times10^{-2}\text{ eV})(2)^2 = 0.38\text{ eV};\\ \psi_2 &= \left(1.00\text{ nm}^{-1/2}\right)\sin\left(3.14\text{ nm}^{-1}\,x\right);\\ E_3 &= (9.42\times10^{-2}\text{ eV})(3)^2 = 0.85\text{ eV};\\ \psi_3 &= \left(1.00\text{ nm}^{-1/2}\right)\sin\left(4.71\text{ nm}^{-1}\,x\right);\\ E_4 &= (9.42\times10^{-2}\text{ eV})(4)^2 = 1.51\text{ eV};\\ \psi_4 &= \left(1.00\text{ nm}^{-1/2}\right)\sin\left(6.28\text{ nm}^{-1}\,x\right).\end{aligned}$$

3–24. An electron is trapped in a rigid box 0.50 nm wide, (a) Determine the energies and wave functions for the four lowest states, (b) Determine the wavelengths of photons emitted for all possible transitions between these states.

SOLUTION. (a) The energy levels for the electron in a rigid box are

$$\begin{aligned}E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2}\\ &= \frac{n^2\pi^2(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{2(9.11\times10^{-31}\text{ kg})(0.50\times10^{-9}\text{ m})^2(1.60\times10^{-19}\text{ J/eV})}\\ &= (1.51\text{ eV})n^2.\end{aligned}$$

The wave functions are

$$\begin{aligned}\psi_n &= \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} = \sqrt{\frac{2}{0.50\text{ nm}}}\sin\frac{n\pi x}{0.50\text{ nm}}\\ &= \left(2.0\text{ nm}^{-1/2}\right)\sin\left(2.0\text{ nm}^{-1}\,n\pi x\right).\end{aligned}$$

Thus we have

$$\begin{aligned}
E_1 &= (1.51\text{ eV})(1)^2 = 1.51\text{ eV};\\
\psi_1 &= \left(2.0\text{ nm}^{-1/2}\right)\sin\left(2.0\text{ nm}^{-1}\,\pi x\right);\\
E_2 &= (1.51\text{ eV})(2)^2 = 6.03\text{ eV};\\
\psi_2 &= \left(2.0\text{ nm}^{-1/2}\right)\sin\left(4.0\text{ nm}^{-1}\,\pi x\right);\\
E_3 &= (1.51\text{ eV})(3)^2 = 13.6\text{ eV};\\
\psi_3 &= \left(2.0\text{ nm}^{-1/2}\right)\sin\left(6.0\text{ nm}^{-1}\,\pi x\right);\\
E_4 &= (1.51\text{ eV})(4)^2 = 24.1\text{ eV};\\
\psi_4 &= \left(2.0\text{ nm}^{-1/2}\right)\sin\left(8.0\text{ nm}^{-1}\,\pi x\right).
\end{aligned}$$

(b) The wavelength of a photon is determined by the energy change:

$$\lambda = \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{\Delta E}.$$

For all possible transitions we have

$$\begin{aligned}
\lambda_{2\to1} &= \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{6.03\text{ eV}-1.51\text{ eV}} = 274\text{ nm};\\
\lambda_{3\to1} &= \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{13.6\text{ eV}-1.51\text{ eV}} = 103\text{ nm};\\
\lambda_{3\to2} &= \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{13.6\text{ eV}-6.03\text{ eV}} = 164\text{ nm};\\
\lambda_{4\to1} &= \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{24.1\text{ eV}-1.51\text{ eV}} = 55\text{ nm};\\
\lambda_{4\to2} &= \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{24.1\text{ eV}-6.03\text{ eV}} = 69\text{ nm};\\
\lambda_{4\to3} &= \frac{1.24\times10^3\text{ eV}\cdot\text{nm}}{24.1\text{ eV}-13.6\text{ eV}} = 118\text{ nm}.
\end{aligned}$$

3–25. Consider an atomic nucleus to be a rigid box of width 10^{-14} m. What would be the ground-state energy for (a) an electron, (b) a neutron, and (c) a proton?

SOLUTION. If we consider a one-dimensional system, the energy level of the ground state is

$$E = \frac{\pi^2\hbar^2}{2mL^2}.$$

(a) For an electron we have

$$\begin{aligned} E_{1e} &= \frac{\pi^2\hbar^2}{2m_eL^2} = \frac{\pi^2(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(9.11\times10^{-31}\ \text{kg})(10^{-14}\ \text{m})^2(1.60\times10^{-19}\ \text{J/eV})} \\ &= 3.8\times10^9\ \text{eV} = 4\ \text{GeV}. \end{aligned}$$

(b) For a proton we have

$$\begin{aligned} E_{1p} &= \frac{\pi^2\hbar^2}{2m_pL^2} = \frac{\pi^2(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(1.67\times10^{-27}\ \text{kg})(10^{-14}\ \text{m})^2(1.60\times10^{-19}\ \text{J/eV})} \\ &= 2.1\times10^6\ \text{eV} = 2\ \text{MeV}. \end{aligned}$$

(c) For a neutron we have

$$\begin{aligned} E_{1n} &= \frac{\pi^2\hbar^2}{2m_nL^2} = \frac{\pi^2(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(1.67\times10^{-27}\ \text{kg})(10^{-14}\ \text{m})^2(1.60\times10^{-19}\ \text{J/eV})} \\ &= 2.1\times10^6\ \text{eV} = 2\ \text{MeV}. \end{aligned}$$

3–26. Suppose that the walls of an infinite square well are at $x = -L/2$ and $x = L/2$. What will be the wave functions for the four lowest energy levels? Sketch the wave functions and probability densities.

SOLUTION. Because the origin of the x-axis is at the center of the well, the wavefunctions will alternate between sines and cosines (Fig. 3.5):

$$\psi_1 = \sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}; \quad \psi_2 = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L};$$
$$\psi_3 = \sqrt{\frac{2}{L}}\cos\frac{3\pi x}{L}; \quad \psi_4 = \sqrt{\frac{2}{L}}\sin\frac{4\pi x}{L}.$$

3–27. An electron is trapped in an infinitely deep potential well of width L. Determine the probability of finding the electron within $\frac{1}{4}L$ of either wall if it is (a) in the ground state, (b) in the $n = 4$ state. [*Hint*: Evaluate $\int_0^{L/4}|\psi|^2\,dx + \int_{3L/4}^{L}|\psi|^2\,dx$.] (c) What is the classical prediction?

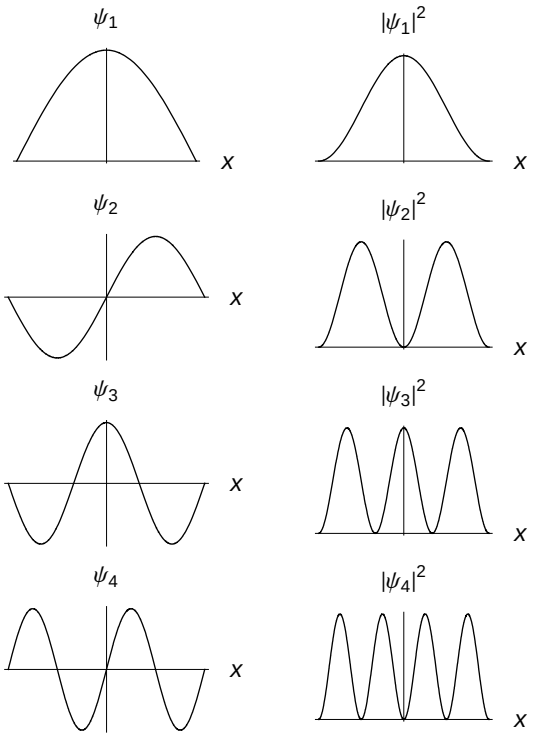


Figure 3.5. Problem 3–26.

SOLUTION. (a) Because the wavefunction is normalized, the probability is

$$\begin{aligned} P &= \int_0^{L/4} |\psi_1|^2 \, dx + \int_{3L/4}^{L} |\psi_1|^2 \, dx \\ &= \frac{2}{L} \int_0^{L/4} \left(\sin \frac{\pi x}{L} \right)^2 dx + \frac{2}{L} \int_{3L/4}^{L} \left(\sin \frac{\pi x}{L} \right)^2 dx. \end{aligned}$$

If we change variable to $\theta = \pi x/L$, so $d\theta = (\pi/L)\,dx$, we have

$$\begin{aligned} P &= \frac{2}{L}\left(\frac{L}{\pi}\right)\int_0^{\pi/4} (\sin\theta)^2\, d\theta + \frac{2}{L}\left(\frac{L}{\pi}\right)\int_{3\pi/4}^{\pi} (\sin\theta)^2\, d\theta \\ &= \frac{2}{\pi}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/4} + \frac{2}{\pi}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Big|_{3\pi/4}^{\pi} \\ &= \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{1}{2}\right) = 0.18. \end{aligned}$$

(b) For the $n = 4$ state the probability is

$$\begin{aligned} P &= \int_0^{L/4} |\psi_4|^2\, dx + \int_{3L/4}^{L} |\psi_4|^2\, dx \\ &= \frac{2}{L}\int_0^{L/4} \left(\sin\frac{4\pi x}{L}\right)^2 dx + \frac{2}{L}\int_{3L/4}^{L} \left(\sin\frac{4\pi x}{L}\right)^2 dx. \end{aligned}$$

If we change variable to $\theta = 4\pi x/L$, so $d\theta = (4\pi/L)\,dx$, we have

$$\begin{aligned} P &= \frac{2}{L}\left(\frac{L}{4\pi}\right)\int_0^{\pi} \sin^2\theta\, d\theta + \frac{2}{L}\left(\frac{L}{4\pi}\right)\int_{3\pi}^{4\pi} \sin^2\theta\, d\theta \\ &= \frac{2}{4\pi}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi} + \frac{2}{4\pi}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Big|_{3\pi}^{4\pi} \\ &= \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 0.50. \end{aligned}$$

c) Classically the electron has equal probability of being anywhere in the well. Thus the classical prediction is

$$\frac{(\frac{1}{4}L - 0) + (L - \frac{3}{4}L)}{L} = 0.50.$$

We see that the probability approaches the classical value for large n.

3–28. An electron is trapped in a 1.00-nm-wide rigid box. Determine the probability of finding the electron within 0.10 nm of the center of the box (on either side of center) for (a) $n = 1$, (b) $n = 5$, and (c) $n = 20$. (d) Compare to the classical prediction.

SOLUTION. Because the wavefunction is normalized, the probability is

$$P = \int_{x_1}^{x_2} |\psi_n|^2 \, dx = \frac{2}{L} \int_{x_1}^{x_2} \sin^2 \frac{n\pi x}{L} \, dx.$$

If we change variable to $\theta = n\pi x/L$, so $d\theta = (n\pi/L)\, dx$, we have

$$P = \frac{2}{L} \left(\frac{L}{n\pi} \right) \int_{0.40\, n\pi}^{0.60\, n\pi} \sin^2 \theta \, d\theta = \frac{2}{n\pi} \left(\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right) \Bigg|_{0.40\, n\pi}^{0.60\, n\pi}.$$

(a) For the $n = 1$ state we have

$$P = \frac{2}{\pi} \left(\frac{1}{2}(0.60\pi - 0.40\pi) - \frac{1}{4}(\sin 1.20\pi - \sin 0.80\pi) \right) = 0.39.$$

(b) For the $n = 5$ state we have

$$\begin{aligned} P &= \frac{2}{5\pi} \left(\frac{1}{2}(0.60(5)\pi - 0.40(5)\pi) - \frac{1}{4}(\sin 1.20(5)\pi - \sin 0.80(5)\pi) \right) \\ &= 0.20. \end{aligned}$$

(c) For the $n = 20$ state we have

$$\begin{aligned} P &= \frac{2}{20\pi} \left(\frac{1}{2}(0.60(20)\pi - 0.40(20)\pi) \right. \\ &\qquad \left. - \frac{1}{4}(\sin 1.20(20)\pi - \sin 0.80(20)\pi) \right) \\ &= 0.20. \end{aligned}$$

(d) Classically the electron has equal probability of being anywhere in the well, so the probability is

$$P = \frac{\Delta x}{L} = \frac{0.20 \text{ nm}}{1.00 \text{ nm}} = 0.20.$$

We see that the probabilities approach the classical value for large n.

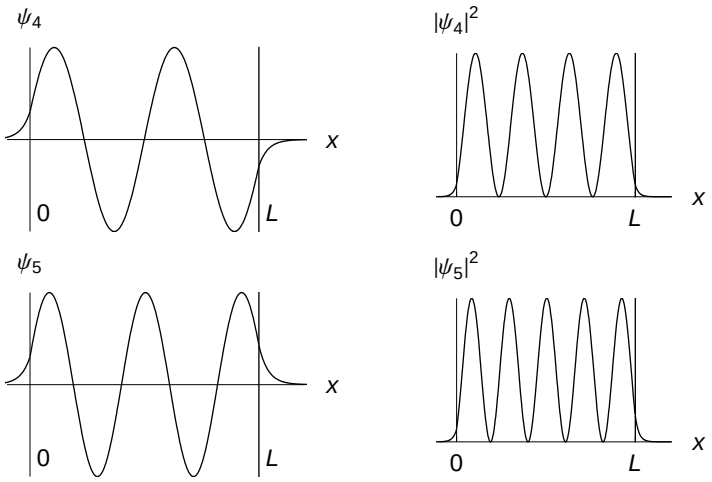


Figure 3.6. Problem 3–29.

3.7 Finite potential well

3–29. Sketch the wave functions and the probability distributions for the $n = 4$ and $n = 5$ states for a particle trapped in a finite square well.

SOLUTION. The wave functions and the probability distributions are shown on Fig. 3.6.

3–30. An electron with 100 eV of kinetic energy in free space passes over a finite potential well 50 eV deep that stretches from $x = 0$ to $x = 0.50$ nm. What is the electron's wavelength (a) in free space, (b) when over the well? (c) Draw a diagram showing the potential energy and total energy as a function of x, and on the diagram sketch a possible wave function.

SOLUTION. We choose the zero of potential energy at the bottom of the well. In free space outside the well the potential energy is $U_0 = 50$ eV. We find the total energy from

$$\begin{aligned} K &= E - U_0; \\ 100 \text{ eV} &= E - 50 \text{ eV}, \end{aligned}$$

which gives $E = 150$ eV.

(a) In free space we have

$$\begin{aligned} K_{\mathrm{a}} = \frac{p_{\mathrm{a}}^2}{2m} &= \frac{2\pi^2\hbar^2}{m\lambda_{\mathrm{a}}^2}; \\ (100\ \mathrm{eV})(1.60\times10^{-19}\ \mathrm{J/eV}) &= \frac{2\pi^2(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}{(9.11\times10^{-31}\ \mathrm{kg})\lambda_{\mathrm{a}}^2}, \end{aligned}$$

which gives $\lambda_{\mathrm{a}} = 1.2\times10^{-10}\ \mathrm{m} = 0.12\ \mathrm{nm}$.

(b) Over the well we have

$$\begin{aligned} K_{\mathrm{b}} = \frac{p_{\mathrm{b}}^2}{2m} &= \frac{2\pi^2\hbar^2}{m\lambda_{\mathrm{b}}^2}; \\ (150\ \mathrm{eV})(1.60\times10^{-19}\ \mathrm{J/eV}) &= \frac{2\pi^2(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}{(9.11\times10^{-31}\ \mathrm{kg})\lambda_{\mathrm{b}}^2}, \end{aligned}$$

which gives $\lambda_{\mathrm{b}} = 1.0\times10^{-10}\ \mathrm{m} = 0.10\ \mathrm{nm}$.

(c) See Fig. 3.7.

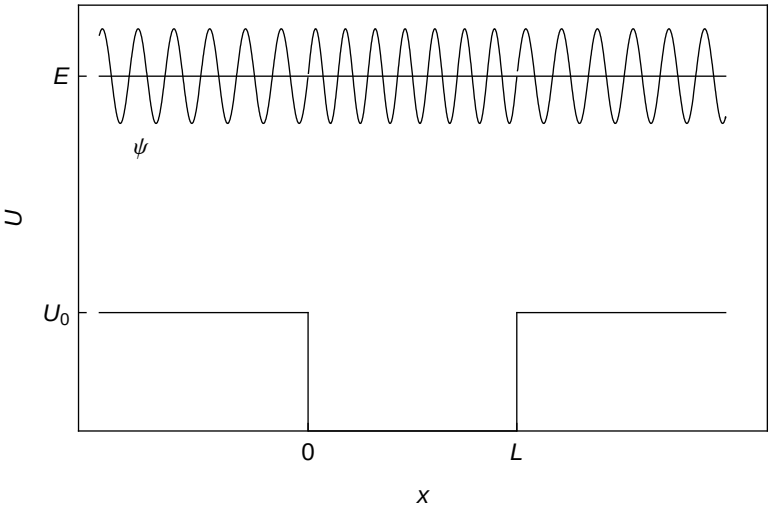


Figure 3.7. Problem 3–30.

3–31. An electron is trapped in a 0.10-nm-wide finite square well of height $U_0 = 1.0$ keV. Estimate at what distance outside the walls of the well the ground state wave function drops to 1 percent of its value at the walls.

SOLUTION. The wavefunction outside the well (negative x) is

$$\psi_{\mathrm{I}} = Ce^{Gx},$$

where

$$G^2 = \frac{2m(U_0 - E)}{\hbar^2}.$$

We approximate the energy as that of the ground state of an infinite well:

$$\begin{aligned} E &\approx \frac{\pi^2\hbar^2}{2mL^2} \\ &= \frac{\pi^2(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}{2(9.11\times10^{-31}\ \mathrm{kg})(0.10\times10^{-9}\ \mathrm{m})^2(1.60\times10^{-19}\ \mathrm{J/eV})} \\ &= 38\ \mathrm{eV}. \end{aligned}$$

Because this is much less than U_0, this should be a good approximation.

We find G from

$$G^2 = \frac{2(9.11\times10^{-31}\ \mathrm{kg})(1.00\times10^{3}\ \mathrm{eV} - 38\ \mathrm{eV})(1.60\times10^{-19}\ \mathrm{J/eV})}{(1.055\times10^{-34}\ \mathrm{J\cdot s})^2},$$

which gives $G = 1.59\times10^{11}\ \mathrm{m}^{-1}$.

Because of the continuity of the wavefunction, the value at the wall ($x = 0$) is $\psi_{\mathrm{wall}} = C$. Thus we have

$$\begin{aligned} \frac{\psi_{\mathrm{I}}}{\psi_{\mathrm{wall}}} &= e^{Gx}; \\ 0.01 &= e^{Gx}, \end{aligned}$$

or

$$\ln(0.01) = (1.59\times10^{11}\ \mathrm{m}^{-1})x,$$

which gives $x = -2.9\times10^{-11}$ m, so $|x| = 0.03$ nm.

3–32. Suppose that a particle of mass m is trapped in a finite potential well that has a rigid wall at $x = 0$ ($U = \infty$ for $x < 0$) and a finite wall

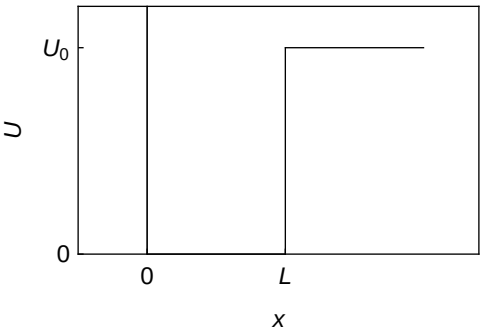


Figure 3.8. Problem 3–32.

of height $U = U_0$ at $x = L$, Fig. 3.8. (a) Sketch the wave functions for the lowest three states. (b) What is the form of the wave function in the ground state in the three regions $x < 0$, $0 < x < L$, $x > L$?

Solution. (a) We assume the lowest 3 states are bound in the well. The corresponding wavefunctions are sketched in Fig. 3.9.

(b) In region I, $x < 0$,

$$\psi_{\mathrm{I}} = 0;$$

In the well, region II, $0 \leqslant x \leqslant L$,

$$\psi_{\mathrm{II}} = A \sin kx,$$

where

$$k^2 = \frac{2mE}{\hbar^2};$$

In region III, $x > L$,

$$\psi_{\mathrm{III}} = Be^{-Gx},$$

where

$$G^2 = \frac{2m(U_0 - E)}{\hbar^2}.$$

3.8 Tunneling through a barrier

3–33. An electron approaches a potential barrier 10 eV high and 0.50 nm wide. If the electron has a 1.0 percent probability of tunneling through the barrier, what must its energy be?

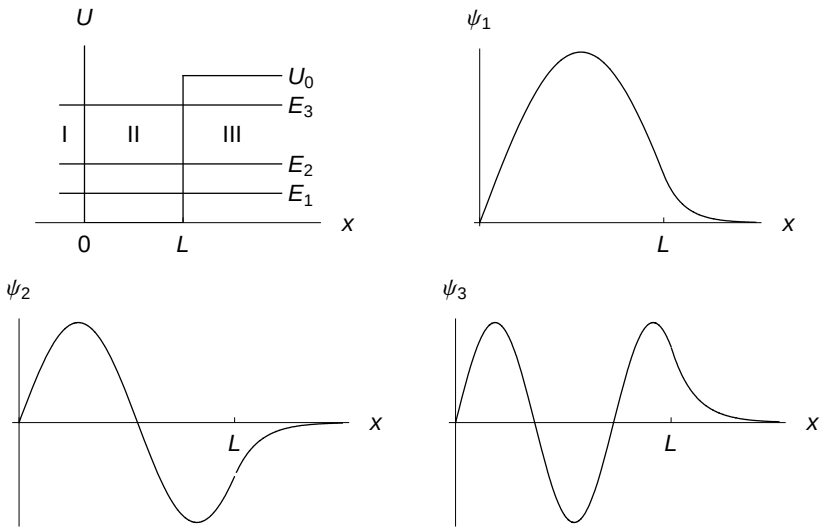


Figure 3.9. Problem 3–32.

SOLUTION. From the transmission we can find the value of G:

$$T = \frac{16E(U_0 - E)}{U_0^2}\, e^{-2GL},$$

or

$$\begin{aligned} \ln T &= \ln\left(\frac{16E(U_0 - E)}{U_0^2}\right) - 2GL; \\ \frac{2\sqrt{2m(U_0 - E)}\, L}{\hbar} &= \ln\left(\frac{16E(U_0 - E)}{U_0^2 T}\right). \end{aligned}$$

One way to solve this equation is to plot each side against E (Fig. 3.10). This yields $E = 9.1$ eV.

3–34. A 1.0-mA current of 1.0-MeV protons strikes a 2.0-MeV-high potential barrier 2.0×10^{-13} m thick. Estimate the transmitted current.

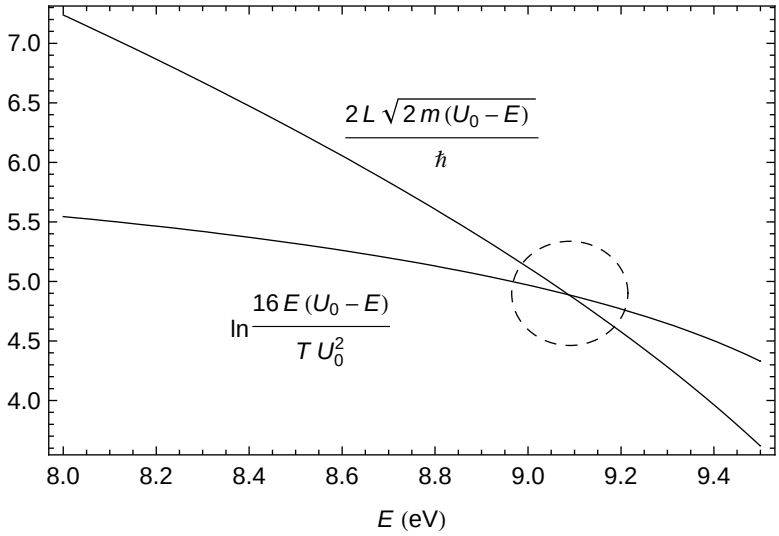


Figure 3.10. Problem 3–33.

SOLUTION. We find the value of G from

$$\begin{aligned} G^2 &= \frac{2m(U_0 - E)}{\hbar^2} \\ &= \frac{2(1.67 \times 10^{-27}\ \text{kg})(2.0\ \text{MeV} - 1.0\ \text{MeV})}{(1.055 \times 10^{-34}\ \text{J} \cdot \text{s})^2} \\ &\quad \times (1.60 \times 10^{-13}\ \text{J/MeV}), \end{aligned}$$

which gives $G = 2.19 \times 10^{14}\ \text{m}^{-1}$.

For the transmission coefficient we have

$$T = \frac{16E(U_0 - E)}{U_0^2}\, e^{-2GL},$$

or

$$\begin{aligned} \ln T &= -2GL + \ln\left(\frac{16E(U_0 - E)}{U_0^2}\right); \\ \ln T &= -(2)(2.19 \times 10^{14}\ \text{m}^{-1})(2.0 \times 10^{-13}\ \text{m}) + 1.39, \end{aligned}$$

which gives $T = 3.4 \times 10^{-38}$.

Thus the transmitted current is

$$I = I_0 T = (1.0 \times 10^{-3}\ \text{A})(3.4 \times 10^{-38}) = 3.4 \times 10^{-41}\ \text{A}.$$

3–35. For part (b) of Example 3–7, what effect will there be on the transmission coefficient if (a) the barrier height is raised 1 percent, (b) the barrier thickness is increased by 1 percent?

SOLUTION. (a) If the relative change in U_0 is dU_0/U_0 then the corresponding relative change in T is

$$\frac{dT}{T} = \frac{U_0}{T}\frac{dT}{dU_0}\frac{dU_0}{U_0}.$$

So if U_0 is changed by 1%, then T will change by

$$\frac{dT}{T} = \frac{U_0}{T}\frac{dT}{dU_0}$$

percent. With the help of Eq. 3.10 we find

$$\begin{aligned}
\frac{dT}{T} &= \frac{-U_0\left(1 + (L/\hbar)\sqrt{2m(U_0 - E)}\right) + 2E}{U_0 - E} \\
&= \frac{-(70\ \text{eV})}{(70\ \text{eV} - 50\ \text{eV})}\left(1 + \frac{(0.1 \times 10^{-9}\ \text{m})}{(0.658 \times 10^{-15}\ \text{eV}\cdot\text{s})}\right. \\
&\quad \left.\times \sqrt{\frac{2(0.511 \times 10^6\ \text{eV})}{(3 \times 10^8\ \text{m/s})^2}(70\ \text{eV} - 50\ \text{eV})}\right) \\
&\quad + \frac{2(50\ \text{eV})}{(70\ \text{eV} - 50\ \text{eV})} \\
&= -6.5.
\end{aligned}$$

Thus T decreases by 6.5%.

(b) We find the small relative change in T from a small relative change in the barrier thickness L:

$$\frac{dT}{T} = \frac{L}{T}\frac{dT}{dL}\frac{dL}{L} = -\frac{2L\sqrt{2m(U_0 - E)}}{\hbar}\frac{dL}{L},$$

or

$$\begin{aligned}\frac{dT}{T} &= -\frac{2(0.1\times10^{-9}\text{ m})}{(0.658\times10^{-15}\text{ eV}\cdot\text{s})} \\ &\quad\times\sqrt{2\left(\frac{0.511\times10^{6}\text{ eV}}{(3\times10^{8}\text{ m/s})^2}\right)(70\text{ eV}-50\text{ eV})}\,(1) \\ &= -4.6.\end{aligned}$$

Thus T decreases by 4.6%.

3–36. Show that the transmission coefficient is given roughly by Eqs. 3.10 for a high or thick barrier, by calculating $|\psi(L)|^2/|\psi(0)|^2$. [*Hint*: Assume that ψ is a decaying exponential inside the barrier.]

SOLUTION. The wavefunction inside the barrier is

$$\psi = Ae^{-Gx} + Be^{+Gx}.$$

If the barrier is high or thick, $\psi(L) \approx 0$, so $B \approx 0$. Thus we have

$$\psi = \psi(0)e^{-Gx},$$

so

$$\frac{|\psi(L)|^2}{|\psi(0)|^2} = e^{-2Gx}.$$

3–37. A uranium-238 nucleus ($Z = 92$) lasts about 5×10^9 years before it decays by emission of an α particle ($Z = 2$, $M = 4M_{\text{proton}}$). (a) Assuming that the α particle is a point, and the nucleus is roughly 8 fm in radius, estimate the height of the Coulomb barrier (the peak in Fig. 3.11). (b) The α particle, when free, has kinetic energy ≈ 4 MeV. Estimate the width of the barrier. (c) Assuming that the square well has $U = 0$ inside (where $U = 0$ far from the nucleus), calculate the speed of the α particle and how often it hits the barrier, and from this (and Eq. 3.10) estimate its lifetime. [*Hint*: Replace the $1/r$ Coulomb barrier with an "averaged" rectangular barrier of width equal to $\frac{1}{3}$ that calculated in (b).]

SOLUTION. (a) We assume the potential energy of the α particle (charge $= 2e$) is produced by the electric field of the remaining nuclear

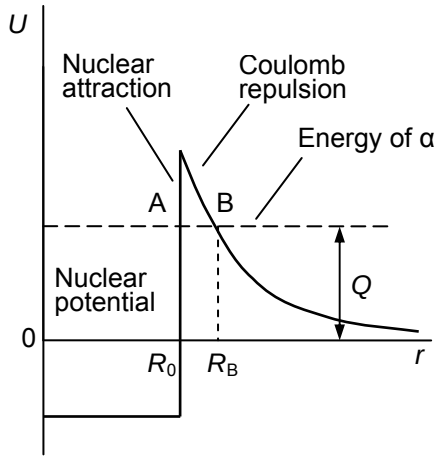


Figure 3.11. Potential energy for α particle and (daughter) nucleus, showing the Coulomb barrier through which the α particle must tunnel to escape (Problems 3–37, 6–34, 6–79 and 7–27).

charge of $90e$. Thus the potential energy at a radius of 8 fm is

$$\begin{aligned} U_{\mathrm{C}} &= k_{\mathrm{e}}\frac{qQ}{r_0} \\ &= \frac{(8.99\times10^{9}\ \mathrm{N\cdot m^2/C^2})(2)(90)(1.60\times10^{-19}\ \mathrm{C})^2}{(8\times10^{-15}\ \mathrm{m})(1.60\times10^{-19}\ \mathrm{J/eV})} \\ &= 3.24\times10^{7}\ \mathrm{eV} = 32.4\ \mathrm{MeV}. \end{aligned}$$

(b) The kinetic energy of the free α particle is also its total energy. At the exit from the barrier, the kinetic energy is zero, so we find the radius where the α particle leaves from

$$\begin{aligned} E &= U_r = k_{\mathrm{e}}\frac{qQ}{r} = k_{\mathrm{e}}\frac{qQ}{r_0}\frac{r_0}{r}; \\ 4\ \mathrm{MeV} &= (32.4\ \mathrm{MeV})\frac{(8\ \mathrm{fm})}{r}, \end{aligned}$$

which gives $r = 64.7$ fm. Thus the width of the barrier at an energy of

4 MeV is

$$64.7 \text{ fm} - 8 \text{ fm} = 56.7 \text{ fm}.$$

(c) If the potential energy is zero inside the barrier, we find the speed of the α particle from

$$\begin{aligned} E = K &= \tfrac{1}{2}mv^2; \\ (4 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV}) &= \tfrac{1}{2}(4)(1.67 \times 10^{-27} \text{ kg})v^2, \end{aligned}$$

which gives $v = 1.4 \times 10^7$ m/s.

Because there is a barrier on both sides of the nucleus, the frequency with which the α particle hits a barrier is

$$f = \frac{v}{2r_0} = \frac{1.4 \times 10^7 \text{ m/s}}{2(8 \times 10^{-15} \text{ m})} = 8.8 \times 10^{20} \text{ s}^{-1}.$$

To find the transmission coefficient, we assume a square barrier. We find the value of G from

$$\begin{aligned} G^2 &= \frac{2m(U_0 - E)}{\hbar^2} \\ &= \frac{2(4)(1.67 \times 10^{-27} \text{ kg})(32.4 \text{ MeV} - 4 \text{ MeV})}{(1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2} \\ &\quad \times (1.60 \times 10^{-13} \text{ J/MeV}), \end{aligned}$$

which gives $G = 2.34 \times 10^{15} \text{ m}^{-1} = 2.34 \text{ fm}^{-1}$.

If we take the height as the value of the Coulomb barrier, the thickness will be less than that found in part (b). The Coulomb potential varies as $1/r$, so the assumed thickness of a square barrier will be less than half that found in part (b). If we use $\frac{1}{3}$ of the width, $L = 18.9$ fm, we have

$$\begin{aligned} T &= \frac{16E(U_0 - E)}{U_0^2} e^{-2GL} \\ &= \frac{16(4 \text{ MeV})(32.4 \text{ MeV} - 4 \text{ MeV})}{(32.4 \text{ MeV})^2} \exp\left(-2(2.34 \text{ fm}^{-1})(18.9 \text{ fm})\right) \\ &= 7.9 \times 10^{-39}. \end{aligned}$$

Each time the α particle hits the barrier, T represents the probability it gets through. If the lifetime is τ, the number of hits is $f\tau$, which has a probability of one:

$$1 = f\tau T,$$

or

$$\tau = \frac{1}{fT} = \frac{1}{(8.8 \times 10^{20}\ \text{s}^{-1})(7.9 \times 10^{-39})} = 1.4 \times 10^{17}\ \text{s} \approx 4.5 \times 10^{9}\ \text{yr}.$$

Note that the result is very sensitive to the value of L.

3.9 The WKB approximation

3–38. The WKB approximation can be used to calculate the energy levels for a harmonic oscillator. In this approximation, the energy levels are the solutions to Eq. 3.12. (a) Determine the classical turning points for a harmonic oscillator with energy E and force constant k. (b) Carry out the integral in the WKB approximation and show that the energy levels in this approximation are $E_n = \hbar\omega$, where $\omega = \sqrt{k/m}$ and $n = 1, 2, 3, \ldots$ [*Hint*: recall that $\hbar = h/2\pi$. A useful standard untegral is $\int \sqrt{A^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{A^2 - x^2} + A^2 \arcsin\left(x/|A|\right)\right)$. Note that the integrand is even, so the integral from $-x$ to x is equal to twice the the integral from 0 to x.] (c) How do the approximate energy levels found in part (b) compare with the true energy levels given by Eq. 5.6? Does the WKB approximation give an underestimate or an overestimate of the energy levels?

SOLUTION. (a) At the turning points $E = \frac{1}{2}kx_{\text{TP}}^2$, so

$$x_{\text{TP}} = \pm\sqrt{\frac{2E}{k}}.$$

(b) We have

$$\int_{-\sqrt{2E/k}}^{+\sqrt{2E/k}} \sqrt{2m\left(E - \tfrac{1}{2}kx^2\right)}\, dx = n\pi\hbar.$$

To evaluate the integral, we want to get it into a form that matches the standard integral given. First we transform the integrand:

$$\sqrt{2m\left(E - \tfrac{1}{2}kx^2\right)} = \sqrt{2mE - mkx^2} = \sqrt{mk}\sqrt{\frac{2E}{k} - x^2}.$$

Then letting $A^2 = 2E/k$, $a = -\sqrt{2E/k}$ and $b = +\sqrt{2E/k}$ we get

$$\begin{aligned}\sqrt{mk}\int_a^b \sqrt{A^2 - x^2}\,dx &= 2\frac{\sqrt{mk}}{2}\left(x\sqrt{A^2 - x^2} + A^2 \arcsin\left(\frac{x}{|A|}\right)\right)\Bigg|_0^b \\ &= \sqrt{mk}\,\frac{2E}{k}\arcsin(1) = 2E\sqrt{\frac{m}{k}}\left(\frac{\pi}{2}\right).\end{aligned}$$

Using WKB, this must be equal to $n\pi\hbar$, so

$$E\sqrt{\frac{m}{k}} = n\hbar.$$

Recall $\omega = \sqrt{k/m}$, so $E = n\hbar\omega$.

(c) We are missing the zero-point-energy offset of $\hbar\omega/2$ [Eq. 5.6 suggests $E = \hbar\omega\left(n + \frac{1}{2}\right)$]. However, our approximation isn't bad at all!

3–39. Protons, neutrons, and many other particles are made of more fundamental particles called *quarks* and *antiquarks* (Chapter 8). A quark and antiquark can form a bound state with a variety of different energy levels, each of which correspond to a different particle observed in the laboratory. As an example, the ψ particle is low-energy bound state of a so-called charm quark and its antiquark, with rest energy of 3 097 MeV; the $\psi(2S)$ particle is an excited state of this same quark-antiquark combination, with a rest energy of 3 686 MeV. A simplified representation of the potential energy of interaction between a quark and an antiquark is $U(x) = A|x|$ where A is a positive constant and x represents the distance between the quark and the antiquark. You can use the WKB approximation to determine the bound-state energy levels for this potential energy function. In the WKB approximation, the energy levels are the solutions to Eq. 3.12. (a) Determine the classical turning points for the potential $U = A|x|$ and for an energy E. (b) Carry out the WKB integral and show that the allowed energy levels in the WKB approximation are given by

$$E_n = \frac{1}{2m}\left(\frac{3\pi mA\hbar}{2}\right)^{2/3} n^{2/3}, \qquad n = 1,\ 2,\ 3,\ \ldots \tag{3.13}$$

(Hint: the integrand is even, so the integral from $-x$ to x is equal to twice the integral from 0 to x.) (c) Does the difference in energy between

successive levels increase, decrease, or remain the same as n increases? How does this compare to the behavior of the energy levels for the harmonic oscillator? For the particle in a box? Can you suggest a simple rule that relates the difference in energy between successive levels to the shape of the potential energy function?

SOLUTION. (a) At the turning points $E = A|x_{\rm TP}|$, whence follows

$$x_{\rm TP} = \pm\frac{E}{A}.$$

(b) We have

$$\int_{-E/A}^{+E/A} \sqrt{2m\,(E - A|x|)}\,dx = 2\int_0^{E/A} \sqrt{2m\,(E - A|x|)}\,dx.$$

Introduce the new variable $y = 2m(E - Ax)$. Then $dy = -2mA\,dx$, and $y = 0$ at $x = E/A$ and $y = 2mE$ at $x = 0$. So,

$$\begin{aligned} n\pi\hbar &= 2\int_0^{E/A} \sqrt{2m\,(E - A|x|)}\,dx = -\frac{1}{mA}\int_{2mE}^{0} \sqrt{y}\,dy \\ &= -\frac{2}{3mA}\,y^{3/2}\Big|_{2mE}^{0} = \frac{2}{3mA}\,(2mE)^{3/2}. \end{aligned}$$

From here follows Eq. 3.13.

(c) The difference in energy decreases between successive levels. For example: $1^{2/3} - 0^{2/3} = 1$, $2^{2/3} - 1^{2/3} = 0.59\ldots$

- A sharp ∞ step gave ever-increasing level differences ($\sim n^2$).
- A parabola ($\sim x^2$) gave evenly spaced levels ($\sim n$).
- Now, a linear potential ($\sim x$) gives ever-decreasing level differences ($\sim n^{2/3}$).

Roughly: If the curvature of the potential ($\sim$ second derivative) is bigger than that of a parabola, then the level differences will increase. If the curvature is less than a parabola, the differences will decrease.

3.10 General problems

3–40. If an electron's position can be measured to an accuracy of 2.0×10^{-8} m, how accurately can its velocity be known?

SOLUTION. We find the uncertainty in the momentum:

$$\Delta p \geqslant \frac{\hbar}{2\,\Delta x} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(2.0 \times 10^{-8}\ \text{m})} = 2.64 \times 10^{-27}\ \text{kg} \cdot \text{m/s}.$$

We find the uncertainty in the velocity from

$$\begin{aligned} \Delta p &= m\,\Delta v; \\ 2.64 \times 10^{-27}\ \text{kg} \cdot \text{m/s} &= (9.11 \times 10^{-31}\ \text{kg})\,\Delta v, \end{aligned}$$

which gives $\Delta v = 2.9 \times 10^{3}$ m/s.

3–41. Estimate the lowest possible energy of a neutron contained in a typical nucleus of radius 1.0×10^{-15} m. [*Hint*: A particle can have an energy at least as large as its uncertainty.]

SOLUTION. We use the radius as the uncertainty in position for the neutron. We find the uncertainty in the momentum from

$$\Delta p \geqslant \frac{\hbar}{2\,\Delta x} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(1.0 \times 10^{-15}\ \text{m})} = 0.53 \times 10^{-19}\ \text{kg} \cdot \text{m/s}.$$

If we assume that the lowest value for the momentum is the least uncertainty, we estimate the lowest possible kinetic energy as

$$E = \frac{(\Delta p)^2}{2m} = \frac{(0.53 \times 10^{-19}\ \text{kg} \cdot \text{m/s})^2}{2(1.67 \times 10^{-27}\ \text{kg})} = 0.83 \times 10^{-12}\ \text{J} = 5.2\ \text{MeV}.$$

3–42. The Z^0 boson, discovered in 1985, is the mediator of the weak nuclear force, and it typically decays very quickly. Its average rest energy is 91.19 GeV, but its short lifetime shows up as an intrinsic width of 2.5 GeV. What is the lifetime of this particle?

SOLUTION. We find the average lifetime of the particle from

$$\Delta t \geqslant \frac{\hbar}{2\,\Delta E} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(2.5\ \text{GeV})(1.60 \times 10^{-10}\ \text{J/GeV})} = 1.3 \times 10^{-25}\ \text{s}.$$

3–43. What is the uncertainty in the mass of a muon ($m = 105.7$ MeV/c^2), specified in eV/c^2, given its average lifetime of 2.20 μs?

SOLUTION. We find the uncertainty in the rest energy of the muon from

$$\Delta E \geqslant \frac{\hbar}{2\,\Delta t} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(2.20 \times 10^{-6}\ \text{s})} = 2.4 \times 10^{-29}\ \text{J} = 1.5 \times 10^{-10}\ \text{eV}.$$

Thus the uncertainty in the mass is

$$\Delta m = \frac{\Delta E}{c^2} = 1.5 \times 10^{-10}\ \text{eV}/c^2.$$

3–44. A free neutron ($m = 1.67 \times 10^{-27}$ kg) has a mean life of 900 s. What is the uncertainty in its mass (in kg)?

SOLUTION. We find the uncertainty in the rest energy of the free neutron from

$$\Delta E \geqslant \frac{\hbar}{2\,\Delta t} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{(2)(900\ \text{s})} = 0.59 \times 10^{-37}\ \text{J}.$$

Thus the uncertainty in the mass is

$$\Delta m = \frac{\Delta E}{c^2} = \frac{0.59 \times 10^{-37}\ \text{J}}{(3.00 \times 10^{8}\ \text{m/s})^2} = 0.65 \times 10^{-54}\ \text{kg}.$$

3–45. Use the uncertainty principle to estimate the position uncertainty for the electron in the ground state of the hydrogen atom. [*Hint*: Determine the momentum using the Bohr model of Chapter 2 and assume the momentum can be anywhere between this value and zero.] How does this compare to the Bohr radius?

SOLUTION. We can relate the momentum to the radius of the orbit from the quantum condition:

$$L = mvr = pr = n\hbar,$$

so

$$p = \frac{n\hbar}{r} = \frac{\hbar}{r_{\text{B}}}$$

for the ground state.

If we assume that this is the uncertainty of the momentum, the uncertainty of the position is

$$\Delta x \geqslant \frac{\hbar}{2\,\Delta p} = \frac{\hbar}{2\,(\hbar/r_{\mathrm{B}})} = \frac{1}{2} r_{\mathrm{B}},$$

which is a half Bohr radius.

3–46. A neutron is trapped in an infinitely deep potential well 2.0 fm in width. Determine (a) the four lowest possible energy states and (b) their wave functions. (c) What is the wavelength and energy of a photon emitted when the neutron makes a transition between the two lowest states? In what region of the EM spectrum does this photon lie? [*Note*: This is a rough model of an atomic nucleus.]

Solution. (a) The energy levels for the neutron in an infinite well are

$$\begin{aligned} E_n &= \frac{n^2\pi^2 h^2}{2mL^2} \\ &= \frac{n^2\pi^2(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}{2(1.67\times10^{-27}\ \mathrm{kg})(2.0\times10^{-15}\ \mathrm{m})^2(1.60\times10^{-13}\ \mathrm{J/MeV})} \\ &= (51.4\ \mathrm{MeV})n^2. \end{aligned}$$

Thus we have

$$E_1 = (51.4\ \mathrm{MeV})(1)^2 = 51\ \mathrm{MeV}; \quad E_2 = (51.4\ \mathrm{MeV})(2)^2 = 210\ \mathrm{MeV};$$
$$E_3 = (51.4\ \mathrm{MeV})(3)^2 = 460\ \mathrm{MeV}; \quad E_4 = (51.4\ \mathrm{MeV})(4)^2 = 820\ \mathrm{MeV}.$$

(b) The wave functions are

$$\begin{aligned} \psi_n &= \sqrt{\frac{2}{L}}\,\sin\frac{n\pi x}{L} = \sqrt{\frac{2}{2.0\ \mathrm{fm}}}\,\sin\left(\frac{n\pi x}{2.0\ \mathrm{fm}}\right) \\ &= (1.0\ \mathrm{fm}^{-1/2})\sin(0.50\ \mathrm{fm}^{-1}\,n\pi x). \end{aligned}$$

Thus we have

$$\begin{aligned} \psi_1 &= (1.0\ \mathrm{fm}^{-1/2})\sin(0.50\ \mathrm{fm}^{-1}\,\pi x); \\ \psi_2 &= (1.0\ \mathrm{fm}^{-1/2})\sin(1.00\ \mathrm{fm}^{-1}\,\pi x); \\ \psi_3 &= (1.0\ \mathrm{fm}^{-1/2})\sin(1.50\ \mathrm{fm}^{-1}\,\pi x); \\ \psi_4 &= (1.0\ \mathrm{fm}^{-1/2})\sin(2.00\ \mathrm{fm}^{-1}\,\pi x). \end{aligned}$$

(c) The energy of the photon is the energy change:

$$2\pi\hbar f = \Delta E = E_2 - E_1 = 205 \text{ MeV} - 51 \text{ MeV} = 150 \text{ MeV}.$$

The wavelength of the photon is

$$\begin{aligned}\lambda_{2\to1} &= \frac{(1.24\times10^3 \text{ eV}\cdot\text{nm})(10^{-6} \text{ MeV/eV})(10^6 \text{ fm/nm})}{\Delta E} \\ &= \frac{1.24\times10^3 \text{ MeV}\cdot\text{fm}}{\Delta E} = \frac{1.24\times10^3 \text{ MeV}\cdot\text{fm}}{150 \text{ MeV}} = 8.3 \text{ fm}.\end{aligned}$$

This is gamma ray.

3–47. Estimate the kinetic energy and speed of an α particle ($q = 2e$, $m = 4$ proton masses) trapped in a nucleus 10^{-14} m wide. Assume an infinitely deep square well potential.

SOLUTION. If we assume the ground state, the kinetic energy is

$$\begin{aligned}K &= E_1 = \frac{n^2\pi^2\hbar^2}{2mL^2} \\ &= \frac{(1)^2\pi^2(1.055\times10^{-34} \text{ J}\cdot\text{s})^2}{2(4)(1.67\times10^{-27} \text{ kg})(10^{-14} \text{ m})^2(1.60\times10^{-19} \text{ J/eV})} \\ &= 5.1\times10^5 \text{ eV} = 0.5 \text{ MeV}.\end{aligned}$$

We find the speed from

$$\begin{aligned}K = \frac{p^2}{2m} &= \frac{mv^2}{2}; \\ (5.1\times10^5 \text{ eV})(1.60\times10^{-19} \text{ J/eV}) &= \frac{(4)(1.67\times10^{-27} \text{ kg})v^2}{2},\end{aligned}$$

which gives $v = 5\times10^6$ m/s.

3–48. Suppose that an electron is trapped not in a square well, but one whose potential energy is that of a simple harmonic oscillator: $U(x) = \frac{1}{2}Cx^2$. That is, if the electron is displaced from $x = 0$, a restoring force $F = -Cx$ acts on it, where C is constant. (a) Sketch this potential energy. (b) Show that $x = Ae^{-Bx^2}$ is a solution to the Schrödinger equation and that the energy of this state is $E = \frac{1}{2}\hbar\omega$, where $\omega = 2\hbar B/m$, and A and B are constants. [*Note*: This is the ground state and this energy $\frac{1}{2}\hbar\omega$ is the

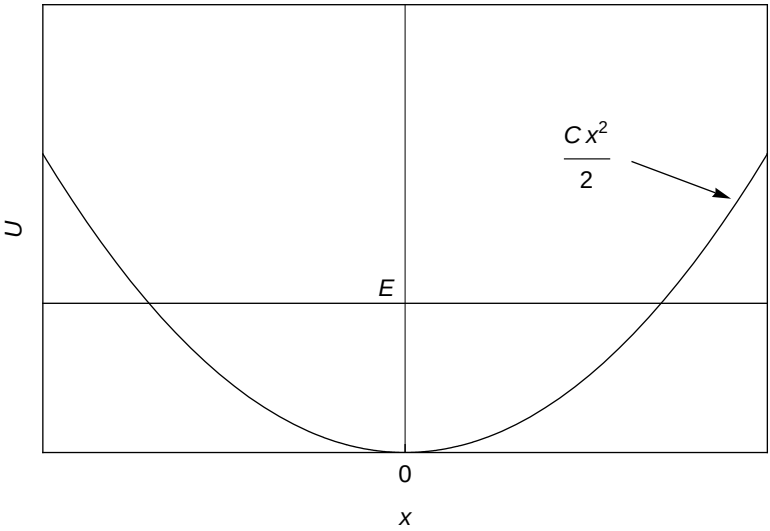


Figure 3.12. Problem 3–48.

zero-point energy for a harmonic oscillator. The energies of higher states are $E_n = (n + 1)\frac{1}{2}\hbar\omega$, where n is an integer.]

SOLUTION. (a) The potential energy is sketched in Fig. 3.12.

(b) The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}Cx^2\psi = E\psi.$$

For the proposed solution $\psi = Ae^{-Bx^2}$, we have

$$\frac{d\psi}{dx} = -2ABxe^{-Bx^2};$$

$$\frac{d^2\psi}{dx^2} = -2ABxe^{-Bx^2} + 4AB^2x^2e^{-Bx^2}.$$

If we substitute these in the equation, we get

$$-\frac{\hbar^2}{2m}\left(-2ABe^{-Bx^2} + 4AB^2x^2e^{-Bx^2}\right) + \frac{1}{2}Cx^2Ae^{-Bx^2} = EAe^{-Bx^2}.$$

When we cancel common factors, we get

$$\left(-\frac{4B^2\hbar^2}{2m}+\frac{1}{2}\,C\right)x^2+\left(\frac{B\hbar^2}{m}-E\right)=0$$

Thus we have a solution if

$$B^2=\frac{mC}{4\hbar^2},$$

and

$$E=\frac{B\hbar^2}{m}=\frac{\hbar\omega}{2},$$

if

$$\omega=\frac{2\hbar B}{m}.$$

3–49. A 10-gram pencil, 20 cm long, is balanced on its point. Classically, this is a configuration of (unstable) equilibrium, so the pencil could remain there forever if it were perfectly placed. A quantum mechanical analysis shows the pencil must fall. (a) Why is this the case? (b) Estimate (within a factor of 2) how long it will take the pencil to hit the table if it is initially positioned as well as possible? [*Hint*: Use the uncertainty principle in its angular form to obtain an expression for the initial angle $\varphi_0\approx\Delta\varphi$.]

Solution. (a) When the pencil, assumed to be a uniform rod, makes an angle φ with the vertical, the torque from its weight about the bottom point (assumed fixed) creates the angular acceleration:

$$\begin{aligned}\tau &= I\alpha;\\ Mg\frac{\ell}{2}\sin\varphi &= \frac{M\ell^2}{3}\,\frac{d^2\varphi}{dt^2}.\end{aligned}$$

Classically, if $\varphi=0$, there is no torque and thus no rotation. From the uncertainty principle we have

$$\Delta L\,\Delta\varphi\geqslant\frac{\hbar}{2},$$

or

$$\Delta\varphi\geqslant\frac{\hbar}{2I\,\Delta\omega}.$$

Thus φ will not always be zero. The resulting torque will cause the pencil to fall.

(b) We need the equation of motion to determine the time of fall. Most of the time will be taken while the angle is small. Thus while $\varphi \ll 1$, we have

$$\frac{d^2\varphi}{dt^2} = \frac{3}{2}\,\frac{g}{\ell}\,\varphi,$$

for which the solution is

$$\varphi = Ae^{+ct} + Be^{-ct}.$$

Because φ increases with time, we use

$$\varphi = Ae^{ct},$$

where

$$c = \sqrt{\frac{3}{2}\,\frac{g}{\ell}} = \sqrt{\frac{3(9.80\ \text{m/s}^2)}{2(0.20\ \text{m})}} = 8.57\ \text{s}^{-1}.$$

If we assume φ has the minimum value of $\Delta\varphi$ at $t = 0$, then $\Delta\varphi = A$.

From the uncertainty principle, the initial angular velocity will be

$$\omega_0 = cA = \Delta\omega = \frac{\hbar}{2I\,\Delta\varphi} = \frac{\hbar}{2\left(\frac{1}{3}M\ell^2\right)A},$$

which gives

$$A^2 = \frac{3\hbar}{2M\ell^2 c} = \frac{3(1.055\times10^{-34}\ \text{J}\cdot\text{s})}{(2)(10\times10^{-3}\ \text{kg})(0.20\ \text{m})^2(8.57\ \text{s}^{-1})} = 4.62\times10^{-32},$$

which gives $A = 2.15\times10^{-16}$ rad.

We find the time to fall through $\frac{1}{2}\,\pi$ rad from $\varphi = Ae^{ct}$, or

$$\begin{aligned} \ln\left(\frac{\varphi}{A}\right) &= ct; \\ \ln\left(\frac{\frac{1}{2}\,\pi\ \text{rad}}{3.0\times10^{-16}\ \text{rad}}\right) &= (8.57\ \text{s}^{-1})t, \end{aligned}$$

which gives $t = 4.3$ s. Because most of the time will be taken while the angle is small, this should be within a factor of 2.

3–50. The probability density for finding a particle at a particular location is $|\psi|^2$. The average value $\langle x\rangle$, the location of a particle averaged over many measurements, is given by

$$\langle x\rangle = \int_0^L x|\psi|^2\,dx$$

for a particle in a rigid box. Calculate the average value of x for any value of n.

SOLUTION. For the average value of the position of a particle in a rigid box, we have

$$\langle x\rangle = \int_0^L x\,|\psi_n|^2\,dx = \frac{2}{L}\int_0^L x\sin^2\frac{n\pi x}{L}\,dx.$$

If we change variable to $\theta = n\pi x/L$, so $d\theta = (n\pi/L)\,dx$, we have

$$\begin{aligned}\langle x\rangle &= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\int_0^{n\pi}\theta\sin^2\theta\,d\theta = \frac{2L}{n^2\pi^2}\left(\frac{\theta^2}{4} - \frac{\theta\sin 2\theta}{4} - \frac{\cos 2\theta}{8}\right)\Bigg|_0^{n\pi}\\ &= \frac{2L}{n^2\pi^2}\left(\frac{n^2\pi^2}{4} - 0 - \frac{1}{8} + \frac{1}{8}\right) = \frac{L}{2}.\end{aligned}$$

This is expected from the symmetry of the probability distributions.

3–51. By how much does the tunneling current through the tip of a scanning tunneling microscope change if the tip rises 0.010 nm from some initial height above a sodium surface with a work function $W_0 = 2.28$ eV? [*Hint*: Let the work function equal the energy needed to raise the electron to the top of the barrier.]

SOLUTION. The potential difference between the surface and the tip creates vacant energy levels on one side of the barrier so tunneling can occur. The change does not appreciably affect the shape of the barrier, so we assume a square barrier of height U_0. If the work function is the energy required to raise the electron to the top of the barrier, then $W_0 = U_0 - E$.

We find the value of G from

$$\begin{aligned}G^2 &= \frac{2m(U_0 - E)}{\hbar^2}\\ &= \frac{2(9.11\times10^{-31}\text{ kg})(2.28\text{ eV})(1.60\times10^{-19}\text{ J/eV})}{(1.055\times10^{-34}\text{ J}\cdot\text{s})^2},\end{aligned}$$

which gives $G = 7.7 \times 10^9\ \mathrm{m}^{-1}$.

We use the distance from the sodium surface to the tip as the width of the barrier, so the transmission coefficient is

$$T = \frac{16E(U_0 - E)}{U_0^2}\, e^{-2GL}.$$

If we find the ratio of the coefficient after the rise of the tip to that before the rise, we have

$$\begin{aligned} \frac{T_2}{T_1} &= \frac{e^{-2GL_2}}{e^{-2GL_1}} = e^{-2G(L_2 - L_1)} \\ &= \exp\left(-2(7.7 \times 10^{+9}\ \mathrm{m}^{-1})(0.010 \times 10^{-9}\ \mathrm{m})\right) = 0.86. \end{aligned}$$

The fractional change is

$$\frac{(T_2 - T_1)}{T_1} = 0.86 - 1 = -0.14.$$

Thus the small change in separation (about a tenth of atomic size) produces a 14% decrease in the tunneling current.

3–52. A small ball of mass 1.0×10^{-6} kg is dropped on a table from a height of 2.0 m. After each bounce the ball rises to 80 percent of its height before the bounce because of its inelastic collision with the table. Estimate how many bounces occur before the uncertainty principle plays a role in the problem. [*Hint*: Determine when the uncertainty in the ball's speed is comparable to its speed of impact on the table.]

Solution. From energy conservation, the speed after falling a height H or to rise to a height H is $v = \sqrt{2gH}$. If H_0 is the initial height and the height after the first bounce is $H_1 = bH_0$, the height after n bounces is $H_n = b^n H_0$. Thus the speed after the nth bounce is

$$v_n = \sqrt{2gb^n H_0}.$$

If we take the height after the nth bounce as the uncertainty in the position of the ball, we find the uncertainty in the speed from

$$\begin{aligned} \Delta p\, \Delta x &\geqslant \frac{\hbar}{2}; \\ m\, \Delta v H_n &\geqslant \frac{\hbar}{2}, \end{aligned}$$

or

$$\Delta v \geqslant \frac{\hbar}{2mH_n}.$$

The uncertainty principle will play a role when the uncertainty in the speed is on the order of the speed:

$$\Delta v \approx v_n = \sqrt{2gb^n H_0} \geqslant \frac{\hbar}{2mb^n H_0},$$

or

$$\begin{aligned} b^{3n} &= \frac{\hbar^2}{8gH_0^3 m^2}; \\ (0.80)^{3n} &= \frac{(1.055 \times 10^{-34}\ \mathrm{J \cdot s})^2}{8(9.80\ \mathrm{m/s^2})(2.0\ \mathrm{m})^3(1.0 \times 10^{-6}\ \mathrm{kg})^2}, \end{aligned}$$

which gives $n = 202$.

4 Quantum Mechanics of Atoms

4.1 Review

In the quantum mechanical view of the atom, the electrons do not have well-defined orbits, but instead exist as a "cloud." Electron clouds can be interpreted as an electron wave spread out in space, or as a *probability distribution* for electrons considered as particles.

For the simplest atom, hydrogen, the (time-independent) Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = E\psi \tag{4.1}$$

contains the potential energy

$$U = -k_{\mathrm{e}}\frac{e^2}{r},$$

where r is the radial distance from the proton to the electron. The solutions give the same values of energy as the old Bohr theory:

$$E_n = -\frac{(13.6\ \mathrm{eV})}{n^2} \quad n = 1,\ 2,\ 3,\ \ldots\ . \tag{4.2}$$

According to quantum mechanics, the state of an electron in an atom is specified by four *quantum numbers*: n, ℓ, m_ℓ, and m_s.

The *principal quantum number*, n, can take on any integer value (1, 2, 3, ...) and corresponds to the quantum number of the old Bohr theory.

The *orbital quantum number*, ℓ, can take on values from 0 up to $n-1$. The actual magnitude of the orbital angular momentum L is related to the quantum number ℓ by

$$L = \sqrt{\ell(\ell+1)}\,\hbar. \tag{4.3}$$

The energy levels in the hydrogen atom depend on n, whereas in other atoms they depend on n and ℓ.

The *magnetic quantum number*, m_ℓ, can take on integer values from $-\ell$ to $+\ell$. In quantum mechanics, the direction of the angular momentum

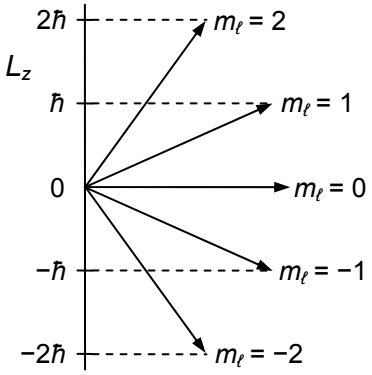


Figure 4.1. Quantization of angular momentum direction for $\ell = 2$.

is usually specified by giving its component along the (arbitrary) z axis. Then L_z is related to m_ℓ by the equation:

$$L_z = m_\ell \hbar. \tag{4.4}$$

This limitation on the direction of **L** is called *space quantization*. For example, if $\ell = 2$, then m_ℓ can be -2, -1, 0, $+1$, $+2$, and the five different directions allowed can be represented by the diagram of Fig. 4.1. The values of L_x and L_y are not definite, however.

When an external magnetic field is applied, the spectral lines are split (the *Zeeman effect*), indicating that the energy depends also on m_ℓ in this case (Fig. 4.2a).

The *spin quantum number*, m_s, can be $+\frac{1}{2}$ or $-\frac{1}{2}$.

Even in the absence of a magnetic field, precise measurements of spectral lines show a tiny splitting of the lines called *fine structure*, whose explanation is that the energy depends very slightly on m_ℓ and m_s.

Example 4–1. How many different states are possible for an electron whose principal quantum number is $n = 3$?

SOLUTION. For $n = 3$, ℓ can have the values $\ell = 2,\ 1,\ 0$. For $\ell = 2$, m_ℓ can be $2,\ 1,\ 0,\ -1,\ -2$, which is five different possibilities. For each of these, m_s can be either up or down ($+\frac{1}{2}$ or $-\frac{1}{2}$), so for $\ell = 2$ there are $2 \times 5 = 10$ states. For $\ell = 1$, m_ℓ can be $1,\ 0,\ -1$, and since m_s can be $+\frac{1}{2}$

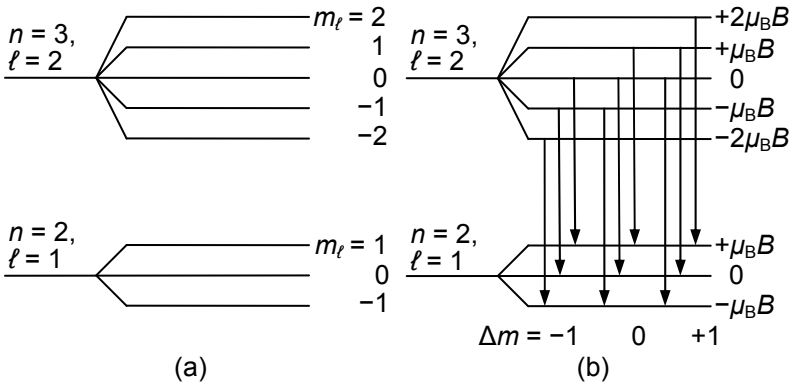


Figure 4.2. (a) The Zeeman effect. When a magnetic field is applied, an $n = 3$, $\ell = 2$ energy level is split into 5 separate levels, corresponding to 5 different values of m_ℓ. An $n = 2$, $\ell = 1$ level is split into 3 levels. (b) Problem 4–46.

or $-\frac{1}{2}$ for each of these, we have 6 more possible states. Finally, for $\ell = 0$, m_ℓ can only be 0, and there are only 2 states corresponding to $m_s = +\frac{1}{2}$ and $-\frac{1}{2}$. The total number of states is $10 + 6 + 2 = 18$, as detailed in the following table:

n	ℓ	m_ℓ	m_s	n	ℓ	m_ℓ	m_s
3	2	+2	$+\frac{1}{2}$	3	2	−2	$-\frac{1}{2}$
3	2	+2	$-\frac{1}{2}$	3	1	+1	$+\frac{1}{2}$
3	2	+1	$+\frac{1}{2}$	3	1	+1	$-\frac{1}{2}$
3	2	+1	$-\frac{1}{2}$	3	1	0	$+\frac{1}{2}$
3	2	0	$+\frac{1}{2}$	3	1	0	$-\frac{1}{2}$
3	2	0	$-\frac{1}{2}$	3	1	−1	$+\frac{1}{2}$
3	2	−1	$+\frac{1}{2}$	3	2	−1	$-\frac{1}{2}$
3	2	−1	$-\frac{1}{2}$	3	0	0	$+\frac{1}{2}$
3	2	−2	$+\frac{1}{2}$	3	0	0	$-\frac{1}{2}$

Example 4–2. Determine (a) the energy and (b) the orbital angular

momentum for each of the states in Example 4–1.

SOLUTION. (a) The energy of a state depends only on n, except for the very small corrections mentioned above, which we will ignore. Since $n = 3$ for all these states, they all have the same energy,

$$E_3 = -\frac{(13.6\text{ eV})}{3^2} = -1.51\text{ eV}.$$

(b) For $\ell = 0$,

$$L = \sqrt{\ell(\ell+1)}\,\hbar = \sqrt{0(0+1)}\,\hbar = 0.$$

For $\ell = 1$,

$$L = \sqrt{1(1+1)}\,\hbar = \sqrt{2}\,\hbar = 1.49 \times 10^{-34}\text{ J}\cdot\text{s}.$$

Atomic angular momenta are generally given as a multiple of $\hbar$ ($\sqrt{2}\,\hbar$ in this case), rather than in SI units. But note that for $\ell = 1$, L is on the order of 10^{-34} J · s. This means that macroscopic angular momenta will have such extremely high quantum numbers that the quantization of angular momentum will not be detectable: L will appear continuous, in accordance with the correspondence principle. Finally, for $\ell = 2$,

$$L = \sqrt{2(2+1)}\,\hbar = \sqrt{6}\,\hbar.$$

Transitions between states that obey the selection rule $\Delta\ell = \pm 1$ are far more probable than other so-called "forbidden" transitions.

The ground-state wave function in hydrogen has spherical symmetry, as do other $\ell = 0$ states:

$$\psi_{100} = \frac{1}{\sqrt{\pi r_{\text{B}}^3}}\, e^{-r/r_{\text{B}}}, \tag{4.5}$$

where $r_{\text{B}} = \hbar^2/k_{\text{e}}m_{\text{e}}e^2 = 0.0529$ nm is the Bohr radius. The subscript 100 on ψ represents the quantum numbers n, ℓ, m_ℓ: $\psi_{n\ell m_\ell}$.

The probability density, $|\psi|^2$, and the *radial probability density*,

$$P_{\text{r}} = 4\pi r^2 |\psi|^2,$$

are both useful to illustrate the spatial extent of the electron cloud. For the ground state of hydrogen, the probability density is

$$|\psi_{100}|^2 = \frac{1}{\pi r_{\mathrm{B}}^3}\, e^{-2r/r_{\mathrm{B}}}, \tag{4.6}$$

while the radial probability density becomes

$$P_{\mathrm{r}} = 4\frac{r^2}{r_{\mathrm{B}}^3}\, e^{-2r/r_{\mathrm{B}}} \tag{4.7}$$

and is plotted in Fig. 4.3.

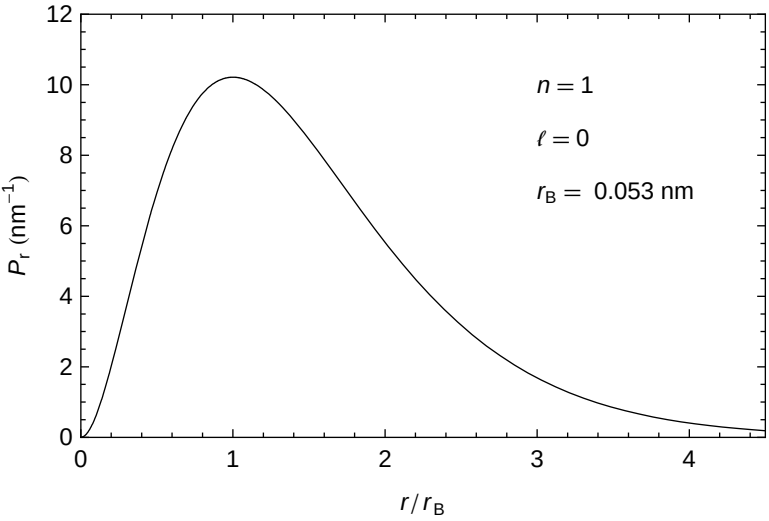


Figure 4.3. The radial probability distribution P_{r} for the ground state of hydrogen, $n = 1$, $\ell = 0$. The peak occurs at $r = r_{\mathrm{B}}$, the Bohr radius.

Example 4–3. Determine the most probable distance r from the nucleus at which to find the electron in the ground state of hydrogen.

Solution. The peak of the curve in Fig. 4.3 corresponds to the most probable value of r. At this point the curve has zero slope, so we take the

derivative of Eq. 4.7 and set it equal to zero:

$$\begin{aligned} \frac{d}{dr}\left(4\,\frac{r^2}{r_{\rm B}^3}\,e^{-2r/r_{\rm B}}\right) &= 0, \\ \left(\frac{8r}{r_{\rm B}^3}-\frac{8r^2}{r_{\rm B}^4}\right)e^{-2r/r_{\rm B}} &= 0. \end{aligned}$$

Since $e^{-2r/r_{\rm B}}$ goes to zero only at $r=\infty$, it is the term in parentheses that must be zero:

$$\frac{8r}{r_{\rm B}^3}-\frac{8r^2}{r_{\rm B}^4}=0.$$

Therefore,

$$r=r_{\rm B}.$$

The most probable radial distance of the electron from the nucleus according to quantum mechanics is at the Bohr radius, an interesting coincidence.

Example 4–4. Determine the probability of finding the electron in the ground state of hydrogen within two Bohr radii of the nucleus.

SOLUTION. We need to integrate $P_{\rm r}$, from $r=0$ out to $r=2r_{\rm B}$: that is, we want to find

$$P_{\rm r}=\int_{r=0}^{2r_{\rm B}}|\psi|^2\,dV=\int_{r=0}^{2r_{\rm B}}4\frac{r^2}{r_{\rm B}^3}\,e^{-2r/r_{\rm B}}\,dr.$$

We first make the substitution

$$x=2\frac{r}{r_{\rm B}}$$

and then integrate by parts ($\int u\,dv=uv-\int v\,du$) letting $u=x^2$ and $dv=e^{-x}\,dx$ (and note that $dx=2\,dr/r_{\rm B}$):

$$P_{\rm r}=\frac{1}{2}\int_0^4 x^2e^{-x}\,dx=\frac{1}{2}\left(-x^2e^{-x}+\int 2xe^{-x}\,dx\right)\Bigg|_0^4.$$

The second term we also integrate by parts with $u=2x$ and $dv=e^{-x}\,dx$:

$$\begin{aligned} P_{\rm r} &= \frac{1}{2}\left(-x^2e^{-x}-2xe^{-x}+2\int e^{-x}\,dx\right)\Bigg|_0^4 \\ &= \left(-\frac{1}{2}x^2-x-1\right)e^{-x}\Bigg|_0^4. \end{aligned}$$

We evaluate this at $x = 0$ and at $x = 2(2r_{\rm B})/r_{\rm B} = 4$:

$$P_{\rm r} = (-8 - 4 - 1)e^{-4} + e^{0} = 0.76$$

or 76 percent. Thus the electron would be found 76 percent of the time within 2 Bohr radii of the nucleus and 24 percent of the time farther away. Note that this result depends on our wave function being properly normalized, which it is, as is readily shown by letting $r \to \infty$ and integrating over all space: $\int_0^\infty |\psi|^2\, dV = 1$.

The first excited state in hydrogen has $n = 2$. For $\ell = 0$, the solution of the Schrödinger equation (Eq. 4.1) is a wave function that is again spherically symmetric:

$$\psi_{200} = \frac{1}{\sqrt{32\pi r_{\rm B}^3}} \left(2 - \frac{r}{r_{\rm B}}\right) e^{-r/2r_{\rm B}}, \tag{4.8}$$

Fig. 4.4 shows a plot of the radial probability distribution

$$P_{\rm r} = \frac{1}{8}\frac{r^2}{r_{\rm B}^3} \left(2 - \frac{r}{r_{\rm B}}\right)^2 e^{-r/2r_{\rm B}}.$$

States with $\ell > 0$ have some directionality in space. For the state with $n = 2$, $\ell = 1$, there are three possible wave functions corresponding to $m_\ell = +1, 0, -1$:

$$\begin{aligned} \psi_{210} &= \frac{z}{\sqrt{32\pi r_{\rm B}^5}}\, e^{-r/2r_{\rm B}}, \\ \psi_{211} &= \frac{x + iy}{\sqrt{64\pi r_{\rm B}^5}}\, e^{-r/2r_{\rm B}}, \\ \psi_{21-1} &= \frac{x - iy}{\sqrt{64\pi r_{\rm B}^5}}\, e^{-r/2r_{\rm B}}. \end{aligned} \tag{4.9}$$

These wave functions are not spherically symmetric. All three of these states are equally likely, and they all have the same energy.

The arrangement of electrons in multi-electron atoms is governed by the *Pauli exclusion principle*, which states that no two electrons can occupy the same quantum state—that is, they cannot have the same set of quantum numbers n, ℓ, m_ℓ, and m_s.

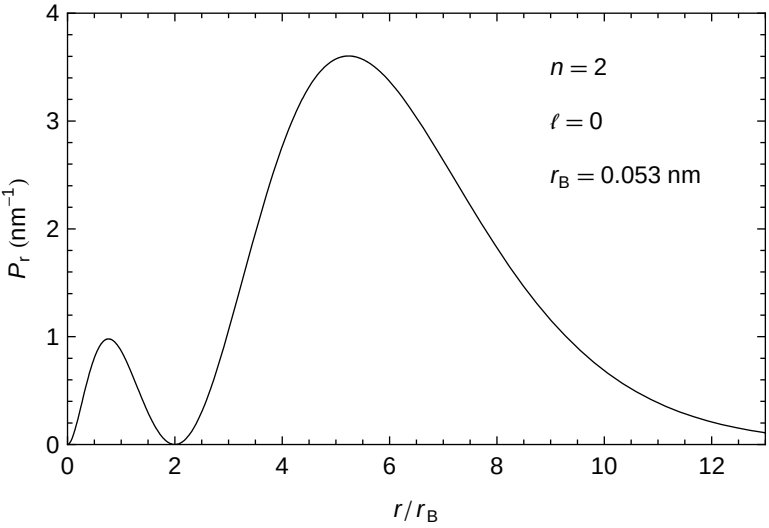


Figure 4.4. The radial probability distribution for the $n = 2$, $\ell = 0$ state in hydrogen.

As a result, electrons in multi-electron atoms are grouped into *shells* (according to the value of n) and *subshells* (according to ℓ).

Electron configurations are specified using the numerical values of n, and using letters for ℓ: s, p, d, f, etc., for $\ell = 0,\ 1,\ 2,\ 3$, and so on, plus a superscript for the number of electrons in that subshell. Thus, the ground state of hydrogen is $1s^1$, whereas that for oxygen is $1s^2\,2s^2\,2p^4$. The *periodic table* (see Fig. C.1) arranges the elements in horizontal rows according to increasing atomic number (number of electrons in the neutral atom). The shell structure gives rise to a periodicity in the properties of the elements, so that each vertical column can contain elements with similar chemical properties.

Example 4–5. Which of the following electron configurations are possible, and which forbidden? (a) $1s^2\,2s^2\,2p^6\,3s^3$; (b) $1s^2\,2s^2\,2p^6\,3s^2\,3p^5\,4s^2$; (c) $1s^2\,2s^2\,2p^6\,2d^1$.

Solution. (a) This is not allowed, because there are too many electrons in the s subshell in the M ($n = 3$) shell. The s subshell has $m_\ell = 0$,

with two slots only: for "spin up" and "spin down" electrons.

(b) This is allowed, but it is an excited state. One of the electrons in the $3p$ subshell has jumped up to the $4s$ subshell. Since there are 19 electrons, the element is potassium.

(c) This is not allowed, because there is no d ($\ell = 2$) subshell in the $n = 2$ shell. The outermost electron will have to be (at least) in the $n = 3$ shell.

X-rays, which are a form of electromagnetic radiation of very short wavelength, are produced when high-speed electrons strike a target. The spectrum of X-rays so produced consists of two parts, a continuous spectrum produced when the electrons are decelerated by atoms of the target, and peaks representing photons emitted by atoms of the target after being excited by collision with the high-speed electrons (Fig. 4.5). When the electrons accelerated by the high voltage of the tube collide with the atoms of the target, they can knock out one of the very tightly held inner electrons. The *characteristic X-rays* (represented by the peaks) are photons emitted when an electron in the upper state drops down to fill the vacated lower state. The K lines result from transitions into the K shell ($n = 1$). The K_α line is a transition that originates from the $n = 2$ (L) shell, the K_β line from the $n = 3$ (M) shell. An L line is due to a transition into the L shell, and so on.

The peaks are are characteristic of the material used. Measurement of these peaks allows determination of inner energy levels of atoms and determination of Z. Indeed, the wavelength of the shortest X-rays emitted will be inversely proportional to Z^2. Actually, for an electron jumping from, say, the $n = 2$ to the $n = 1$ level, the wavelength is inversely proportional to $(Z - 1)^2$ because the nucleus is shielded by the one electron that still remains in the $1s$ level. In 1914, H. Moseley found that a plot of $\lambda^{-1/2}$ versus Z produced a straight line, Fig. 4.6.

Example 4–6. Estimate the wavelength for an $n = 2$ to $n = 1$ transition in molybdenum ($Z = 42$). What is the energy of such a photon?

SOLUTION. We use the Bohr formula with Z^2 replaced by $(Z - 1)^2 = (41)^2$. Or, more simply, we can use the result of Example 2–13 for the $n = 2$ to $n = 1$ transition in hydrogen ($Z = 1$). Since

$$\lambda \propto \frac{1}{(Z-1)^2},$$

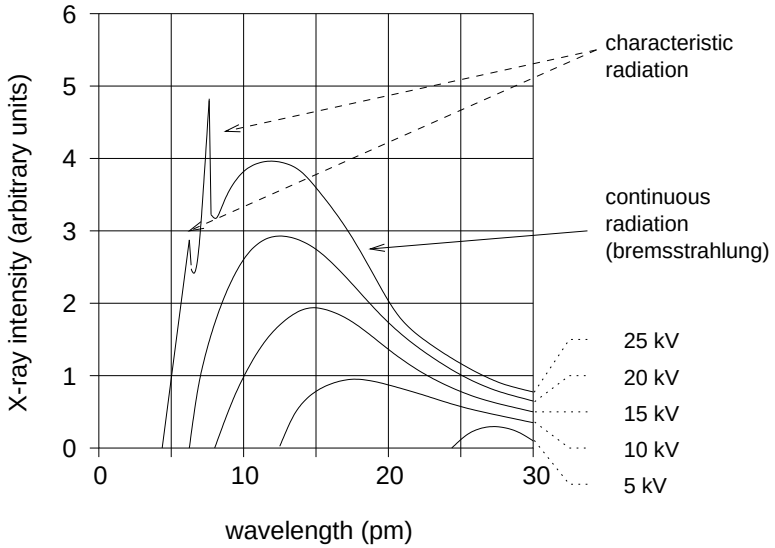


Figure 4.5. Spectrum of X-rays emitted from a molybdenum target in a high-voltage X-ray tube. The characteristic radiation is represented by two sharp peaks: the higher one corresponds to K_α line, while the lower one to K_β line.

we will have

$$\lambda = \frac{(1.22 \times 10^{-7}\ \text{m})}{(41)^2} = 0.073\ \text{nm}.$$

This is close to the measured value (Fig. 4.5) of 0.071 nm. Each of these photons would have energy (in eV) of:

$$E = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{(7.3 \times 10^{-11}\ \text{m})(1.60 \times 10^{-19}\ \text{J/eV})} = 17\ \text{keV}.$$

Example 4–7. High-energy photons are used to bombard an unknown material. The strongest peak is found for X-rays emitted with an energy of 66 keV. Guess what the material is.

Solution. The strongest X-rays are generally for the K_α line (see Fig. 4.5) which occurs when photons knock out K shell electrons (the

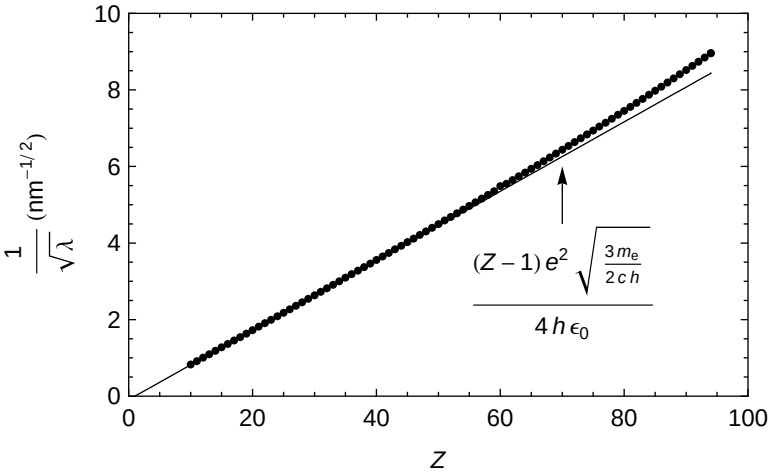


Figure 4.6. The plot of $\lambda^{-1/2}$ vs. Z for K_α X-ray lines (dots). The straight line with an intercept at $Z = 1$ represents Mosley's law (see Problem 4–62).

innermost orbit) and then their place is taken by electrons from the L shell. We use the Bohr model, and assume the electrons "see" a nuclear charge of $Z-1$ (screened by one electron). The hydrogen transition $n = 2$ to $n = 1$ would yield about 10.2 eV (see Example 2–13). Then since energy E is proportional to Z^2 (Eq. 2.10), or rather $(Z-1)^2$ as we've just discussed, we can write

$$\frac{(Z-1)^2}{1^2} = \frac{66 \times 10^3 \text{ eV}}{10.2 \text{ eV}} = 6.5 \times 10^3,$$

so $Z - 1 = \sqrt{6\,500} = 81$, and $Z = 82$, which makes it lead.

The continuous part of an X-ray spectrum is due to deceleration of electrons by interaction with atoms of the target (*Bremsstrahlung* process). The continuous spectrum has minimum wavelength (corresponding to maximum frequency and photon energy) determined by the accelerating voltage, V, in the X-ray tube:

$$\lambda_0 = \frac{2\pi\hbar c}{eV}. \tag{4.10}$$

The cutoff wavelength λ_0 in the bremsstrahlung process do not depend on the target material.

Example 4–8. What is the shortest-wavelength X-ray photon emitted in an X-ray tube subjected to 25 kV?

SOLUTION. From Eq. 4.10,

$$\lambda = \frac{(6.6 \times 10^{-34}\ \text{J} \cdot \text{s})(3.0 \times 10^{8}\ \text{m/s})}{(1.6 \times 10^{-19}\ \text{C})(2.5 \times 10^{4}\ \text{V})} = 5.0 \times 10^{-11}\ \text{m} = 0.050\ \text{nm}.$$

This agrees well with experiment, Fig. 4.5.

If $\mathbf{L}$ is the orbital angular momentum of an electron, the associated *magnetic dipole moment* is

$$\boldsymbol{\mu} = -\frac{1}{2}\frac{e}{m}\mathbf{L}. \tag{4.11}$$

A magnetic dipole moment in a magnetic field $\mathbf{B}$ experiences a torque, and the potential energy of such a system depends on $\mathbf{B}$ and the orientation of $\boldsymbol{\mu}$ relative to $\mathbf{B}$:

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}.$$

If $\mathbf{B}$ is in the z direction, then $U = \mu_z B_z$ and from Eq. 4.4 and Eq. 4.11 we have

$$\mu_z = -\frac{e\hbar}{2m} m_\ell.$$

The quantity

$$\mu_\text{B} = \frac{e\hbar}{2m} = 9.27 \times 10^{-24}\ \text{J/T} \tag{4.12}$$

is called the *Bohr magneton*. Then we can write

$$\mu_z = -\mu_\text{B} m_\ell. \tag{4.13}$$

An atom placed in a magnetic field would have its energy split into levels that differ by $\Delta U = \mu_\text{B} B$; this is the *Zeeman effect* (Fig. 4.2).

The first evidence of space quantization came in 1922 in a famous experiment known as the *Stern–Gerlach experiment*. A narrow beam of neutral silver atoms (and later others) was made to pass through a nonuniform magnetic field. The field was deliberately made nonhomogeneous

so that it would exert a force on atomic magnetic moments: the potential energy $-\boldsymbol{\mu} \cdot \mathbf{B}$ must change in space if there is to be a force. Suppose the beam of atoms is directed parallel to the y-axis. If $\mathbf{B}$ has a gradient along the z axis, then the force is along z:

$$F_z = -\frac{dU}{dz} = \mu_z \frac{dB_z}{dz}.$$

Thus the silver atoms would be deflected up or down depending to the value of μ of each atom. If there were only orbital angular momentum, the deflections would split the beam into an odd number $(2\ell + 1)$ of different components. But Stern and Gerlach saw instead two distinct lines for silver. Why there are two lines was eventually explained by the concept of electron spin. With a spin of $\frac{1}{2}$, the electron spin dipole can have only two orientations in space. Hence the magnetic dipole moment associated with spin would have only two positions. Thus, the two states for silver seen in the Stern–Gerlach experiment must be due to the spin of its one valence electron. Silver atoms must thus have zero orbital angular momentum but a total spin of $\frac{1}{2}$ due to this one valence electron (of its 47 electrons, the spins of the first 46 cancel). Two lines were found also for the H atom in its ground state, again due to the spin $\frac{1}{2}$ of its electron since the orbital angular momentum is zero. The z component of the associated spin magnetic moment turns out to be related to m_s by

$$\mu_z = -g\mu_{\mathrm{B}} m_s, \tag{4.14}$$

where g ($= 2.0023\ldots$ for a free electron) is called the *g-factor* or *gyromagnetic ratio*. Equation 4.14 is the same as Eq. 4.13 for orbital angular momentum with m_s replacing m_ℓ. But for the orbital case, $g = 1$.

An atom can have both orbital and spin angular momenta. The *total angular momentum* is the vector sum of the orbital angular momentum $\mathbf{L}$ and the spin $\mathbf{S}$:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

According to quantum mechanics, the magnitude of the total angular momentum $\mathbf{J}$ is quantized:

$$J = \sqrt{j(j+1)}\,\hbar. \tag{4.15}$$

For a single electron in the H atom, quantum mechanics gives the result that j can be $j = \ell + s = \ell + \frac{1}{2}$ or $j = \ell - s = \ell - \frac{1}{2}$ but never less than zero, just as for ℓ and s. For the $1s$ state, $\ell = 0$ and $j = \frac{1}{2}$ is the only possibility. For p states, say the $2p$ state, $\ell = 1$ and j can be either $\frac{3}{2}$ or $\frac{1}{2}$. The z component for j is quantized in a usual way:

$$m_j = j,\ j-1, \cdots,\ -j.$$

For a $2p$ state with $j = \frac{1}{2}$, m_j can be $\frac{1}{2}$ or $-\frac{1}{2}$; for $j = \frac{3}{2}$, m_j can be $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$, for a total of four states. Note that the state of a single electron can be specified by giving n, ℓ, m_ℓ, m_s, or by giving n, j, ℓ, m_j (only one of these descriptions at a time).

The interactions of magnetic fields with atoms, as in the Zeeman effect and the Stern–Gerlach experiment, involves the *total* angular momentum. Thus the Stern–Gerlach experiment on H atoms in the ground state shows two lines (for $m_j = +\frac{1}{2}$ and $-\frac{1}{2}$, but for the first excited state it shows four lines corresponding to the four possible m_j values $\left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$.

The state of an atom, including the total angular momentum quantum number j, can be specified using the following *spectroscopic notation*. For a single electron state we can write nL_j, where the value of L is specified using the same letters *spdfg*, but in upper case:

$$\begin{array}{rcccccccc} L & = & 0 & 1 & 2 & 3 & 4 & \cdots \\ \text{letter} & = & S & P & D & F & G & \cdots . \end{array}$$

So the $2P_{3/2}$ state has $n = 2$, $\ell = 1$, $j = \frac{3}{2}$, whereas $1S_{1/2}$ specifies the ground state in hydrogen.

The spin magnetic dipole moment also gives splitting of energy levels even when there is no external field. The interaction energy U can be expressed in terms of the scalar product of the angular-momentum vectors $\mathbf{L}$ and $\mathbf{S}$:

$$\Delta U \propto \mathbf{L} \cdot \mathbf{S}.$$

This interaction, which produces a fine structure of spectral lines, is called the *spin-orbit coupling*. Its magnitude is related to a dimensionless constant known as the *fine structure constant*,

$$\alpha = \frac{k_e e^2}{\hbar c} \approx \frac{1}{137}.$$

For example, in the H atom, the $2P \rightarrow 1S$ transition is split into two lines corresponding to $2P_{1/2} \rightarrow 1S_{1/2}$ and $2P_{3/2} \rightarrow 1S_{1/2}$. The difference in energy between these two is only about 5×10^{-5} eV, which is very small compared to the $2P \rightarrow 1S$ transition energy of $13.6\ \text{eV} - 3.4\ \text{eV} = 10.2\ \text{eV}$.

A laser is a device that can produce a very narrow intense beam of monochromatic coherent light. The action of a laser is based on quantum theory. A photon can be absorbed by an atom if (and only if) its energy $2\pi\hbar f$ corresponds to the energy difference between an occupied energy level of the atom and an available excited state. This is, in a sense, a resonant situation. If the atom is already in the excited state, it may of course jump spontaneously to the lower state with the emission of a photon. However, if a photon with this same energy strikes the excited atom, it can stimulate the atom to make the transition sooner to the lower state. This phenomenon is called *stimulated emission*, and it can be seen that not only do we still have the original photon, but also a second one of the same frequency as a result of the atom's transition. And these two photons are exactly in phase, and they are moving in the same direction.

The natural population of atoms in thermal equilibrium at any temperature T is given by the *Boltzmann distribution*

$$N_n = Ce^{-E_n/k_\text{B}T}, \tag{4.16}$$

where N_n is the number of atoms in the state with energy E_n. For two states n and n', the ratio of the number of atoms in the two states is

$$\frac{N_n}{N_{n'}} = Ce^{-(E_n - E_{n'})/k_\text{B}T}. \tag{4.17}$$

Thus most atoms are in the ground state unless the temperature is very high. In a two-level system most atoms are normally in the lower state, so the majority of incident photons will be absorbed. In order to obtain the coherent light from stimulated emission, two conditions must be satisfied. First, atoms must be excited to the higher state, so that an *inverted population* (or *population inversion*) is produced, one in which more atoms are in the upper state than in the lower one. Then the emission of photons will dominate over absorbtion. Hence the system will not be in thermal equilibrium. And second, the higher state must be a *metastable state*—a state in which the electrons remain longer than usual so that the transition to the lower state occurs by stimulated emission rather than spontaneously.

The excitation of the atoms in a laser can be done in several ways to produce the necessary inverted population. In a ruby laser, the lasing material is a ruby rod consisting of Al_2O_3 with small percentage of aluminum (Al) atoms replaced by chromium (Cr) atoms. The Cr atoms are the ones involved in lasing. The atoms are excited by strong flashes of 550-nm light, which corresponds to a photon energy of 2.2 eV. As shown in Fig. 4.7, the atoms are excited from state E_0 to state E_2. This process is called *optical pumping*. The atoms quickly decay either back to E_0 or to the intermediate state E_1, which is metastable with a lifetime of about 3×10^{-3} s (compared to 10^{-8} s for ordinary levels). With strong pumping action, more atoms can be forced into the E_1 state than are in the E_0 state. Thus we have the inverted population needed for lasing. As soon as a few atoms in the E_1 state jump down to E_0, the emit photons that produce stimulated emission of the other atoms and the lasing action begins. A ruby laser thus emits a beam whose photons have energy 1.8 eV and a wavelength of 694.3 nm.

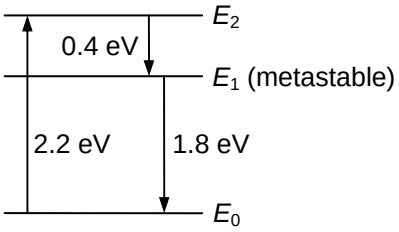


Figure 4.7. Energy levels of chromium in a ruby crystal. Photons of energy 2.2 eV pump atoms from E_0 to E_2, followed by decay to the metastable state E_1. Lasing action occurs by stimulated emission of photons in transition from E_1 to E_0.

4.2 Hydrogen atom: Schrödinger equation and quantum numbers

4–1. For $n = 6$, what values can ℓ have?

SOLUTION. The value of ℓ can range from 0 to $n - 1$. Thus for $n = 6$, we have

$$\ell = 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

4–2. For $n = 5$, $\ell = 3$, what are the possible values of m_ℓ and m_s?

SOLUTION. The value of m_ℓ can range from $-\ell$ to $+\ell$. Thus for $\ell = 3$, we have

$$m_\ell = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3.$$

The possible values of m_s are $-\frac{1}{2}$, $+\frac{1}{2}$.

4–3. How many different states are possible for an electron whose principal quantum number is $n = 4$? Write down the quantum numbers for each state.

SOLUTION. The value of ℓ can range from 0 to $n - 1$. Thus for $n = 4$, we have $\ell = 0,\ 1,\ 2,\ 3$. For each ℓ the value of m_ℓ can range from $-\ell$ to $+\ell$, or $2\ell + 1$ values. For each of these there are two values of m_s. Thus the total number for each ℓ is $2(2\ell + 1)$.

The number of states is

$$N = 2(0 + 1) + 2(2 + 1) + 2(4 + 1) + 2(6 + 1) = 32 \text{ states.}$$

We start with $\ell = 0$, and list the quantum numbers in the order (n, ℓ, m_ℓ, m_s):

$(4,\ 0,\ \ 0,\ -\frac{1}{2}),\quad (4,\ 0,\ \ 0,\ +\frac{1}{2}),\quad (4,\ 1,\ -1,\ -\frac{1}{2}),\quad (4,\ 1,\ -1,\ +\frac{1}{2}),$

$(4,\ 1,\ \ 0,\ -\frac{1}{2}),\quad (4,\ 1,\ \ 0,\ +\frac{1}{2}),\quad (4,\ 1,\ \ 1,\ -\frac{1}{2}),\quad (4,\ 1,\ \ 1,\ +\frac{1}{2}),$

$(4,\ 2,\ -2,\ -\frac{1}{2}),\quad (4,\ 2,\ -2,\ +\frac{1}{2}),\quad (4,\ 2,\ -1,\ -\frac{1}{2}),\quad (4,\ 2,\ -1,\ +\frac{1}{2}),$

$(4,\ 2,\ \ 0,\ -\frac{1}{2}),\quad (4,\ 2,\ \ 0,\ +\frac{1}{2}),\quad (4,\ 2,\ \ 1,\ -\frac{1}{2}),\quad (4,\ 2,\ \ 1,\ +\frac{1}{2}),$

$(4,\ 2,\ \ 2,\ -\frac{1}{2}),\quad (4,\ 2,\ \ 2,\ +\frac{1}{2}),\quad (4,\ 3,\ -3,\ -\frac{1}{2}),\quad (4,\ 3,\ -3,\ +\frac{1}{2}),$

$(4,\ 3,\ -2,\ -\frac{1}{2}),\quad (4,\ 3,\ -2,\ +\frac{1}{2}),\quad (4,\ 3,\ -1,\ -\frac{1}{2}),\quad (4,\ 3,\ -1,\ +\frac{1}{2}),$

$(4,\ 3,\ \ 0,\ -\frac{1}{2}),\quad (4,\ 3,\ \ 0,\ +\frac{1}{2}),\quad (4,\ 3,\ \ 1,\ -\frac{1}{2}),\quad (4,\ 3,\ \ 1,\ +\frac{1}{2}),$

$(4,\ 3,\ \ 2,\ -\frac{1}{2}),\quad (4,\ 3,\ \ 2,\ +\frac{1}{2}),\quad (4,\ 3,\ \ 3,\ -\frac{1}{2}),\quad (4,\ 3,\ \ 3,\ +\frac{1}{2}).$

4–4. If a hydrogen atom has $m_\ell = -3$, what are the possible values of n, ℓ, m_s?

SOLUTION. The value of m_ℓ can range from $-\ell$ to $+\ell$, so we have $\ell \geqslant 3$. The value of ℓ can range from 0 to $n-1$. Thus we have $n \geqslant \ell+1$ (minimum 4).

There are two values of m_s: $m_s = -\frac{1}{2},\ +\frac{1}{2}$.

4–5. A hydrogen atom is known to have $\ell = 4$. What are the possible values for n, m_ℓ, and m_s?

SOLUTION. The value of ℓ can range from 0 to $n-1$. Thus for $\ell = 4$, we have $n \geqslant 5$.

For each ℓ the value of m_ℓ can range from $-\ell$ to $+\ell$:

$$m_\ell = -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4.$$

There are two values of m_s: $m_s = -\frac{1}{2}, +\frac{1}{2}$.

4–6. Calculate the magnitude of the angular momentum of an electron in the $n = 4$, $\ell = 2$ state of hydrogen.

SOLUTION. The magnitude of the angular momentum depends on ℓ only:

$$\begin{aligned} L &= \hbar\sqrt{\ell(\ell+1)} = \hbar\sqrt{(2)(2+1)} = (1.055 \times 10^{-34}\ \text{J}\cdot\text{s})\sqrt{6} \\ &= 2.58 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}. \end{aligned}$$

4–7. A hydrogen atom is in the $6g$ state. Determine (a) the principal quantum number, (b) the energy of the state, (c) the orbital angular momentum and its quantum number ℓ, and (d) the possible values for the magnetic quantum number.

SOLUTION. (a) The principal quantum number is $n = 6$.

(b) The energy of the state is

$$E_6 = -\frac{(13.6\ \text{eV})}{n^2} = -\frac{(13.6\ \text{eV})}{6^2} = -0.378\ \text{eV}.$$

(c) From *spdfgh*, we see that the "g" subshell has $\ell = 4$. The magnitude of the angular momentum depends on ℓ only:

$$\begin{aligned} L &= \hbar\sqrt{\ell(\ell+1)} = (1.055 \times 10^{-34}\ \text{J}\cdot\text{s})\sqrt{(4)(4+1)} = \sqrt{20}\,\hbar \\ &= 4.72 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}. \end{aligned}$$

(d) For each ℓ the value of m_ℓ can range from $-\ell$ to $+\ell$:

$$m_\ell = -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4.$$

4–8. (a) Show that the number of different states possible for a given value of ℓ is equal to $2(2\ell+1)$. (b) What is this number for $\ell = 0$, 1, 2, 3, 4, 5, and 6?

SOLUTION. (a) For each ℓ the value of m_ℓ can range from $-\ell$ to $+\ell$, or $2\ell+1$ values. For each of these there are two values of m_s. Thus the total number of electrons allowed in a subshell is $N = 2(2\ell+1)$.

(b) For the values of ℓ we have

$$\begin{aligned}
\ell &= 0, \quad N = 2\,(2(0)+1) = 2;\\
\ell &= 1, \quad N = 2\,(2(1)+1) = 6;\\
\ell &= 2, \quad N = 2\,(2(2)+1) = 10;\\
\ell &= 3, \quad N = 2\,(2(3)+1) = 14;\\
\ell &= 4, \quad N = 2\,(2(4)+1) = 18;\\
\ell &= 5, \quad N = 2\,(2(5)+1) = 22;\\
\ell &= 6, \quad N = 2\,(2(6)+1) = 26.
\end{aligned}$$

4–9. Show that the number of different electron states possible for a given value of n is $2n^2$. (See Problem 4–8.)

SOLUTION. From Problem 4–8 the total number of electrons allowed in a subshell with quantum number ℓ is $2(2\ell+1)$. For a state with quantum number n the value of ℓ can range from 0 to $n-1$. Thus we have

$$N = 2\sum_{\ell=0}^{n-1}(2\ell+1) = 4\sum_{\ell=0}^{n-1}\ell + 2\sum_{\ell=0}^{n-1}1 = 4\frac{(n-1)n}{2} + 2n = 2n^2.$$

4–10. An excited H atom is in a $6d$ state. (a) Name all the states (n, ℓ) to which the atom is "allowed" to jump with the emission of a photon, (b) How many different wavelengths are there (ignoring fine structure)?

SOLUTION. (a) The $6d$ subshell has $\ell = 2$. Allowed transitions will be to any lower value of n with $\Delta\ell = \pm 1$. Thus the final value of ℓ can be 1 (p) or 3 (f), so we have

$$5f,\ 5p,\ 4f,\ 4p,\ 3p,\ 2p.$$

(b) If we ignore fine structure, the energy change depends only on the two values of n. The allowed transitions from $n = 6$ are to $n = 5,\ 4,\ 3,\ 2$, so there are 4 wavelengths.

4.3 Hydrogen atom wave functions

4–11. Show that the ground-state wave function, Eq. 4.5 is normalized. [*Hint*: See Example 4–4.]

SOLUTION. To see if the ground-state wave function is normalized, we integrate the radial probability density over all radii:

$$\int_0^\infty 4\pi r^2 |\psi_{100}|^2 \, dr = \int_0^\infty 4\pi r^2 \frac{1}{\pi r_{\mathrm{B}}^3} e^{-2r/r_{\mathrm{B}}} \, dr.$$

We change variable to $x = 2r/r_{\mathrm{B}}$ and use the result for the integration in Example 4–4:

$$\tfrac{1}{2} \int_0^\infty x^2 e^{-x} \, dx = \left(-\tfrac{1}{2}x^2 - x - 1\right) e^{-x} \Big|_0^\infty = 0 - (-1e^0) = 1.$$

4–12. For the ground state of hydrogen, what is the value of (a) ψ, (b) $|\psi|^2$, and (c) P_{r}, at $r = r_{\mathrm{B}}$?

SOLUTION. The ground-state wave function is

$$\psi_{100} = \frac{1}{\sqrt{\pi r_{\mathrm{B}}^3}} e^{-r/r_{\mathrm{B}}}.$$

(a) The wave function is

$$\psi_{100}(r_{\mathrm{B}}) = \frac{1}{\sqrt{\pi r_{\mathrm{B}}^3}} e^{-1}.$$

(b) The probability density is

$$|\psi_{100}(r_{\mathrm{B}})|^2 = \frac{1}{\pi r_{\mathrm{B}}^3} e^{-2}.$$

(c) The radial probability density is

$$P_{\mathrm{r}} = 4\pi r^2 |\psi_{100}(r)|^2.$$

At $r = r_{\mathrm{B}}$ we have

$$P_{\mathrm{r}}(r_{\mathrm{B}}) = \frac{4}{r_{\mathrm{B}}} e^{-2}.$$

4–13. For the $n = 2$, $\ell = 0$ state of hydrogen, what is the value of (a) ψ, (b) $|\psi|^2$, and (c) P_r at $r = 5\,r_\mathrm{B}$?

SOLUTION. The $n = 2$, $\ell = 0$ wave function is

$$\psi_{200} = \frac{1}{\sqrt{32\pi r_\mathrm{B}^3}} \left(2 - \frac{r}{r_\mathrm{B}}\right) e^{-r/2r_\mathrm{B}}.$$

(a) The wave function is

$$\psi_{200}(5\,r_\mathrm{B}) = \frac{1}{\sqrt{32\pi r_\mathrm{B}^3}} (2-5)\, e^{-5/2} = -\frac{3}{\sqrt{32\pi r_\mathrm{B}^3}}\, e^{-5/2}.$$

(b) The probability density is

$$|\psi_{200}(5\,r_\mathrm{B})|^2 = \frac{9}{32\pi r_\mathrm{B}^3}\, e^{-5}.$$

(c) The radial probability density is

$$P_\mathrm{r} = 4\pi r^2 |\psi_{200}(r)|^2.$$

At $r = 5\,r_\mathrm{B}$ we have

$$P_\mathrm{r}(5\,r_\mathrm{B}) = 4\pi (5\,r_\mathrm{B})^2 \frac{9}{32\pi r_\mathrm{B}^3}\, e^{-5} = \frac{225}{8\,r_\mathrm{B}}\, e^{-5}.$$

4–14. Show that ψ_{200} as given by Eq. 4.8 is normalized.

SOLUTION. To see if the wave function is normalized, we integrate the radial probability density over all radii:

$$\int_0^\infty 4\pi r^2 |\psi_{200}|^2\, dr = \int_0^\infty 4\pi r^2 \frac{1}{32\pi r_\mathrm{B}^3} \left(2 - \frac{r}{r_\mathrm{B}}\right)^2 e^{-r/r_\mathrm{B}}\, dr.$$

We change variable to $x = r/r_\mathrm{B}$ and expand:

$$\begin{aligned} &\frac{1}{2}\int_0^\infty x^2 e^{-x}\, dx - \frac{1}{2}\int_0^\infty x^3 e^{-x}\, dx + \frac{1}{8}\int_0^\infty x^4 e^{-x}\, dx \\ &\qquad = \tfrac{1}{2}(2!) - \tfrac{1}{2}(3!) + \tfrac{1}{8}(4!) = 1 - 3 + 3 = 1. \end{aligned}$$

4–15. By what factor is it more likely to find the electron in the ground state of hydrogen at the Bohr radius ($r_{\rm B}$) than at twice the Bohr radius ($2r_{\rm B}$)?

SOLUTION. We form the ratio of the radial probability densities:

$$\frac{P_{\rm r}(r_{\rm B})}{P_{\rm r}(2r_{\rm B})} = \frac{4\pi\left(r_{\rm B}^2/\pi r_{\rm B}^3\right)\, e^{-2r_{\rm B}/r_{\rm B}}}{4\pi\left((2r_{\rm B})^2/\pi r_{\rm B}^3\right)\, e^{-4r_{\rm B}/r_{\rm B}}} = \frac{e^{-2}}{4\,e^{-4}} = 1.85.$$

4–16. For the ground state of hydrogen, what is the probability of finding the electron within a spherical shell of inner radius $0.99\,r_{\rm B}$ and outer radius $1.01\,r_{\rm B}$?

SOLUTION. Because the change in r is small, we can use the radial probability density at $r = r_{\rm B}$:

$$P = P_{\rm r}(r_{\rm B})\,\Delta r = \frac{4\pi r_{\rm B}^2}{\pi r_{\rm B}^3}\, e^{-2r_{\rm B}/r_{\rm B}}(0.02\,r_{\rm B}) = 0.08\,e^{-2} = 0.011 = 1.1\%.$$

4–17. For the $n = 2$, $\ell = 0$ state of hydrogen, what is the probability of finding the electron within a spherical shell of inner radius $4.00\,r_{\rm B}$ and outer radius $5.00\,r_{\rm B}$?

SOLUTION. The $n = 2$, $\ell = 0$ wave function is

$$\psi_{200} = \frac{1}{\sqrt{32\pi r_{\rm B}^3}}\left(2 - \frac{r}{r_{\rm B}}\right) e^{-r/2r_{\rm B}}.$$

The radial probability density is

$$P_{\rm r}(r) = 4\pi r^2|\psi_{200}|^2 = \frac{r^2}{8r_{\rm B}^3}\left(2 - \frac{r}{r_{\rm B}}\right)^2 e^{-r/r_{\rm B}}.$$

To find the probability we integrate the radial probability density:

$$P = \int_{4.00\,r_{\rm B}}^{5.00\,r_{\rm B}} \frac{r^2}{8r_{\rm B}^3}\left(2 - \frac{r}{r_{\rm B}}\right)^2 e^{-r/r_{\rm B}}\,dr.$$

We change variable to $x = r/r_{\rm B}$ and expand:

$$\begin{aligned} P &= \int_{4.00}^{5.00} \frac{1}{8}(4x^2 - 4x^3 + x^4)e^{-x}\,dx \\ &= \left(-\tfrac{1}{2}x^2 - x - 1 - \tfrac{1}{8}x^4\right) e^{-x}\Big|_{4.00}^{5.00} \\ &= 0.173 = 17.3\%. \end{aligned}$$

4–18. (a) Show that the probability of finding the electron in the ground state of hydrogen at less than one Bohr radius from the nucleus is 32 percent. (b) What is the probability of finding a $1s$ electron between $r = r_{\mathrm{B}}$ and $r = 2r_{\mathrm{B}}$?

Solution. (a) To find the probability we integrate the radial probability density:

$$P = \int_0^{r_{\mathrm{B}}} \frac{4\pi r^2}{\pi r_{\mathrm{B}}^3}\, e^{-2r/r_{\mathrm{B}}}\, dr.$$

We change variable to $x = 2r/r_{\mathrm{B}}$ and use the result for the integration in Example 4–4:

$$\begin{aligned} P &= \frac{1}{2}\int_0^2 x^2 e^{-x}\, dx = \left(-\tfrac{1}{2}x^2 - x - 1\right) e^{-x}\Big|_0^2 \\ &= (-2-2-1)e^{-2} - (-1) = 0.32 = 32\%. \end{aligned}$$

(b) To find the probability we integrate the radial probability density:

$$P = \int_{r_{\mathrm{B}}}^{2r_{\mathrm{B}}} \frac{4\pi r^2}{\pi r_{\mathrm{B}}^3}\, e^{-2r/r_{\mathrm{B}}}\, dr.$$

We change variable to $x = 2r/r_{\mathrm{B}}$ and use the result for the integration in Example 4–4:

$$\begin{aligned} P &= \frac{1}{2}\int_2^4 x^2 e^{-x}\, dx = \left(-\tfrac{1}{2}x^2 - x - 1\right) e^{-x}\Big|_2^4 \\ &= (-8-4-1)e^{-4} - (-2-2-1)e^{-2} = 0.44 = 44\%. \end{aligned}$$

4–19. Determine the radius r of a sphere centered on the nucleus within which the probability of finding the electron for the ground state of hydrogen is (a) 50 percent, (b) 90 percent, (c) 99 percent.

Solution. To find the probability for the electron to be within a sphere of radius r, we integrate the radial probability density:

$$P = \int_0^{r} \frac{4\pi r'^2}{\pi r_{\mathrm{B}}^3}\, e^{-2r'/r_{\mathrm{B}}}\, dr'.$$

We change variable to $x' = 2r'/r_{\rm B}$ and use the result for the integration in Example 4–4:

$$\begin{aligned} P &= \frac{1}{2}\int_0^x x'^2 e^{-x'}\,dx' = \left(-\tfrac{1}{2}x'^2 - x' - 1\right)e^{-x'}\Big|_0^x \\ &= 1 - \left(\tfrac{1}{2}x^2 + x + 1\right)e^{-x}. \end{aligned}$$

(a) For the probability to be 50% we have

$$\left(\tfrac{1}{2}x^2 + x + 1\right)e^{-x} = 0.50.$$

A numerical calculation gives $x = 2.68$, so $r = 1.34\,r_{\rm B}$.

(b) For the probability to be 90% we have

$$\left(\tfrac{1}{2}x^2 + x + 1\right)e^{-x} = 0.10.$$

A numerical calculation gives $x = 5.32$, so $r = 2.7\,r_{\rm B}$.

(c) For the probability to be 99% we have

$$\left(\tfrac{1}{2}x^2 + x + 1\right)e^{-x} = 0.01.$$

A numerical calculation gives $x = 8.40$, so $r = 4.2\,r_{\rm B}$.

4–20. (a) Estimate the probability of finding an electron, in the ground state of hydrogen, within the nucleus assuming it to be a sphere of radius $r = 1.1$ fm. (b) What would be the probability if the electron were replaced with a muon, which is very similar to an electron (Chapter 8) except that its mass is 207 times greater?

SOLUTION. (a) Because $r \ll r_{\rm B}$, we assume the probability density is constant and equal to the value at $r = 0$:

$$\begin{aligned} P &= \frac{1}{\pi r_{\rm B}^3}\,e^{-0}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\left(\frac{r}{r_{\rm B}}\right)^3 = \frac{4}{3}\left(\frac{1.1\times10^{-15}\ \text{m}}{0.529\times10^{-10}\ \text{m}}\right)^3 \\ &= 1.2\times10^{-14}. \end{aligned}$$

(b) From the Bohr theory, we have $r_{\rm B} \propto 1/m$. Thus $P_{\rm r} \propto 1/r_{\rm B}^3 \propto m^3$. If we form the ratio we have

$$\begin{aligned} \frac{P_{\rm muon}}{P_{\rm electron}} &= \left(\frac{m_{\rm muon}}{m_{\rm electron}}\right)^3; \\ \frac{P_{\rm muon}}{(1.2\times10^{-14})} &= (207)^3, \end{aligned}$$

which gives $P_{\text{muon}} = 1.1 \times 10^{-7}$.

4–21. Determine the average radial probability distribution P_{r} for the $n = 2$, $\ell = 1$ state in hydrogen by calculating

$$P_{\text{r}} = 4\pi r^2 \left(\tfrac{1}{3}|\psi_{210}|^2 + \tfrac{1}{3}|\psi_{211}|^2 + \tfrac{1}{3}|\psi_{21-1}|^2 \right).$$

SOLUTION. The three wave functions are

$$\begin{aligned} \psi_{210} &= \frac{z}{\sqrt{32\pi r_{\text{B}}^5}}\, e^{-r/2r_{\text{B}}}, \\ \psi_{211} &= \frac{x+iy}{\sqrt{64\pi r_{\text{B}}^5}}\, e^{-r/2r_{\text{B}}}, \\ \psi_{21-1} &= \frac{x-iy}{\sqrt{64\pi r_{\text{B}}^5}}\, e^{-r/2r_{\text{B}}}. \end{aligned}$$

The radial probability density is

$$\begin{aligned} P_{\text{r}} &= 4\pi r^2 \left(\tfrac{1}{3}|\psi_{210}|^2 + \tfrac{1}{3}|\psi_{211}|^2 + \tfrac{1}{3}|\psi_{21-1}|^2 \right) \\ &= \frac{r^2}{24 r_{\text{B}}^5} \left(z^2 + \frac{1}{2}(x+iy)(x-iy) + \frac{1}{2}(x-iy)(x+iy) \right) e^{-r/r_{\text{B}}} \\ &= \frac{r^2}{24 r_{\text{B}}^5} \left(z^2 + x^2 + y^2 \right) e^{-r/r_{\text{B}}} = \frac{r^4}{24 r_{\text{B}}^5}\, e^{-r/r_{\text{B}}}. \end{aligned}$$

4–22. Use the result of Problem 4–21 to show that the most probable distance r from the nucleus for an electron in the $2p$ state of hydrogen is $r = 4r_{\text{B}}$, which is just the second Bohr radius.

SOLUTION. From the result for Problem 4–21 the radial probability density is

$$P_{\text{r}} = \frac{r^4}{24 r_{\text{B}}^5}\, e^{-r/r_{\text{B}}}.$$

We find the most probable distance by setting the first derivative equal to zero:

$$\begin{aligned} \frac{dP_{\text{r}}}{dr} &= \frac{1}{24 r_{\text{B}}^5} \left(4r^3 - \frac{r^4}{r_{\text{B}}} \right) e^{-r/r_{\text{B}}} = 0; \\ 4r^3 &= \frac{r^4}{r_{\text{B}}}, \end{aligned}$$

which gives $r = 4r_{\mathrm{B}}$.

4–23. Show that the mean value of r for an electron in the ground state of hydrogen is $\overline{r} = \frac{3}{2}r_{\mathrm{B}}$, by calculating

$$\overline{r} = \int_{\text{all space}} r|\psi_{100}|^2\, dV = \int_0^\infty r|\psi_{100}|^2 4\pi r^2\, dr.$$

SOLUTION. We find the mean value of r from

$$\overline{r} = \int_0^\infty r|\psi_{100}|^2 4\pi r^2\, dr = \int_0^\infty 4\pi r^3 \frac{1}{\pi r_{\mathrm{B}}^3}\, e^{-2r/r_{\mathrm{B}}}\, dr.$$

We change variable to $x = 2r/r_{\mathrm{B}}$:

$$\overline{r} = \frac{r_{\mathrm{B}}}{4}\int_0^\infty x^3 e^{-x}\, dx = \frac{r_{\mathrm{B}}}{4}\, 3! = \frac{3}{2}r_{\mathrm{B}}.$$

4–24. Show that the probability of finding the electron within 1 Bohr radius of the nucleus in the hydrogen atom is (a) 3. 4 percent for the $n = 2$, $\ell = 0$ state, and (b) 0. 37 percent for the $n = 2$, $\ell = 1$ state.

SOLUTION. (a) The $n = 2$, $\ell = 0$ wave function is

$$\psi_{200} = \frac{1}{\sqrt{32\pi r_{\mathrm{B}}^3}}\left(2 - \frac{r}{r_{\mathrm{B}}}\right) e^{-r/2r_{\mathrm{B}}}.$$

The radial probability density is

$$P_{\mathrm{r}}(r) = 4\pi r^2|\psi_{200}|^2 = \frac{r^2}{8r_{\mathrm{B}}^3}\left(2 - \frac{r}{r_{\mathrm{B}}}\right)^2 e^{-r/r_{\mathrm{B}}}.$$

To find the probability for $0 < r < r_{\mathrm{B}}$, we integrate the radial probability density:

$$P = \int_0^{r_{\mathrm{B}}} \frac{r^2}{8r_{\mathrm{B}}^3}\left(2 - \frac{r}{r_{\mathrm{B}}}\right)^2 e^{-r/r_{\mathrm{B}}}\, dr.$$

We change variable to $x = r/r_{\mathrm{B}}$ and expand:

$$\begin{aligned} P &= \int_0^1 \frac{1}{8}(4x^2 - 4x^3 + x^4)e^{-x}\, dx = \left(-\tfrac{1}{2}x^2 - x - 1 - \tfrac{1}{8}x^4\right)e^{-x}\Big|_0^1 \\ &= 1 + \left(-\tfrac{1}{2} - 1 - 1 - \tfrac{1}{8}\right)e^{-1} = 0.034 = 3.4\%. \end{aligned}$$

(b) From the result for Problem 4–21 the radial probability density is

$$P_{\mathrm{r}} = \frac{r^4}{24 r_{\mathrm{B}}^5}\, e^{-r/r_{\mathrm{B}}}.$$

To find the probability for $0 < r < r_{\mathrm{B}}$, we integrate the radial probability density:

$$P = \int_0^{r_{\mathrm{B}}} \frac{r^4}{24 r_{\mathrm{B}}^5}\, e^{-r/r_{\mathrm{B}}}\, dr.$$

We change variable to $x = r/r_{\mathrm{B}}$:

$$\begin{aligned} P &= \frac{1}{24}\int_0^1 x^4 e^{-x}\, dx = \tfrac{1}{24}\left(-x^4 - 4x^3 - 12x^2 - 24x - 24\right) e^{-x}\Big|_0^1 \\ &= 1 + \tfrac{1}{24}(-1 - 4 - 12 - 24 - 24)e^{-1} = 0.0037 = 0.37\%. \end{aligned}$$

4–25. Show that ψ_{100} (Eq. 4.5) satisfies the Schrödinger equation (Eq. 4.1) with the Coulomb potential, for energy

$$E = -\frac{k_{\mathrm{e}}^2 e^4 m}{2\hbar^2}.$$

Solution. The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - \frac{k_{\mathrm{e}} e^2}{r}\psi = E\psi,$$

where $r^2 = x^2 + y^2 + z^2$. For later use we find a partial derivative:

$$2r\frac{\partial r}{\partial x} = 2x, \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}.$$

We find the partial derivatives of the ground-state wave function ψ_{100}:

$$\begin{aligned} \frac{\partial\psi}{\partial x} &= \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x} = -\frac{e^{-r/r_{\mathrm{B}}}}{r_{\mathrm{B}}\sqrt{\pi r_{\mathrm{B}}^3}}\,\frac{x}{r}; \\ \frac{\partial^2\psi}{\partial x^2} &= \frac{e^{-r/r_{\mathrm{B}}}}{r_{\mathrm{B}}^2\sqrt{\pi r_{\mathrm{B}}^3}}\,\frac{\partial r}{\partial x}\frac{x}{r} - \frac{e^{-r/r_{\mathrm{B}}}}{r_{\mathrm{B}}\sqrt{\pi r_{\mathrm{B}}^3}}\left(\frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}\right) \\ &= \frac{e^{-r/r_{\mathrm{B}}}}{r r_{\mathrm{B}}\sqrt{\pi r_{\mathrm{B}}^3}}\left(\frac{x^2}{r r_{\mathrm{B}}} - 1 + \frac{x^2}{r^2}\right) \\ &= -\frac{e^{-r/r_{\mathrm{B}}}}{r r_{\mathrm{B}}\sqrt{\pi r_{\mathrm{B}}^3}}\left(1 - x^2\left(\frac{1}{r r_{\mathrm{B}}} + \frac{1}{r^2}\right)\right). \end{aligned}$$

By exchanging x with y and z, we have

$$\begin{aligned}\frac{\partial^2\psi}{\partial y^2} &= -\frac{e^{-r/r_{\rm B}}}{rr_{\rm B}\sqrt{\pi r_{\rm B}^3}}\left(1-y^2\left(\frac{1}{rr_{\rm B}}+\frac{1}{r^2}\right)\right);\\ \frac{\partial^2\psi}{\partial z^2} &= -\frac{e^{-r/r_{\rm B}}}{rr_{\rm B}\sqrt{\pi r_{\rm B}^3}}\left(1-z^2\left(\frac{1}{rr_{\rm B}}+\frac{1}{r^2}\right)\right).\end{aligned}$$

When we put these in the Schrödinger equation and use $r^2 = x^2+y^2+z^2$, we get

$$\frac{\hbar^2\, e^{-r/r_{\rm B}}}{2mrr_{\rm B}\sqrt{\pi r_{\rm B}^3}}\left(3-r^2\left(\frac{1}{rr_{\rm B}}+\frac{1}{r^2}\right)\right)-\frac{k_{\rm e}e^2}{r}\frac{e^{-r/r_{\rm B}}}{\sqrt{\pi r_{\rm B}^3}} = E\,\frac{e^{-r/r_{\rm B}}}{\sqrt{\pi r_{\rm B}^3}}.$$

If we cancel common factors, we have

$$-\frac{\hbar^2}{2mr_{\rm B}^2}+\frac{\hbar^2}{mrr_{\rm B}}-\frac{k_{\rm e}e^2}{r} = E.$$

If we look at the second and third terms and use $r_{\rm B} = \hbar^2/k_{\rm e}e^2m$, we get

$$\frac{\hbar^2}{mr\left(\hbar^2/k_{\rm e}e^2m\right)}-\frac{k_{\rm e}e^2}{r} = 0.$$

Thus we have

$$E = -\frac{\hbar^2}{2m\left(\hbar^2/k_{\rm e}e^2m\right)^2} = -\frac{k_{\rm e}^2e^4m}{2\hbar^2}.$$

4–26. For the $n = 2$, $\ell = 0$ state in hydrogen, what is the probability of finding the electron within the smaller peak in the radial probability distribution, Fig. 4.4?

SOLUTION. From Fig. 4.4 we see that the first peak is from $r = 0$ to $r = 2r_{\rm B}$. The $n = 2$, $\ell = 0$ wave function is

$$\psi_{200} = \frac{1}{\sqrt{32\pi r_{\rm B}^3}}\left(2-\frac{r}{r_{\rm B}}\right)e^{-r/2r_{\rm B}}.$$

The radial probability density is

$$P_{\rm r}(r) = 4\pi r^2|\psi_{200}|^2 = \frac{r^2}{8r_{\rm B}^3}\left(2-\frac{r}{r_{\rm B}}\right)^2 e^{-r/r_{\rm B}}.$$

To find the probability for $0 < r < 2\,r_{\rm B}$, we integrate the radial probability density:

$$P = \int_0^{2\,r_{\rm B}} \frac{r^2}{8r_{\rm B}^3}\left(2 - \frac{r}{r_{\rm B}}\right)^2 e^{-r/r_{\rm B}}\,dr.$$

We change variable to $x = r/r_{\rm B}$ and expand:

$$\begin{aligned}\int_0^2 \frac{1}{8}(4x^2 - 4x^3 + x^4)e^{-x}\,dx &= \left(-\tfrac{1}{2}x^2 - x - 1 - \tfrac{1}{8}x^4\right)e^{-x}\Big|_0^2 \\ &= 1 - 0.947 = 0.053 = 5.3\%.\end{aligned}$$

4–27. The wave function for the $n = 3$, $\ell = 0$ state in hydrogen is

$$\psi_{300} = \frac{1}{\sqrt{27\pi r_{\rm B}^3}}\left(1 - \frac{2r}{3r_{\rm B}} + \frac{2r^2}{27r_{\rm B}^2}\right)e^{-r/3r_{\rm B}}.$$

(a) Determine the radial probability distribution $P_{\rm r}$ for this state, and (b) draw the curve for it on a graph. (c) Determine the most probable distance from the nucleus for an electron in this state.

SOLUTION. (a) The $n = 3$, $\ell = 0$ wave function is

$$\psi_{300} = \frac{1}{\sqrt{27\pi r_{\rm B}^3}}\left(1 - \frac{2r}{3r_{\rm B}} + \frac{2r^2}{27r_{\rm B}^2}\right)e^{-r/3r_{\rm B}}.$$

The radial probability distribution is

$$P_{\rm r}(r) = 4\pi r^2|\psi_{300}|^2 = \frac{4r^2}{27r_{\rm B}^3}\left(1 - \frac{2r}{3r_{\rm B}} + \frac{2r^2}{27r_{\rm B}^2}\right)^2 e^{-2r/3r_{\rm B}}.$$

(b) See Fig. 4.8.

(c) To find the most probable distance from the nucleus, we find the maxima by setting the first derivative equal to zero. If we change variable to $x = r/r_{\rm B}$, we have

$$P_{\rm r}(x) = \frac{4}{27r_{\rm B}}\left(1 - \frac{2x}{3} + \frac{2x^2}{27}\right)^2 x^2 e^{-2x/3}.$$

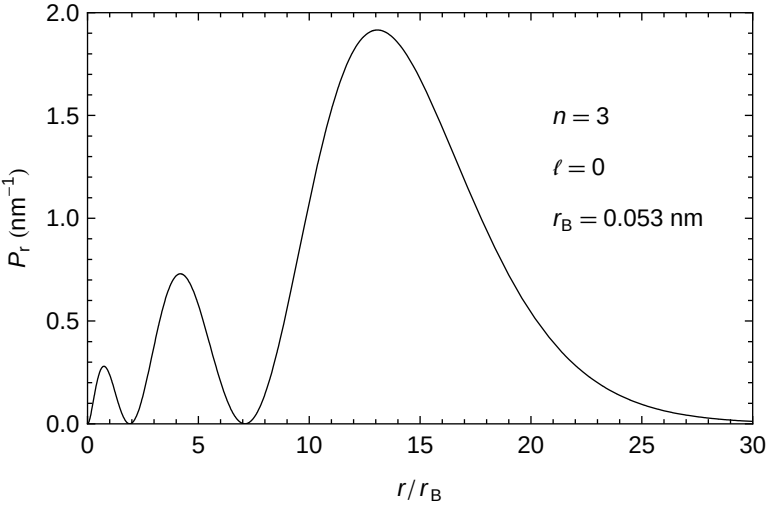


Figure 4.8. Problem 4–27.

If we suppress the constant in front, the derivative is

$$\begin{aligned}\frac{dP_r}{dx} &= \left(1-\frac{2x}{3}+\frac{2x^2}{27}\right)^2\left(2x-\frac{2x^2}{3}\right)e^{-2x/3} \\ &\quad + 2x^2e^{-2x/3}\left(1-\frac{2x}{3}+\frac{2x^2}{27}\right)\left(-\frac{2}{3}+\frac{4x}{27}\right)\end{aligned}$$

$$\begin{aligned}&= 2xe^{-2x/3}\left(1-\frac{2x}{3}+\frac{2x^2}{27}\right) \\ &\quad \times\left(\left(1-\frac{x}{3}\right)\left(1-\frac{2x}{3}+\frac{2x^2}{27}\right)+x\left(-\frac{2}{3}+\frac{4x}{27}\right)\right) \\ &= 2xe^{-2x/3}\left(1-\frac{2x}{3}+\frac{2x^2}{27}\right)\left(1-\frac{5x}{3}+\frac{12x^2}{27}-\frac{2x^3}{81}\right)=0.\end{aligned}$$

Thus we have

$$\text{(i)} \quad x = 0,$$

which is the minimum at $r = 0$;

$$\left(1 - \frac{2x}{3} + \frac{2x^2}{27}\right) = 0.$$

Because this is a factor in ψ, its two solutions are the other minima at $x = 1.90, 7.10$;

$$\text{(ii)} \quad 1 - \frac{5x}{3} + \frac{12x^2}{27} - \frac{2x^3}{81} = 0,$$

which can be solved numerically for the three solutions, which are $x = 0.74$, 4.19, 13.07. These correspond to the three peaks in the distribution. The highest peak is the most probable distance: $r = 13\, r_{\mathrm{B}}$.

4.4 Complex atoms. The exclusion principle. The periodic table

4–28. List the quantum numbers for each electron in the ground state of nitrogen ($Z = 7$).

SOLUTION. We start with the $n = 1$ shell, and list the quantum numbers in the order (n, ℓ, m_ℓ, m_s):

$$\begin{array}{llll} (1,0,0,-\tfrac{1}{2}), & (1,0,0,+\tfrac{1}{2}), & (2,0,0,-\tfrac{1}{2}), & (2,0,0,+\tfrac{1}{2}), \\ (2,1,-1,-\tfrac{1}{2}), & (2,1,-1,+\tfrac{1}{2}), & (2,1,0,-\tfrac{1}{2}). & \end{array}$$

Note that, without additional information, there are other possibilities for the last three electrons.

4–29. List the quantum numbers for each electron in the ground state of (a) carbon ($Z = 6$), (b) magnesium ($Z = 12$).

SOLUTION. (a) We start with the $n = 1$ shell, and list the quantum numbers in the order (n, ℓ, m_ℓ, m_s):

$$\begin{array}{llll} (1,0,0,-\tfrac{1}{2}), & (1,0,0,+\tfrac{1}{2}), & (2,0,0,-\tfrac{1}{2}), & (2,0,0,+\tfrac{1}{2}), \\ (2,1,-1,-\tfrac{1}{2}), & (2,1,-1,+\tfrac{1}{2}). & & \end{array}$$

Note that, without additional information, there are other possibilities for the last two electrons.

(b) We start with the $n = 1$ shell, and list the quantum numbers in the order $(n,\ \ell,\ m_\ell,\ m_s)$:

$$\begin{array}{cccc} (1,0,0,-\tfrac{1}{2}), & (1,0,0,+\tfrac{1}{2}), & (2,0,0,-\tfrac{1}{2}), & (2,0,0,+\tfrac{1}{2}), \\ (2,1,-1,-\tfrac{1}{2}), & (2,1,-1,+\tfrac{1}{2}), & (2,1,0,-\tfrac{1}{2}), & (2,1,0,+\tfrac{1}{2}), \\ (2,1,1,-\tfrac{1}{2}), & (2,1,1,+\tfrac{1}{2}), & (3,0,0,-\tfrac{1}{2}), & (3,0,0,+\tfrac{1}{2}). \end{array}$$

4–30. How many electrons can be in the $n = 6$, $\ell = 3$ subshell?

SOLUTION. The number of electrons in the subshell is determined by the value of ℓ. For each ℓ the value of m_ℓ can range from $-\ell$ to $+\ell$, or $2\ell + 1$ values. For each of these there are two values of m_s. Thus the total number for $\ell = 3$ is

$$N = 2(2\ell + 1) = 2\,(2(3) + 1) = 14$$

electrons.

4–31. What is the full electron configuration for (a) selenium (Se), (b) gold (Au), (c) uranium (U)? [*Hint*: See Appendix C.]

SOLUTION. (a) Selenium has $Z = 34$:

$$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^{10}\,4s^2\,4p^4.$$

(b) Gold has $Z = 79$:

$$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^{10}\,4s^2\,4p^6\,4d^{10}\,4f^{14}\,5s^2\,5p^6\,5d^{10}\,6s^1.$$

(c) Uranium has $Z = 92$:

$$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^{10}\,4s^2\,4p^6\,4d^{10}\,4f^{14}\,5s^2\,5p^6\,5d^{10}\,6s^2\,6p^6\,5f^3\,6d^1\,7s^2.$$

4–32. For each of the following atomic transitions, state whether the transition is *allowed* or *forbidden*, and why: (a) $4p \to 3p$; (b) $2p \to 1s$; (c) $3d \to 2d$; (d) $4d \to 3s$; (e) $4s \to 2p$.

SOLUTION. (a) The $4p \to 3p$ transition is forbidden, because $\Delta\ell \neq \pm 1$.

(b) The $2p \to 1s$ transition is allowed, $\Delta\ell = -1$.

(c) The $3d \to 2d$ transition is forbidden, because $\Delta\ell \neq \pm 1$.

(d) The $4d \to 3s$ transition is forbidden, because $\Delta\ell \neq \pm 1$.

(e) The $4s \to 2p$ transition is allowed, $\Delta\ell = +1$.

4–33. Using the Bohr formula for the radius of an electron orbit (Eq. 2.8), estimate the average distance from the nucleus for an electron in the innermost ($n = 1$) orbit in uranium ($Z = 92$). Approximately how much energy would be required to remove this innermost electron?

SOLUTION. From the Bohr formula for the radius, we see that $r \propto 1/Z$, so

$$r_{\mathrm{U}} = \frac{r_1}{Z} = \frac{(0.529 \times 10^{-10}\ \mathrm{m})}{92} = 5.8 \times 10^{-13}\ \mathrm{m}.$$

The innermost electron would "see" a nucleus with charge Ze. Thus we use the energy of the hydrogen atom:

$$E_1 = -\frac{(13.6\ \mathrm{eV})Z^2}{n^2} = -\frac{(13.6\ \mathrm{eV})(92)^2}{1^2} = -1.15 \times 10^5\ \mathrm{eV},$$

so the binding energy is 0.115 MeV.

4–34. Estimate the binding energy of the third electron in lithium using Bohr theory. [*Hint*: This electron has $n = 2$ and "sees" a net charge of approximately $+1e$.] The measured value is 5.36 eV.

SOLUTION. The third electron in lithium is in the $2s$ subshell, which is outside the more tightly bound filled $1s$ shell. This makes it appear as if there is a "nucleus" with a net charge of $+1e$. Thus we use the energy of the hydrogen atom:

$$E_2 = -\frac{(13.6\ \mathrm{eV})}{n^2} = -\frac{(13.6eV)}{2^2} = -3.4\ \mathrm{eV},$$

so the binding energy is 3.4 eV.

Our assumption of complete shielding of the nucleus by the $1s$ electrons is probably not correct.

4–35. Show that the total angular momentum is zero for a filled subshell.

SOLUTION. In a filled subshell, we have $2(2\ell + 1)$ electrons. All of the m_ℓ values

$$-\ell,\ -\ell + 1,\ \ldots\ ,\ 0,\ \ldots\ ,\ \ell - 1,\ \ell$$

are filled, so their sum is zero. For each m_ℓ value, both values of m_s are filled, so their sum is also zero. Thus the total angular momentum is zero.

4–36. Let us apply the exclusion principle to an infinitely high square well (Chapter 3). Let there be five electrons confined to this rigid box

whose width is L. Find the lowest energy state of this system, by placing the electrons in the lowest available levels, consistent with the Pauli exclusion principle.

SOLUTION. The energy levels of the infinite square well of width L are

$$E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}.$$

An electron in the well has two quantum numbers: n and $m_s = \pm\frac{1}{2}$. To be consistent with the Pauli exclusion principle, a maximum of two electrons can be in each level. The lowest energy state will have two electrons in the $n = 1$ state, two electrons in the $n = 2$ state, and one electron in the $n = 3$ state. The total energy is

$$E = 2E_1 + 2E_2 + E_3 = 2\frac{\pi^2\hbar^2 1^2}{2mL^2} + 2\frac{\pi^2\hbar^2 2^2}{2mL^2} + \frac{\pi^2\hbar^2 3^2}{2mL^2} = \frac{19\pi^2\hbar^2}{2mL^2}.$$

4.5 X-ray spectra and atomic number

4–37. What are the shortest-wavelength X-rays emitted by electrons striking the face of a 30-kV TV picture tube? What are the longest wavelengths?

SOLUTION. The shortest wavelength X-ray has the most energy, which is the maximum kinetic energy of the electron in the tube:

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{E} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{30 \times 10^3 \text{ eV}} = 0.041 \text{ nm}.$$

The longest wavelength of the continuous spectrum would be at the limit of the X-ray region of the electromagnetic spectrum, generally on the order of 1 nm.

4–38. If the shortest-wavelength bremsstrahlung X-rays emitted from an X-ray tube have $\lambda = 0.029$ nm, what is the voltage across the tube?

SOLUTION. The shortest wavelength X-ray has the most energy, which is the maximum kinetic energy of the electron in the tube:

$$E = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{0.029 \text{ nm}} = 4.3 \times 10^4 \text{ eV} = 43 \text{ keV}.$$

Thus the operating voltage of the tube is 43 kV.

4–39. Show that the cutoff wavelength λ_0 is given by

$$\lambda_0 = \frac{(1\,240 \text{ nm})}{V},$$

where V is the X-ray tube voltage in volts.

SOLUTION. The energy of the photon with the shortest wavelength must equal the maximum kinetic energy of an electron:

$$2\pi\hbar f_0 = \frac{2\pi\hbar c}{\lambda_0} = eV,$$

or

$$\begin{aligned} \lambda_0 &= \frac{2\pi\hbar c}{eV} = \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})(3.00 \times 10^8 \text{ m/s})(10^9 \text{ nm/m})}{(1.60 \times 10^{-19} \text{ J/eV})(1 \text{ e})V} \\ &= (1.24 \times 10^3 \text{ V} \cdot \text{nm})/V. \end{aligned}$$

4–40. Use the result of Example 4–6 to estimate the X-ray wavelength emitted when a cobalt atom ($Z = 27$) jumps from $n = 2$ to $n = 1$.

SOLUTION. With the shielding provided by the remaining $n = 1$ electron, we use the energies of the hydrogen atom with Z replaced by $Z - 1$. The energy of the photon is

$$2\pi\hbar f = \Delta E = -(13.6 \text{ eV})(27-1)^2 \left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 6.90 \times 10^3 \text{ eV}.$$

The wavelength of the photon is

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\Delta E} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{6.90 \times 10^3 \text{ eV}} = 0.18 \text{ nm}.$$

4–41. Estimate the wavelength for an $n = 2$ to $n = 1$ transition in iron ($Z = 26$).

SOLUTION. With the shielding provided by the remaining $n = 1$ electron, we use the energies of the hydrogen atom with Z replaced by $Z - 1$. The energy of the photon is

$$2\pi\hbar f = \Delta E = -(13.6 \text{ eV})(26-1)^2 \left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 6.38 \times 10^3 \text{ eV}.$$

The wavelength of the photon is

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\Delta E} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{6.38 \times 10^3 \text{ eV}} = 0.19 \text{ nm}.$$

4–42. Use Bohr theory to estimate the wavelength for an $n = 3$ to $n = 1$ transition in molybdenum. The measured value is 0.063 nm. Why do we not expect perfect agreement?

SOLUTION. If we assume that the shielding is provided by the remaining $n = 1$ electron, we use the energies of the hydrogen atom with Z replaced by $Z - 1$. The energy of the photon is

$$2\pi\hbar f = \Delta E = -(13.6 \text{ eV})(42 - 1)^2 \left(\frac{1}{3^2} - \frac{1}{1^2}\right) = 2.03 \times 10^4 \text{ eV}.$$

The wavelength of the photon is

$$\lambda = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\Delta E} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{2.03 \times 10^4 \text{ eV}} = 0.061 \text{ nm}.$$

We do not expect perfect agreement because there is some partial shielding provided by the $n = 2$ shell, which was ignored when we replaced Z by $Z - 1$.

4–43. A mixture of iron and an unknown material are bombarded with electrons. The wavelength of the K_α lines are 194 pm for iron and 229 pm for the unknown. What is the unknown material?

SOLUTION. The K_α line is from the $n = 2$ to $n = 1$ transition. We use the energies of the hydrogen atom with Z replaced by $Z - 1$. Thus we have

$$2\pi\hbar f = \Delta E \propto (Z - 1)^2,$$

so

$$\lambda \propto (Z - 1)^{-2}.$$

When we form the ratio for the two materials, we get

$$\frac{\lambda_X}{\lambda_{\text{iron}}} = \frac{(Z_{\text{iron}} - 1)^2}{(Z_X - 1)^2}; \quad \frac{229 \text{ pm}}{194 \text{ pm}} = \frac{(26 - 1)^2}{(Z_X - 1)^2},$$

which gives $Z_X = 24$, so the material is chromium.

4–44. Use conservation of energy and momentum to show that a moving electron cannot give off an X-ray photon unless there is a third body present, such as an atom or nucleus.

SOLUTION. For the emission of a photon, energy and momentum are conserved in any reference frame. If we consider the frame in which the electron is initially at rest, its momentum is 0 and its energy is mc^2. The emitted photon will have momentum $2\pi\hbar/\lambda = 2\pi\hbar f/c$ and energy $2\pi\hbar f$. The final momentum of the electron will be opposite to that of the photon with magnitude p. Thus we have

$$\begin{aligned} \text{momentum conservation}: \quad 0 &= p - \frac{2\pi\hbar f}{c}; \\ \text{energy conservation}: \quad mc^2 &= \sqrt{p^2c^2 + m^2c^4} + 2\pi\hbar f. \end{aligned}$$

When we combine the two equations to eliminate p, we get

$$(mc^2)^2 - 4\pi\hbar f mc^2 + (2\pi\hbar f)^2 = (2\pi\hbar f)^2 + m^2c^4,$$

which gives $\hbar f = 0$. Thus no photon is emitted. If a third body is present, some momentum can be transferred to it; a photon can be emitted.

4.6 Magnetic dipole moments. Total angular momentum

4–45. Verify that the Bohr magneton has the value $\mu_B = 9.27 \times 10^{-24}$ J/T (see Eq. 4.12).

SOLUTION. For the Bohr magneton we have

$$\mu_B = \frac{e\hbar}{2m} = \frac{(1.60 \times 10^{-19}\ \text{C})(1.055 \times 10^{-34}\ \text{J}\cdot\text{s})}{2(9.11 \times 10^{-31}\ \text{kg})} = 9.27 \times 10^{-24}\ \text{J/T}.$$

4–46. Suppose that the splitting of energy levels shown in Fig. 4.2a was produced by a 2.0-T magnetic field. (a) What is the separation in energy between adjacent m_ℓ levels for the same ℓ? (b) How many different wavelengths will there be for $3d$ to $2p$ transitions, if m_ℓ can change only by ± 1 or 0? (c) What is the wavelength for each of these transitions?

SOLUTION. (a) The additional energy term in a magnetic field is

$$U = -\mu_z B = -\mu_B m_\ell B.$$

Thus the separation of energy levels is

$$\begin{aligned}\Delta U &= \Delta m_\ell \mu_{\rm B} B = (1)(9.27\times10^{-24}\ {\rm J/T})(2.0\ {\rm T}) = 1.9\times10^{-23}\ {\rm J}\\ &= 1.2\times10^{-4}\ {\rm eV}.\end{aligned}$$

(b) The $3d$ level, with $\ell = 2$, will split into 5 levels. The $2p$ level, with $\ell = 1$, will split into 3 levels. With the restriction that $\Delta m_\ell = 0, \pm1$, there will be only 3 wavelengths from the 9 transitions, as shown on the diagram (Fig. 4.2b).

(c) If we assume the hydrogen energy levels, we find the wavelength for the $n' = 3$ to $n = 2$ transition, which is also the wavelength for $\Delta m_\ell = 0$, from

$$\frac{1}{\lambda_0} = R_\infty\left(\frac{1}{n^2} - \frac{1}{n'^2}\right) = (1.0974\times10^7\ {\rm m}^{-1})\left(\frac{1}{2^2} - \frac{1}{3^2}\right),$$

which gives $\lambda_0 = 656.10\times10^{-9}$ m $= 656.10$ nm. Because the splitting of the levels is much smaller than their separation, we can find the wavelength change from

$$\begin{aligned}E &= \frac{2\pi\hbar c}{\lambda};\\ \Delta E &= -\frac{2\pi\hbar c}{\lambda^2}\Delta\lambda = -\frac{2\pi\hbar c}{\lambda}\frac{\Delta\lambda}{\lambda};\\ \pm1.2\times10^{-4}\ {\rm eV} &= -\frac{(1.24\times10^3\ {\rm eV\cdot nm})}{(656.10\ {\rm nm})}\frac{\Delta\lambda}{(656.10\ {\rm nm})},\end{aligned}$$

which gives $\Delta\lambda = \pm0.04$ nm. Thus the wavelengths are 656.06 nm, 656.10 nm, 656.14 nm.

4–47. In a Stern–Gerlach experiment, Ag atoms issued from the oven with an average speed of 700 m/s and passed through a magnetic field gradient $dB/dz = 1.5\times10^3$ T/m for a distance of 4.0 cm. (a) What is the separation of the two beams as they emerge from the magnet? (b) What would the separation be if the g-factor were 1 for electron spin?

Solution. (a) We take the original direction of the atomic beam as the x-axis and the direction of the force from the magnetic field gradient as the z-axis. The force is

$$F_z = \mu_z\frac{dB}{dz} = -g\mu_{\rm B}m_s\frac{dB}{dz}.$$

This constant force will produce a constant acceleration. The time to traverse the field is

$$t = \frac{x}{v_0} = \frac{(4.0 \times 10^{-2}\ \text{m})}{(700\ \text{m/s})} = 5.71 \times 10^{-5}\ \text{s},$$

so the deflection of one of the beams is

$$\begin{aligned} z &= \frac{at^2}{2} = \frac{F_z t^2}{2m} = \frac{g\mu_{\text{B}} m_s\,(dB/dz)\,t^2}{2m} \\ &= \frac{(2.0023)(9.27 \times 10^{-24}\ \text{J/T})(\frac{1}{2})(1.5 \times 10^{3}\ \text{T/m})(5.71 \times 10^{-5}\ \text{s})^2}{2(107.9)(1.66 \times 10^{-27}\ \text{kg})} \\ &= 1.26 \times 10^{-4}\ \text{m}. \end{aligned}$$

Thus the separation is

$$2z = 2.52 \times 10^{-4}\ \text{m} = 0.25\ \text{mm}.$$

(b) Because the separation is proportional to g, we have

$$2z' = \frac{2.52 \times 10^{-4}\ \text{m}}{2.0023} = 1.26 \times 10^{-4}\ \text{m} = 0.13\ \text{mm}.$$

4–48. (a) Write down the quantum numbers for each electron in the aluminum atom, (b) Which subshells are filled? (c) The last electron is in the $3p$ state; what are the possible values of the total angular momentum quantum number, j, for this electron? (d) Explain why the angular momentum of this last electron also represents the total angular momentum for the entire atom (ignoring any angular momentum of the nucleus). (e) How could you use a Stern–Gerlach experiment to determine which value of j the atom has?

SOLUTION. (a) Aluminum has $Z = 13$. We start with the $n = 1$ shell, and list the quantum numbers in the order $(n,\ \ell,\ m_\ell,\ m_s)$:

$$\begin{array}{ccccc} (1,0,0,-\frac{1}{2}), & (1,0,0,+\frac{1}{2}), & (2,0,0,-\frac{1}{2}), & (2,0,0,+\frac{1}{2}), & (2,1,-1,-\frac{1}{2}), \\ (2,1,-1,+\frac{1}{2}), & (2,1,0,-\frac{1}{2}), & (2,1,0,+\frac{1}{2}), & (2,1,1,-\frac{1}{2}), & (2,1,1,+\frac{1}{2}), \\ (3,0,0,-\frac{1}{2}), & (3,0,0,+\frac{1}{2}), & (3,1,0,-\frac{1}{2}). & & \end{array}$$

Note that, without additional information, there are other possibilities for the last electron.

(b) The filled subshells are $n = 1$, $\ell = 0$; $n = 2$, $\ell = 0$; $n = 2$, $\ell = 1$; $n = 3$, $\ell = 0$.

(c) The possible values of the total angular momentum of the last electron are

$$j = \ell + s = 1 \pm \frac{1}{2} = \frac{1}{2},\ \frac{3}{2}.$$

(d) The $3p$ electron is the only electron not in a filled subshell. The angular momentum of a filled subshell is zero, so the total angular momentum of the atom is the angular momentum of the $3p$ electron.

(e) When the beam passes through the magnetic field gradient, the deflecting force will be proportional to m_j. If $j = \frac{1}{2}$, the values of m_j are $\pm\frac{1}{2}$, and there will be two lines. If $j = \frac{3}{2}$, the values of m_j are $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, and there will be four lines. Thus the number of lines shows the value of m_j.

4–49. What are the possible values of j for an electron in (a) the $3p$, (b) the $4f$, and (c) the $4d$ state of hydrogen? What is J in each case?

SOLUTION. (a) For the $3p$ state, $\ell = 1$ and $s = \frac{1}{2}$. Thus the values of j are

$$j = \ell \pm s = 1 \pm \tfrac{1}{2} = \tfrac{1}{2},\ \tfrac{3}{2}.$$

The values of J are

$$\begin{aligned} J &= \sqrt{j(j+1)}\,\hbar = \sqrt{\tfrac{1}{2}\,\tfrac{3}{2}}\,\hbar = \tfrac{1}{2}\sqrt{3}\,\hbar; \\ J &= \sqrt{j(j+1)}\,\hbar = \sqrt{\tfrac{3}{2}\,\tfrac{5}{2}}\,\hbar = \tfrac{1}{2}\sqrt{15}\,\hbar. \end{aligned}$$

(b) For the $4f$ state, $\ell = 3$ and $s = \frac{1}{2}$. Thus the values of j are

$$j = \ell \pm s = 3 \pm \tfrac{1}{2} = \tfrac{5}{2},\ \tfrac{7}{2}.$$

The values of J are

$$\begin{aligned} J &= \sqrt{j(j+1)}\,\hbar = \sqrt{\tfrac{5}{2}\,\tfrac{7}{2}}\,\hbar = \tfrac{1}{2}\sqrt{35}\,\hbar; \\ J &= \sqrt{j(j+1)}\,\hbar = \sqrt{\tfrac{7}{2}\,\tfrac{9}{2}}\,\hbar = \tfrac{1}{2}\sqrt{63}\,\hbar. \end{aligned}$$

(c) For the $4d$ state, $\ell = 2$ and $s = \frac{1}{2}$. Thus the values of j are

$$j = \ell \pm s = 2 \pm \tfrac{1}{2} = \tfrac{3}{2},\ \tfrac{5}{2}.$$

The values of J are

$$\begin{aligned} J &= \sqrt{j(j+1)}\,\hbar = \sqrt{\tfrac{3}{2}\,\tfrac{5}{2}}\,\hbar = \tfrac{1}{2}\sqrt{15}\,\hbar; \\ J &= \sqrt{j(j+1)}\,\hbar = \sqrt{\tfrac{5}{2}\,\tfrac{7}{2}}\,\hbar = \tfrac{1}{2}\sqrt{35}\,\hbar. \end{aligned}$$

4–50. For an electron in a $5g$ state, what are all the possible values of j, m_j, J, and J_z?

Solution. For the $5g$ state, $\ell = 4$ and $s = \frac{1}{2}$. Thus the values of j are

$$j = \ell \pm s = 4 \pm \tfrac{1}{2} = \tfrac{7}{2},\ \tfrac{9}{2}.$$

For $j = \frac{7}{2}$, we have

$$\begin{aligned} m_j &= -\tfrac{7}{2},\ -\tfrac{5}{2},\ -\tfrac{3}{2},\ -\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{2},\ \tfrac{5}{2},\ \tfrac{7}{2}; \\ J &= \sqrt{j(j+1)}\,\hbar = \sqrt{\tfrac{7}{2}\,\tfrac{9}{2}}\,\hbar = \tfrac{3}{2}\sqrt{7}\,\hbar; \\ J_z &= m_j\hbar = -\tfrac{7}{2}\hbar,\ -\tfrac{5}{2}\hbar,\ -\tfrac{3}{2}\hbar,\ -\tfrac{1}{2}\hbar,\ \tfrac{1}{2}\hbar,\ \tfrac{3}{2}\hbar,\ \tfrac{5}{2}\hbar,\ \tfrac{7}{2}\hbar. \end{aligned}$$

For $j = \frac{9}{2}$, we have

$$\begin{aligned} m_j &= -\tfrac{9}{2},\ -\tfrac{7}{2},\ -\tfrac{5}{2},\ -\tfrac{3}{2},\ -\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{2},\ \tfrac{5}{2},\ \tfrac{7}{2},\ \tfrac{9}{2}; \\ J &= \sqrt{j(j+1)}\,\hbar = \sqrt{\tfrac{9}{2}\,\tfrac{11}{2}}\,\hbar = \tfrac{3}{2}\sqrt{11}\,\hbar; \\ J_z &= m_j\hbar = -\tfrac{9}{2}\hbar,\ -\tfrac{7}{2}\hbar,\ -\tfrac{5}{2}\hbar,\ -\tfrac{3}{2}\hbar,\ -\tfrac{1}{2}\hbar,\ \tfrac{1}{2}\hbar,\ \tfrac{3}{2}\hbar,\ \tfrac{5}{2}\hbar,\ \tfrac{7}{2}\hbar,\ \tfrac{9}{2}\hbar. \end{aligned}$$

4–51. The difference between the $2P_{3/2}$ and $2P_{1/2}$ energy levels in hydrogen is about 5×10^5 eV, due to the spin-orbit interaction. (a) Taking the electron's (orbital) magnetic moment to be 1 Bohr magneton, estimate the internal magnetic field due to the electron's orbital motion, (b) Estimate the internal magnetic field using a simple model of the nucleus revolving in a circle about the electron.

Solution. (a) The additional energy term for the spin-orbit interaction is

$$U = -\mu_s B_n = -g\mu_{\mathrm{B}} m_s B_n.$$

Thus the separation of energy levels is

$$\begin{aligned} \Delta U &= \Delta m_s g\mu_{\mathrm{B}} B_n; \\ (5\times10^{-5}\ \mathrm{eV})(1.60\times10^{-19}\ \mathrm{J/eV}) &= \left(\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right)(2.0023) \\ &\quad \times (9.27\times10^{-24}\ \mathrm{J/T})B_n, \end{aligned}$$

which gives $B_n = 0.4$ T.

(b) If we consider the nucleus to be a charge e revolving in a circle of radius r, the effective current is

$$I = \frac{e}{(2\pi r/v)} = \frac{ev}{2\pi r} = \frac{m_e evr}{2\pi m_e r^2} = \frac{eL}{2\pi m_e r^2},$$

where L is the orbital angular momentum of the electron, because the nucleus has the same v and r. The electron is at the center of this circular current, so the magnetic field is

$$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 eL}{4\pi m_e r^3}.$$

If we use the Bohr quantization, $L = n\hbar$, and $r = n^2 r_B$, we have

$$\begin{aligned} B &= \frac{\mu_0 en\hbar}{4\pi m_e n^6 r_0^3} = \frac{(\mu_0/4\pi)\, e\hbar}{m_e n^5 r_0^3} \\ &= \frac{(10^{-7}\ \text{T}\cdot\text{m/A})(1.60\times10^{-19}\ \text{C})(1.055\times10^{-34}\ \text{J}\cdot\text{s})}{(9.11\times10^{-31}\ \text{kg})(2)^5(0.529\times10^{-10}\ \text{m})^3} \\ &= 0.4\ \text{T}. \end{aligned}$$

This is consistent with the result from part (a).

4.7 Lasers

4–52. A laser used to weld detached retinas puts out 25-ms-long pulses of 640-nm light which average 0.64 W output during a pulse. How much energy can be deposited per pulse and how many photons does each pulse contain?

SOLUTION. The energy of a pulse is

$$E = P\Delta t = (0.65\ \text{W})(25\times10^{-3}\ \text{s}) = 0.016\ \text{J}.$$

The energy of a photon is

$$2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{1.24\times10^3\ \text{eV}\cdot\text{nm}}{\lambda} = \frac{1.24\times10^3\ \text{eV}\cdot\text{nm}}{640\ \text{nm}} = 1.94\ \text{eV}.$$

Thus the number of photons in a pulse is

$$N = \frac{E}{2\pi\hbar f} = \frac{0.016\ \text{J}}{(1.94\ \text{eV})(1.6\times10^{-19}\ \text{J/eV})} = 5.2\times10^{16}\ \text{photons}.$$

4–53. Estimate the angular spread of a laser beam due to diffraction if the beam emerges through a 3.0-mm-diameter mirror. Assume that $\lambda = 694$ nm. What would be the diameter of this beam if it struck a satellite 300 km above the Earth?

SOLUTION. We find the angular half width of the beam from

$$\Delta\theta = \frac{1.22\,\lambda}{d} = \frac{1.22(694 \times 10^{-9}\text{ m})}{3.0 \times 10^{-3}\text{ m}} = 2.8 \times 10^{-4}\text{ rad},$$

so the angular width is $\theta = 5.6 \times 10^{-4}$ rad. The diameter of the beam when it reaches the satellite is

$$D = r\theta = (300 \times 10^{3}\text{ m})(5.6 \times 10^{-4}\text{ rad}) = 1.7 \times 10^{2}\text{ m}.$$

4–54. Suppose that the energy level system in Fig. 4.7 is not not being pumped and is in thermal equilibrium. Determine the fraction of atoms in levels E_2 and E_1 relative to E_0 at $T = 300$ K.

SOLUTION. In thermal equilibrium we find the fraction of atoms relative to those in E_0 from

$$\begin{aligned}
\frac{N_2}{N_0} &= e^{-\Delta E/k_{\text{B}}T} = \exp\left(-\frac{(2.2\text{ eV})(1.60 \times 10^{-19}\text{ J/eV})}{(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}\right) \\
&= 1.2 \times 10^{-37}; \\
\frac{N_1}{N_0} &= e^{-\Delta E/k_{\text{B}}T} = \exp\left(-\frac{(1.8\text{ eV})(1.60 \times 10^{-19}\text{ J/eV})}{(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}\right) \\
&= 6.1 \times 10^{-31}.
\end{aligned}$$

4–55. To what temperature would the system in Fig. 4.7 have to be raised (see Problem 4–54) so that in thermal equilibrium the level E_2 would have half as many atoms as E_0. (Note that pumping mechanisms do not maintain thermal equilibrium.)

SOLUTION. We find the temperature from

$$\begin{aligned}
\frac{N_2}{N_0} &= e^{-\Delta E/k_{\text{B}}T}; \\
\frac{1}{2} &= \exp\left(-\frac{(2.2\text{ eV})(1.60 \times 10^{-19}\text{ J/eV})}{(1.38 \times 10^{-23}\text{ J/K})T}\right),
\end{aligned}$$

which gives $T = 3.7 \times 10^4$ K.

4–56. Show that a population inversion for two levels (as in a pumped laser) corresponds to a negative Kelvin temperature in the Boltzmann distribution. Explain why such a situation does not contradict the idea that negative Kelvin temperatures cannot be reached in the normal sense of temperature.

SOLUTION. If we assume the population follows the Boltzmann distribution, we have

$$\frac{N_2}{N_1} = e^{-\Delta E/k_{\rm B}T};$$

or

$$\ln\left(\frac{N_2}{N_1}\right) = -\frac{\Delta E}{k_{\rm B}T}.$$

If $N_2/N_1 > 1$, then $\ln(N_2/N_1) > 0$. Thus $T < 0$. There is no contradiction because the system is not in thermal equilibrium.

4.8 General problems

4–57. The ionization (binding) energy of the outermost electron in boron is 8.26 eV. (a) Use the Bohr model to estimate the "effective charge," $Z_{\rm eff}$, seen by this electron. (b) Estimate the average orbital radius.

SOLUTION. (a) Boron has $Z = 4$, so the outermost electron has $n = 2$. We use the Bohr result with an effective Z:

$$\begin{aligned} E_2 &= -\frac{(13.6\text{ eV})(Z_{\rm eff})^2}{n^2}; \\ -8.26\text{ eV} &= -\frac{(13.6\text{ eV})(Z_{\rm eff})^2}{2^2}, \end{aligned}$$

which gives $Z_{\rm eff} = 1.56$. Note that this indicates some shielding by the second electron in the $n = 2$ shell.

(b) We find the average radius from

$$r = \frac{n^2 r_1}{Z_{\rm eff}} = \frac{2^2(0.529 \times 10^{-10}\text{ m})}{1.56} = 1.4 \times 10^{-10}\text{ m}.$$

4–58. Show that there can be 18 electrons in a "g" subshell.

SOLUTION. From *spdfg*, we see that the "g" subshell has $\ell = 4$, so the number of electrons is

$$N = 2(2\ell + 1) = 2\,(2(4) + 1) = 18 \text{ electrons}.$$

4–59. What is the full electron configuration in the ground state for elements with Z equal to (a) 27, (b) 36, (c) 38? [*Hint*: See Appendix C.]

SOLUTION. (a) For $Z = 27$ we start with hydrogen and fill the levels as indicated in the periodic table:

$$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^7\,4s^2.$$

Note that the $4s^2$ level is filled before the $3d$ level is completed.

(b) For $Z = 36$ we have

$$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^{10}\,4s^2\,4p^6.$$

(c) For $Z = 38$ we have

$$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,3d^{10}\,4s^2\,4p^6\,5s^2.$$

Note that the $5s^2$ level is filled before the $4d$ level is started.

4–60. What are the largest and smallest possible values for the angular momentum L of an electron in the $n = 5$ shell?

SOLUTION. The value of ℓ can range from 0 to $n - 1$. Thus for $n = 5$, we have $\ell = 4$. The largest magnitude of L is

$$\begin{aligned} L &= \hbar\sqrt{\ell(\ell+1)} = (1.055 \times 10^{-34}\ \mathrm{J\cdot s})\sqrt{(4)(4+1)} \\ &= 4.72 \times 10^{-34}\ \mathrm{kg\cdot m^2/s}. \end{aligned}$$

The smallest value of L is 0.

4–61. Estimate (a) the quantum number ℓ for the orbital angular momentum of the Earth about the Sun, and (b) the number of possible orientations for the plane of Earth's orbit.

SOLUTION. (a) We find the quantum number for the orbital angular momentum from

$$\begin{aligned} L = M_{\mathrm{earth}} v R = \frac{M_{\mathrm{earth}} 2\pi R^2}{T} &= \hbar\sqrt{\ell(\ell+1)}; \\ \frac{(5.98\times10^{24}\ \mathrm{kg})2\pi(1.50\times10^{11}\ \mathrm{m})^2}{(3.16\times10^{7}\ \mathrm{s})} &= (1.055\times10^{-34}\ \mathrm{J\cdot s})\sqrt{\ell(\ell+1)}, \end{aligned}$$

which gives $\ell = 2.5 \times 10^{74}$.

(b) The value of m_ℓ can range from $-\ell$ to $+\ell$, or $2\ell + 1$ values, so the number of orientations is

$$N = 2\ell + 1 = 2(2.5 \times 10^{74}) + 1 = 5.0 \times 10^{74}.$$

4–62. Use the Bohr theory (especially Eq. 2.11) to show that the Moseley plot (Fig. 4.6) can be written

$$\lambda^{-1/2} = a(Z - b),$$

where $b \approx 1$, and evaluate a.

SOLUTION. The wavelength of the photon emitted for the transition from n' to n is

$$\frac{1}{\lambda} = \frac{Z^2 k_e^2 e^4 m_e}{4\pi\hbar^3 c}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right) = R_\infty Z^2\left(\frac{1}{n^2} - \frac{1}{n'^2}\right).$$

The K_α line is from the $n' = 2$ to $n = 1$ transition, and the other $n = 1$ electron shields the nucleus, so the effective Z is $Z - 1$:

$$\frac{1}{\lambda} = R_\infty Z_{\text{eff}}^2\left(\frac{1}{n^2} - \frac{1}{n'^2}\right) = R_\infty (Z-1)^2\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = \frac{3}{4}R_\infty (Z-1)^2.$$

Thus we have

$$\frac{1}{\sqrt{\lambda}} = (Z-1)\sqrt{\frac{3R_\infty}{4}},$$

which is the equation of a straight line, as in the Moseley plot, with $b = 1$. The value of a is

$$a = \sqrt{\tfrac{3}{4}R_\infty} = \sqrt{\tfrac{3}{4}(1.0974 \times 10^7\ \text{m}^{-1})} = 2\,869\ \text{m}^{-1/2}.$$

4–63. Determine the most probable distance from the nucleus of an electron in the $n = 2$, $\ell = 0$ state of hydrogen.

SOLUTION. The $n = 2$, $\ell = 0$ wave function is

$$\psi_{200} = \frac{1}{\sqrt{32\pi r_B^3}}\left(2 - \frac{r}{r_B}\right)e^{-r/2r_B}.$$

The radial probability density is

$$P_{\rm r}(r) = 4\pi r^2 |\psi_{200}|^2 = \frac{r^2}{8r_{\rm B}^3}\left(2 - \frac{r}{r_{\rm B}}\right)^2 e^{-r/r_{\rm B}}.$$

To find the most probable distance from the nucleus, we find the maxima by setting the first derivative equal to zero. If we change variable to $x = r/r_{\rm B}$, we have

$$P_{\rm r}(x) = \frac{1}{8r_{\rm B}} x^2 (2-x)^2 e^{-x}.$$

If we suppress the constant in front, the derivative is

$$\begin{aligned} \frac{dP_{\rm r}}{dx} &= 2x(2-x)^2 e^{-x} - 2x^2(2-x)e^{-x} - x^2(2-x)^2 e^{-x} \\ &= x(2-x)(x^2 - 6x + 4)e^{-x} = 0. \end{aligned}$$

Thus we have

$$x = 0,$$

which is the minimum at $r = 0$;

$$2 - x = 0.$$

Because this is a factor in ψ, this is the other minimum at $x = 2$.

$$x^2 - 6x + 4 = 0,$$

with solutions $x = 0.76,\ 5.24$. These correspond to the two peaks in the distribution. The highest peak is the most probable distance:

$$r = 5.24\, r_{\rm B}.$$

4–64. In the so-called *vector model* of the atom, space quantization of angular momentum (Fig. 4.1) is illustrated as shown in Fig. 4.9. The angular momentum vector of magnitude $L = \hbar\sqrt{\ell(\ell+1)}$ is thought of as precessing around the z axis (like a spinning top or gyroscope) in such a way that the z component of angular momentum, $L_z = m_\ell \hbar$, also stays constant. Calculate the possible values for the angle θ between $\mathbf{L}$ and the z axis (a) for $\ell = 1$, (b) $\ell = 2$, and (c) $\ell = 3$. (d) Determine the

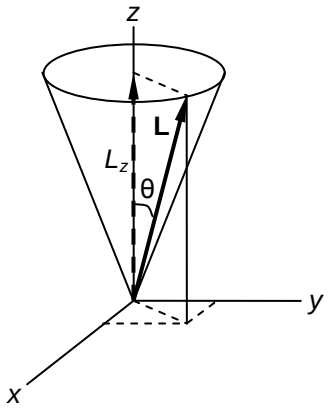


Figure 4.9. Problems 4–64 and 4–65. The vector model for orbital angular momentum. The orbital momentum vector **L** is imagined to precess about the z axis; L and L_z remain constant, but L_x and L_y continually change.

minimum value of θ for $\ell = 100$ and $\ell = 10^6$. Is this consistent with the correspondence principle?

SOLUTION. The magnitude of the angular momentum is given by

$$L = \hbar\sqrt{\ell(\ell+1)}\,,$$

and the z-component is given by

$$L_z = m_\ell \hbar.$$

Thus the angle between **L** and the z-axis is given by

$$\cos\theta = \frac{L_z}{L} = \frac{m_\ell \hbar}{\sqrt{\ell(\ell+1)}\,\hbar} = \frac{m_\ell}{\sqrt{\ell(\ell+1)}}\,.$$

For each ℓ the value of m_ℓ can range from $-\ell$ to $+\ell$, so there are $2\ell + 1$ values for θ.

(a) For $\ell = 1$ the magnitude of **L** is

$$L = \hbar\sqrt{(1)(1+1)} = \sqrt{2}\,\hbar.$$

The angles for the 3 values of m_ℓ are

$$\begin{aligned} \cos\theta_{1,3} &= \tfrac{\pm 1}{\sqrt{2}} = \pm\tfrac{1}{\sqrt{2}}, \qquad \text{or } \theta_1 = 45^\circ,\ \theta_3 = 135^\circ; \\ \cos\theta_2 &= \tfrac{0}{\sqrt{2}} = 0, \qquad \text{or } \theta_2 = 90^\circ. \end{aligned}$$

Thus the possible values for $\ell = 1$ are 45°, 90°, 135°.

(b) For $\ell = 2$ the magnitude of $\mathbf{L}$ is

$$L = \hbar\sqrt{(2)(2+1)} = \sqrt{6}\,\hbar.$$

The angles for the 5 values of m_ℓ are

$$\begin{aligned} \cos\theta_{1,5} &= \tfrac{\pm 2}{\sqrt{6}} = \pm\sqrt{\tfrac{2}{3}}, \qquad \text{or } \theta_1 = 35.3^\circ,\ \theta_5 = 144.7^\circ; \\ \cos\theta_{2,4} &= \tfrac{\pm 1}{\sqrt{6}} = \pm\sqrt{\tfrac{1}{6}}, \qquad \text{or } \theta_2 = 65.9^\circ,\ \theta_4 = 114.1^\circ; \\ \cos\theta_3 &= \tfrac{0}{\sqrt{6}} = 0, \qquad \text{or } \theta_3 = 90^\circ. \end{aligned}$$

Thus the possible values for $\ell = 2$ are 35.3°, 65.9°, 90°, 114.1°, 144.7°.

(c) For $\ell = 3$ the magnitude of $\mathbf{L}$ is

$$L = \hbar\sqrt{(3)(3+1)} = \sqrt{12}\,\hbar.$$

The angles for the 7 values of m_ℓ are

$$\begin{aligned} \cos\theta_{1,7} &= \tfrac{\pm 3}{\sqrt{12}} = \pm\sqrt{\tfrac{3}{4}}, \qquad \text{or } \theta_1 = 30.0^\circ,\ \theta_7 = 150.0^\circ; \\ \cos\theta_{2,6} &= \tfrac{\pm 2}{\sqrt{12}} = \pm\sqrt{\tfrac{1}{3}}, \qquad \text{or } \theta_2 = 54.7^\circ,\ \theta_6 = 125.3^\circ; \\ \cos\theta_{3,5} &= \tfrac{\pm 1}{\sqrt{12}} = \pm\sqrt{\tfrac{1}{12}}, \qquad \text{or } \theta_3 = 73.2^\circ,\ \theta_5 = 106.8^\circ; \\ \cos\theta_4 &= \tfrac{0}{\sqrt{12}} = 0, \qquad \text{or } \theta_4 = 90^\circ. \end{aligned}$$

Thus the possible values for $\ell = 3$ are 30.0°, 54.7°, 73.2°, 90°, 106.8°, 125.3°, 150.0°.

(d) The minimum value of θ has the largest value of m_ℓ. For $\ell = 100$, we have

$$\cos\theta_{\min} = \frac{100}{\sqrt{(100)(100+1)}} = 0.995,$$

or

$$\theta_{\min} = 5.71^\circ.$$

For $\ell = 10^6$, we have

$$\begin{aligned} \cos\theta_{\min} &= \frac{10^6}{\sqrt{(10^6)(10^6+1)}} = \frac{1}{\sqrt{1+10^{-6}}} \\ &\approx 1 - \tfrac{1}{2}(10^{-6}) = 0.9999995, \end{aligned}$$

or

$$\theta_{\min} = 0.0573^\circ.$$

This is consistent with the correspondence principle; **L** could align with the z-axis when $n \to \infty$.

4–65. The vector model (Problem 4–64) gives some insight into the uncertainty principle for angular momentum, which is

$$\Delta L_z \, \Delta\varphi \geqslant \frac{\hbar}{2}$$

for the z component. Here φ is the angular position measured in the plane perpendicular to the z axis. Once m_ℓ for an atom is known, L_z is known precisely, so $\Delta L_z = 0$. (a) What does this tell us about φ? (b) What can you say about L_x and L_y, which are not quantized (only L and L_z are)? (c) Show that although L_x and L_y are not quantized, nonetheless $\sqrt{L_x^2 + L_y^2} = \hbar\sqrt{\ell(\ell+1) - m_\ell^2}$ is.

Solution. (a) From $\Delta L_z \, \Delta\varphi \geqslant \hbar/2$, we see that if $\Delta L_z = 0$, $\Delta\varphi \to \infty$. Thus the angle φ is unknown.

(b) If φ is unknown, then the direction of the component of **L** perpendicular to the z-axis is unknown, so L_x and L_y are unknown.

(c) From $L^2 = L_x^2 + L_y^2 + L_z^2$, we have

$$\ell(\ell+1)\hbar^2 = L_x^2 + L_y^2 + m_\ell^2\hbar^2,$$

or

$$\sqrt{L_x^2 + L_y^2} = \hbar\sqrt{\ell(\ell+1) - m_\ell^2}\,.$$

4–66. Show that the diffractive angular spread of a laser beam, $\approx \lambda/a$, is precisely what you might expect from the uncertainty principle. [*Hint*:

Since the beam's width is constrained by the dimension of the aperture a, the component of the light's momentum perpendicular to the laser axis is uncertain.]

SOLUTION. We find the uncertainty in the momentum component perpendicular to the motion, when the width of the beam is constrained to a dimension a:

$$\Delta p_y \geqslant \frac{\hbar}{2\,\Delta y} = \frac{\hbar}{2a}.$$

The half angle of the beam is given by the direction of the velocity (or momentum):

$$\sin\theta = \frac{p_y}{p}.$$

We assume that the angle is small: $\sin\theta \approx \theta$, and we take the minimum uncertainty to be the perpendicular momentum, so we have

$$\theta \approx \frac{p_y}{p} = \frac{(\hbar/2a)}{(2\pi\hbar/\lambda)} = \frac{\lambda}{4\pi a}.$$

The angular spread is

$$2\theta \approx \frac{\lambda}{2\pi a} \sim \frac{\lambda}{a}.$$

4–67. Silver atoms (spin $= \frac{1}{2}$) are placed in a 1.0-T magnetic field which splits the ground state into two close levels. (a) What is the difference in energy between these two levels, and (b) what wavelength photon could cause a transition from the lower level to the upper one? (c) How would your answer differ if the atoms were hydrogen?

SOLUTION. (a) The additional energy term in a magnetic field is

$$U = -\mu_s B = -g\mu_{\mathrm{B}} m_s B.$$

Thus the separation of energy levels is

$$\begin{aligned}\Delta U &= \Delta m_s g \mu_{\mathrm{B}} B = \left(\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right)(2.0023)(9.27\times10^{-24}\ \mathrm{J/T})(1.0\ \mathrm{T}) \\ &= 1.9\times10^{-23}\ \mathrm{J} = 1.2\times10^{-4}\ \mathrm{eV}.\end{aligned}$$

(b) We find the wavelength from

$$\begin{aligned}\lambda &= \frac{1.24\times10^{3}\ \mathrm{eV\cdot nm}}{\Delta U} = \frac{1.24\times10^{3}\ \mathrm{eV\cdot nm}}{1.2\times10^{-4}\ \mathrm{eV}} \\ &= 1.07\times10^{7}\ \mathrm{nm} = 1.1\ \mathrm{cm}.\end{aligned}$$

(c) From the periodic table we see that the participating electron is in the $5s^1$ state. Thus the splitting for both atoms is of a single s-state electron, so there will be no difference.

4–68. Estimate the angular spread of a laser beam due to diffraction if the beam emerges through a 4.0-mm diameter mirror. Assume that $\lambda = 694$ nm. What would be the diameter of this beam if it struck (a) a satellite 1 000 km above the Earth, or (b) the Moon?

SOLUTION. We find the angular half width of the beam from

$$\Delta\theta = \frac{1.22\,\lambda}{d} = \frac{1.22(694 \times 10^{-9}\ \text{m})}{4.0 \times 10^{-3}\ \text{m}} = 2.1 \times 10^{-4}\ \text{rad},$$

so the angular spread is $\theta = 4.2 \times 10^{-4}$ rad.

(a) The diameter of the beam when it reaches the satellite is

$$D = r\theta = (1\,000 \times 10^{3}\ \text{m})(4.2 \times 10^{-4}\ \text{rad}) = 4.2 \times 10^{2}\ \text{m}.$$

(b) The diameter of the beam when it reaches the Moon is

$$D = r\theta = (3.84 \times 10^{8}\ \text{m})(4.2 \times 10^{-4}\ \text{rad}) = 1.6 \times 10^{5}\ \text{m} = 160\ \text{km}.$$

4–69. *Populations in the H atom.* Use the Boltzmann factor (Eq. 4.16) to estimate the fraction of H atoms in the $n = 2$ and $n = 3$ levels (relative to the ground state) for thermal equilibrium at (a) $T = 300$ K and (b) $T = 6\,000$ K. [*Note*: Since there are eight states with $n = 2$ and only two with $n = 1$, multiply your result for $n = 2$ by $\frac{8}{2} = 4$; do similarly for $n = 3$.] (c) Given 1.0 g of hydrogen, estimate the number of atoms in each state at $T = 6\,000$ K. (d) Estimate the number of $n = 3$ to $n = 1$ and $n = 2$ to $n = 1$ photons that will be emitted per second at $T = 6\,000$ K. Assume that the lifetime of each excited state is 10^{-8} s.

SOLUTION. (a) Because the Boltzmann factor compares two single energy levels, to find the fraction in the 8 $n = 2$ levels relative to the 2 $n = 1$ levels, we multiply by $\frac{8}{2}$:

$$\begin{aligned}
\frac{N_2}{N_1} &= \frac{8}{2} e^{-\Delta E/k_{\text{B}}T} \\
&= 4 \exp\left(-\frac{((-3.4\ \text{eV}) - (-13.6\ \text{eV}))\,(1.60 \times 10^{-19}\ \text{J/eV})}{(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})} \right) \\
&= 3 \times 10^{-171}.
\end{aligned}$$

The $n = 3$ level has $\ell = 0, 1, 2$, so the number of states is $2 + 6 + 10 = 18$. Thus we have

$$\begin{aligned}\frac{N_3}{N_1} &= \frac{18}{2} e^{-\Delta E/k_{\mathrm{B}}T} \\ &= 9 \exp\left(-\frac{((-1.5\ \mathrm{eV}) - (-13.6\ \mathrm{eV}))\,(1.60 \times 10^{-19}\ \mathrm{J/eV})}{(1.38 \times 10^{-23}\ \mathrm{J/K})(300\ \mathrm{K})}\right) \\ &= 7 \times 10^{-203}.\end{aligned}$$

(b) At the new temperature we have

$$\begin{aligned}\frac{N_2}{N_1} &= \frac{8}{2} e^{-\Delta E/k_{\mathrm{B}}T} \\ &= 4 \exp\left(-\frac{((-3.4\ \mathrm{eV}) - (-13.6\ \mathrm{eV}))\,(1.60 \times 10^{-19}\ \mathrm{J/eV})}{(1.38 \times 10^{-23}\ \mathrm{J/K})(6\,000\ \mathrm{K})}\right) \\ &= 1.1 \times 10^{-8};\end{aligned}$$

$$\begin{aligned}\frac{N_3}{N_1} &= \frac{18}{2} e^{-\Delta E/k_{\mathrm{B}}T} \\ &= 9 \exp\left(-\frac{((-1.5\ \mathrm{eV}) - (-13.6\ \mathrm{eV}))\,(1.60 \times 10^{-19}\ \mathrm{J/eV})}{(1.38 \times 10^{-23}\ \mathrm{J/K})(6\,000\ \mathrm{K})}\right) \\ &= 6.3 \times 10^{-10}.\end{aligned}$$

(c) We find the number of hydrogen atoms in 1.0 g from

$$N = \frac{(1.0\ \mathrm{g})(6.02 \times 10^{23}\ \mathrm{atoms/mol})}{1.0\ \mathrm{g/mol}} = 6.02 \times 10^{23}\ \mathrm{atoms}.$$

Because the fraction in the excited states is very small, we use this for N_1:

$$\begin{aligned}N_2 &= (1.1 \times 10^{-8})(6.02 \times 10^{23}) = 6.6 \times 10^{15}; \\ N_3 &= (6.3 \times 10^{-10})(6.02 \times 10^{23}) = 3.8 \times 10^{14}.\end{aligned}$$

(d) If the lifetime of the excited state is 10^{-8} s, the rate at which photons are emitted is

$$\begin{aligned}n_2 &= \frac{N_2}{\tau} = \frac{6.6 \times 10^{15}}{10^{-8}\ \mathrm{s}} \approx 7 \times 10^{23}\ \mathrm{photons/s}; \\ n_3 &= \frac{N_3}{\tau} = \frac{3.8 \times 10^{14}}{10^{-8}\ \mathrm{s}} \approx 4 \times 10^{22}\ \mathrm{photons/s}.\end{aligned}$$

4–70. A very simple model of a "one-dimensional" metal consists of N electrons confined to a rigid box of width L. We neglect the Coulomb interaction between the electrons (this is called the "independent electron model" and does remarkably well in predicting many properties of real metals). The Fermi energy of a metal is the energy of the most energetic electron when the metal is in its ground state ($T = 0$ K). (a) Calculate the Fermi energy for this one-dimensional metal, taking into account the Pauli exclusion principle. You can assume for simplicity that N is even. (b) What is the smallest amount of energy that this metal can absorb? Consider the limit of your answer for large N. (The relatively small value you should obtain is indicative of how metals can readily conduct).

SOLUTION. (a) The energy levels for a rigid box are given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

There are two electrons in each level, corresponding to the two different spin states. With N electrons, the quantum number of the highest filled state is

$$n_{\text{highest}} = \frac{N}{2}.$$

At $T = 0$ K, the highest energy of an electron in the ground state is the energy of the highest filled state:

$$E_{\text{F}} = \frac{N^2\pi^2\hbar^2}{8mL^2}.$$

(b) The smallest energy absorption will raise an electron to the next unfilled state:

$$\begin{aligned}\Delta E &= E_{n_{\text{highest}}+1} - E_{n_{\text{highest}}} = \frac{\left((N/2+1)^2 - (N/2)^2\right)\pi^2\hbar^2}{2mL^2} \\ &= \frac{(N+1)\pi^2\hbar^2}{2mL^2} \approx \frac{N\pi^2\hbar^2}{2mL^2} = \frac{4E_{\text{F}}}{N}.\end{aligned}$$

4–71. If the principal quantum number n were limited to the range from 1 to 6, how many elements would we find in nature?

SOLUTION. Each shell with quantum number n can contain $2n^2$ electrons. Thus the maximum number of electrons in the shells from $n = 1$ to $n = 6$ is

$$N = \sum_{n=1}^{6} 2n^2 = 2\frac{1}{6}n(n+1)(2n+1) = \frac{1}{3}(6)(6+1)(12+1) = 182.$$

Because each electron corresponds to one proton in an atom, there would be a maximum of 182 elements.

4–72. It is possible for atoms to be excited into states with very high values of the principal quantum number. Electrons in these so-called *Rydberg states* have very small ionization energies and huge orbital radii. This makes them particularly sensitive to external perturbation, as would be the case if the atom were in an electric field. Consider the $n = 50$ state of the hydrogen atom. Determine the binding energy, the radius of the orbit, and the effective cross sectional area of this Rydberg state.

SOLUTION. The binding energy of the state is

$$-E_n = \frac{13.6 \text{ eV}}{n^2} = \frac{13.6 \text{ eV}}{(50)^2} = 5.4 \times 10^{-3} \text{ eV}.$$

The radius of the orbit is

$$r = n^2 r_{\text{B}} = (50)^2(0.529 \times 10^{-10} \text{ m}) = 1.32 \times 10^{-7} \text{ m}.$$

The effective cross-section is

$$\sigma = \pi r^2 = \pi(1.32 \times 10^{-7} \text{ m})^2 = 5.5 \times 10^{-14} \text{ m}^2.$$

4–73. In the analytical technique known as *electron spin resonance* (ESR), materials with atoms having one or more unpaired electrons are placed in a variable magnetic field plus a constant radio frequency (RF) electric field. The magnetic field splits the energy of the unpaired electrons according to the two possible spin orientations. If the splitting is the same as the photon energy which is available from the RF radiation, a resonant absorption takes place. This can be detected, and is a measure of the g-factor of the electrons in those atoms. The g-factor for molecules and solids can be different than for electrons in atoms, due to the chemical

interactions in the material. Therefore, ESR provides a fingerprint of the system. In a particular ESR experiment, the RF radiation has a fixed wavelength of 2.0 cm. As the magnetic field is varied, a resonance is observed at 0.476 T. What is the g-factor for the unpaired electrons in the sample?

SOLUTION. The additional energy term in a magnetic field is

$$U = -\mu_s B = -g\mu_{\mathrm{B}} m_s B.$$

For resonance the separation of energy levels is the energy in the photon:

$$\begin{aligned} 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} &= \Delta m_s g \mu_{\mathrm{B}} B; \\ \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{(2.0 \text{ cm})(107 \text{ nm/cm})} & \\ \times (1.60 \times 10^{-19} \text{ J/eV}) &= \left(\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right) g \\ & \quad \times (9.27 \times 10^{-24} \text{ J/T})(0.476 \text{ T}), \end{aligned}$$

which gives $g = 2.25$.

5 Molecules and Solids

5.1 Review

Quantum mechanics explains the bonding together of atoms to form molecules. In a *covalent bond*, the electron clouds of two or more atoms overlap because of constructive interference between the electron waves. The positive nuclei are attracted to this concentration of negative charge between them, forming the bond. An *ionic bond* is extreme case of a covalent bond in which one or more electrons from one atom spend much more time around the other atom than around their own. The atoms then act as oppositely charged ions that attract each other, forming the bond. These *strong bonds* hold molecules together, and also hold atoms and molecules together in solids.

One description of the potential energy of a diatomic molecule is given by the *Morse potential*,

$$U(r) = D\left(e^{-2\alpha(r-r_0)/r_0} - 2e^{-\alpha(r-r_0)/r_0}\right) \tag{5.1}$$

where r is the separation of the atoms. The function $U(r)$ contains two parameters D and α that are determined from experiments. For instance, for the H_2 molecule $D = 4.74744$ eV, $\alpha = 1.440$. There is an optimal separation of the atoms, r_0 in Fig. 5.1, at which the energy is lowest. This is the point of greatest stability for the hydrogen molecule, and r_0 is the average separation of atoms in the H_2 molecule. The bottom of the potential-energy curve is the *binding energy*. For the H_2 molecule, the binding energy is ≈ 4.75 eV, and $r_0 = 0.074$ nm.

Example 5–1. In the potassium iodide molecule, assume that the K and I atoms are bond ionically by the transfer of one electron from K to I. (a) The ionization energy for K is 4.34 eV, and the electron affinity of I is 3.06 eV. What energy is needed to transfer an electron from K to I, to form K^+ and I^- ions from neutral atoms? This is sometimes called the *activation energy* E_a. (b) A model potential-energy function for the KI molecule is the Lennard–Jones potential:

$$U(r) = 4\varepsilon\left(\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right) + E_a \tag{5.2}$$

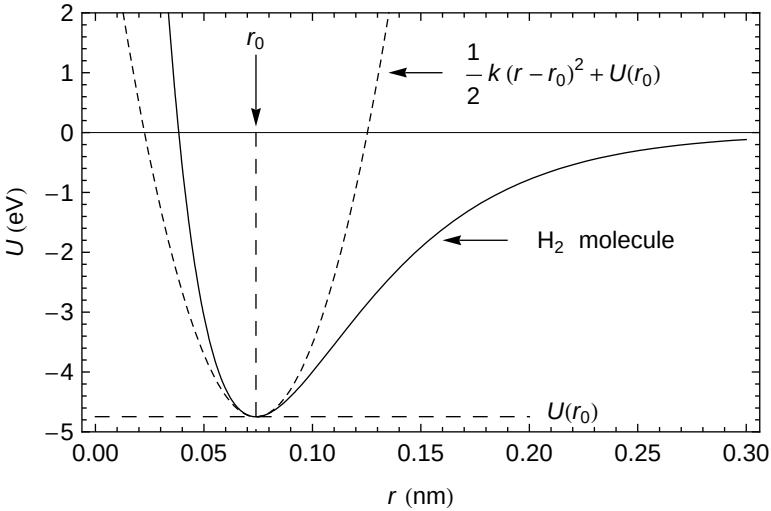


Figure 5.1. Potential-energy for H_2 molecule and for a simple harmonic oscillator ($U_{\rm SHO} = \frac{1}{2}kx^2$, with $x = r - r_0$).

where σ and ε are adjustable parameters. The $E_{\rm a}$ term is added to ensure the correct asymptotic behavior at large r. At the equilibrium separation distance, $r = r_0 = 0.305$ nm, $U(r)$ is a minimum. Now $U(r_0)$ is the negative of the dissociation energy: $U(r_0) = -3.37$ eV. Evaluate σ and ε. (c) Calculate the force needed to break up a KI molecule.

SOLUTION. (a) We add the reactions

$$\begin{aligned} \mathrm{K} + 4.34\ \mathrm{eV} &\rightarrow \mathrm{K}^+ + \mathrm{e}^- \\ \mathrm{I} + \mathrm{e}^- &\rightarrow \mathrm{I}^- + 3.06\ \mathrm{eV} \end{aligned}$$

to obtain

$$\mathrm{K} + \mathrm{I} + (4.34 - 3.06) \rightarrow \mathrm{K}^+ + \mathrm{I}^-.$$

The activation energy is 1.28 eV.

(b) Here is the derivative:

$$\frac{dU}{dr} = \frac{4\varepsilon}{\sigma}\left(-12\left(\frac{\sigma}{r}\right)^{13} + 6\left(\frac{\sigma}{r}\right)^{7}\right).$$

At $r = r_0$ we have $dU/dr = 0$. Hence

$$\left(\frac{\sigma}{r_0}\right)^{13} = \frac{1}{2}\left(\frac{\sigma}{r_0}\right)^{7},$$

so $\sigma = 2^{-1/6}(0.305)\ \text{nm} = 0.272\ \text{nm}$. Then also

$$\begin{aligned} U(r_0) &= 4\varepsilon\left(\left(\frac{2^{-1/6}r_0}{r_0}\right)^{12} - \left(\frac{2^{-1/6}r_0}{r_0}\right)^{6}\right) + E_\text{a} \\ &= 4\varepsilon\left(\frac{1}{4} - \frac{1}{2}\right) + E_\text{a} = -\varepsilon + E_\text{a}, \end{aligned}$$

whence

$$\varepsilon = E_\text{a} - U(r_0) = 1.28\ \text{eV} + 3.37\ \text{eV} = 4.65\ \text{eV}.$$

A potential-energy diagram for the KI ionic bond is shown in Fig. 5.2, where we have set $U = 0$ for free K and I neutral atoms. It takes 4.65 eV to break the KI bond and form the K^+ and I^- ions. The energy to separate the KI into K and I atoms (the binding energy) is 3.37 eV.

(c) The force is given by the equation

$$F(r) = -\frac{dU}{dr} = \frac{4\varepsilon}{\sigma}\left(12\left(\frac{\sigma}{r}\right)^{13} - 6\left(\frac{\sigma}{r}\right)^{7}\right).$$

To find the maximum force we calculate

$$\frac{dF}{dr} = \frac{4\varepsilon}{\sigma^2}\left(-156\left(\frac{\sigma}{r}\right)^{14} + 42\left(\frac{\sigma}{r}\right)^{8}\right) = 0$$

whence

$$\frac{\sigma}{r_\text{rupture}} = \left(\frac{42}{156}\right)^{1/6}.$$

Hence

$$\begin{aligned} F_\text{max} &= \frac{4(4.65\ \text{eV})}{0.272\ \text{nm}}\left(12\left(\frac{42}{156}\right)^{13/6} - 6\left(\frac{42}{156}\right)^{7/6}\right) = -41.0\ \text{eV/nm} \\ &= -(41.0)\frac{(1.6\times10^{-19}\ \text{N}\cdot\text{m})}{(10^{-9}\ \text{m})} = -6.55\ \text{nN}. \end{aligned}$$

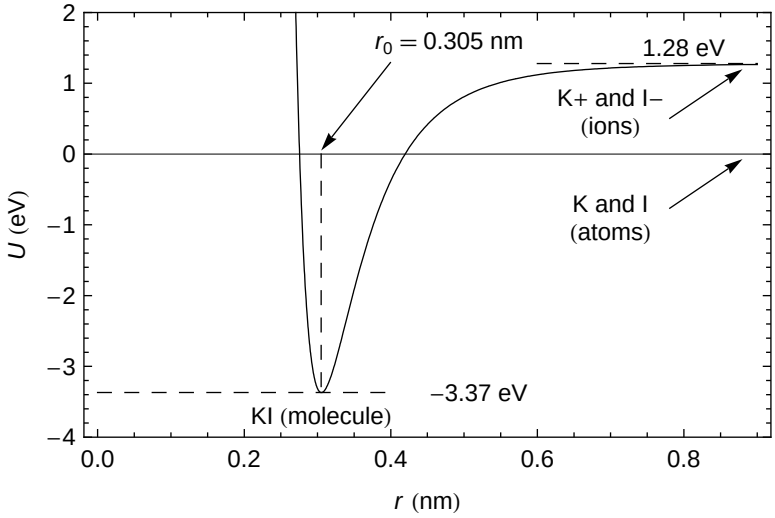


Figure 5.2. Potential-energy diagram for the KI bond.

Therefore the applied force required to rupture the molecule is 6.55 nN away from the center.

Also important are *weak bonds* (or *van der Waals bonds*), which are generally dipole attractions between molecules. For example, Fig. 5.3 shows two molecules, which have permanent dipole moments, attracting one another.

Example 5–2. The DNA molecule consists of two intertwined linear chains. Sticking out from each monomer (link in the chain) is one of four bases (nucleotides): adenine (A), guanine (G), thymine (T), or cytosine(C). In the double helix, each base from one strand bonds to a base in the other strand. The correct matches, A–T and G–C, are more tightly bound than are the improper matches. Calculate the interaction energy between the C=O dipole of thymine and the H−N dipole of adenine, assuming that the two dipoles are lined up as shown in Fig. 5.3. Dipole moment measurements give $q_{\mathrm{H}} = -q_{\mathrm{N}} = 0.19e = 3.0 \times 10^{-20}$ C, and $q_{\mathrm{C}} = -q_{\mathrm{O}} = 0.42e = 6.7 \times 10^{-20}$ C.

Solution. The interaction energy will be equal to the potential energy

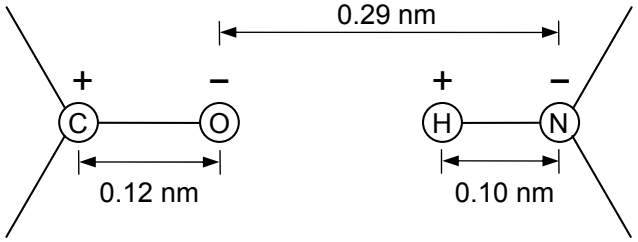


Figure 5.3. The C^+-O^- and H^+-N^- dipoles attract each other. These dipoles may be part of larger molecules.

of one dipole in the presence of the other, since this will be equal to the work needed to pull the two dipoles infinitely far apart. U will consist of four terms:

$$U = U_{\mathrm{CH}} + U_{\mathrm{CN}} + U_{\mathrm{OH}} + U_{\mathrm{ON}},$$

where U_{CH} means the potential energy of C in the presence of H, and similarly for the other terms. We do not have terms corresponding to C and O, or N and H, because the two dipoles are assumed to be stable entities (the bonds C=O and H−N remain intact). The potential energy for two point charges is

$$U_{12} = k_{\mathrm{e}} \frac{q_1 q_2}{r},$$

where r is the distance between them. So,

$$U = k_{\mathrm{e}} \left(\frac{q_{\mathrm{C}} q_{\mathrm{H}}}{r_{\mathrm{CH}}} + \frac{q_{\mathrm{C}} q_{\mathrm{N}}}{r_{\mathrm{CN}}} + \frac{q_{\mathrm{O}} q_{\mathrm{H}}}{r_{\mathrm{OH}}} + \frac{q_{\mathrm{O}} q_{\mathrm{N}}}{r_{\mathrm{ON}}} \right)$$

Using the distances shown in Fig. 5.3, we get:

$$\begin{aligned} U &= (9 \times 10^9 \ \mathrm{N \cdot m^2/C}) \left(\frac{(6.7)(3.0)}{0.31} + \frac{(6.7)(-3.0)}{0.41} + \frac{(-6.7)(3.0)}{0.19} \right. \\ &\quad \left. + \frac{(-6.7)(-3.0)}{0.29} \right) \frac{(10^{-20} \ \mathrm{C})^2}{10^{-9} \ \mathrm{m}} \\ &= -1.82 \times 10^{-20} \ \mathrm{J} = -0.11 \ \mathrm{eV}. \end{aligned}$$

The potential energy is negative, meaning 0.11 eV of work (or energy input) is required to separate the molecules. That is, the binding energy

of this "weak" or *hydrogen bond* is 0.11 eV. This is only an estimate, of course, since other charges in the vicinity would have an influence too.

When atoms combine to form molecules, the energy levels of the outer electrons are altered because they now interact with each other. Additional energy levels also become possible because the atoms can vibrate with respect to each other, and the molecule as a whole can rotate. The energy levels for both vibrational and rotational motion are quantized, and are very close together (typically, 10^{-1} eV to 10^{-3} eV apart). Each atomic energy level thus becomes a set of closely spaced levels corresponding to the vibrational and rotational motions. Transitions from one level to another appear as many very closely spaced lines. The resulting spectra are called *band spectra*.

The quantized rotational energy levels are given by

$$E_{\text{rot}} = L(L+1)\frac{\hbar^2}{2I}, \qquad L = 0,\ 1,\ 2,\ \cdots, \tag{5.3}$$

where I is the moment of inertia of the molecule.

Example 5–3. Show that the moment of inertia of a diatomic molecule rotating about its center of mass can be written

$$I = \mu r^2, \tag{5.4}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the *reduced mass*, and r is the distance between the atoms.

SOLUTION. The moment of inertia of a single particle of mass m a distance r from the rotation axis is $I = mr^2$. Hence for our diatomic molecule (Fig. 5.4)

$$I = m_1 r_1^2 + m_2 r_2^2.$$

Now $r = r_1 + r_2$ and $m_1 r_1 = m_2 r_2$ because the axis of rotation passes through the center of mass. Hence

$$r_1 = r - r_2 = r - \frac{m_1 r_1}{m_2},$$

so

$$r_1 = \frac{r}{1 + (m_1/m_2)} = \frac{m_2 r}{m_1 + m_2}.$$

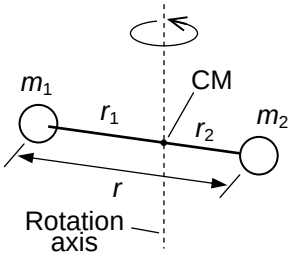


Figure 5.4. Diatomic molecule rotating about a vertical axis.

Similarly,

$$r_2 = \frac{m_1 r}{m_1 + m_2}.$$

Then

$$\begin{aligned} I &= m_1 \left(\frac{m_2 r}{m_1 + m_2} \right)^2 + m_2 \left(\frac{m_1 r}{m_1 + m_2} \right)^2 = \frac{m_1 m_2 (m_1 + m_2) r^2}{(m_1 + m_2)^2} \\ &= \frac{m_1 m_2}{m_1 + m_2} r^2 = \mu r^2, \end{aligned}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2},$$

which is what we wished to show.

Transitions between rotational energy levels are subject to the selection rule: $\Delta L = \pm 1$. The energy of a photon emitted or absorbed for a transition between rotational states with angular momentum quantum number L and $L - 1$ will be

$$\Delta E_{\text{rot}} = E_L - E_{L-1} = \frac{\hbar^2 L}{I}. \tag{5.5}$$

Example 5–4. A rotational transition $L = 1$ to $L = 0$ for the molecule CO has a measured absorption wavelength $\lambda = 2.60$ mm (microwave region). Use this to calculate (a) the moment of inertia of the CO molecule, and (b) the CO bond length, r. (c) Calculate the wavelengths of the next

three rotational transitions, and the energies of the photon emitted for each of these four transitions.

SOLUTION. (a) From Eq. 5.5, we can write

$$\frac{\hbar^2 L}{I} = \Delta E = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda_1}.$$

With $L = 1$ (the upper state) in this case, we solve for I:

$$\begin{aligned} I &= \frac{\hbar^2 L}{2\pi\hbar c}\lambda_1 = \frac{\hbar\lambda_1}{2\pi c} = \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})(2.60\times10^{-3}\ \text{m})}{2\pi(3.00\times10^{8}\ \text{m/s})} \\ &= 1.46\times10^{-46}\ \text{kg}\cdot\text{m}^2. \end{aligned}$$

(b) The masses of C and O are 12.0 and 16.0 u, respectively, where $1\ \text{u} = 1.66\times10^{-27}$ kg. Thus the reduced mass is

$$\begin{aligned} \mu &= \frac{m_1 m_2}{m_1 + m_2} = \frac{(12.0)(16.0)}{28.0}(1.66\times10^{-27}\ \text{kg}) = 1.14\times10^{-26}\ \text{kg} \\ &= 6.86\ \text{u}. \end{aligned}$$

Then, from Eq. 5.4, the bond length is

$$r = \sqrt{\frac{I}{\mu}} = \sqrt{\frac{1.46\times10^{-46}\ \text{kg}\cdot\text{m}^2}{1.14\times10^{-26}\ \text{kg}}} = 1.13\times10^{-10}\ \text{m} = 0.113\ \text{nm}.$$

(c) From Eq. 5.5, $\Delta E \propto L$. Hence $\lambda = c/f = 2\pi\hbar c/\Delta E$ is proportional to L^{-1}. Thus, for $L = 2$ to $L = 1$ transitions, $\lambda_2 = \frac{1}{2}\lambda_1 = 1.30$ mm. For $L = 3$ to $L = 2$, $\lambda_3 = \frac{1}{3}\lambda_2 = 0.87$ mm. And for $L = 4$ to $L = 3$, $\lambda_4 = 0.65$ mm. All are close to measured values. The energies of the photons, $2\pi\hbar f = 2\pi\hbar c/\lambda$, are respectively 4.8×10^{-4} eV, 9.5×10^{-4} eV, 1.4×10^{-3} eV, and 1.9×10^{-3} eV.

The energy levels for vibrational motion are given by

$$E_{\text{vib}} = 2\pi(\nu + \tfrac{1}{2})\hbar f, \qquad \nu = 0,\ 1,\ 2,\ \ldots, \tag{5.6}$$

where f is the classical natural frequency of vibration for the molecule:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}\,. \tag{5.7}$$

Transitions between energy levels are subject to the selection rule $\Delta\nu = \pm 1$, so allowed transitions occur only between adjacent states and all give off photons of energy

$$\Delta E_{\text{vib}} = 2\pi\hbar f. \tag{5.8}$$

Example 5–5. (a) Use Eq. 5.1 to estimate the value of the stiffness constant k for the H_2 molecule, and then (b) estimate the fundamental wavelength for vibrational transitions.

SOLUTION. (a) We set $r = r_0 + x$ where $x/r_0 \ll 1$, and expand $U(r)$ in powers of x/r_0 up to second-order terms:

$$U(r) = D\left(-1 + \left(\frac{\alpha(r - r_0)}{r_0}\right)^2\right) = -D + \tfrac{1}{2}\,kx^2.$$

This yields

$$\begin{aligned} k &= 2D\left(\frac{\alpha}{r_0}\right)^2 \\ &= 2(4.747\text{ eV})(1.6\times10^{-19}\text{ J/eV})\left(\frac{1.440}{0.074\times10^{-9}\text{ m}}\right)^2 \\ &= 575\text{ N/m}. \end{aligned}$$

This value of k would also be reasonable for a macroscopic spring.

(b) The reduced mass is

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1}{2} = \frac{1}{2}(1.0\text{ u})(1.66\times10^{-27}\text{ kg}) = 0.83\times10^{-27}\text{ kg}.$$

Hence, using Eq. 5.7,

$$\lambda = \frac{c}{f} = 2\pi c\sqrt{\frac{\mu}{k}} = 2\pi(3.0\times10^8\text{ m/s})\sqrt{\frac{0.83\times10^{-27}\text{ kg}}{575\text{ N/m}}} = 2\,264\text{ nm},$$

which is in the infrared region of the spectrum.

Example 5–6. Given that the hydrogen molecule emits infrared radiation of wavelength around 2 300 nm, (a) what is the separation in energy

between different vibrational levels, and (b) what is the lowest vibrational energy state?

SOLUTION.

$$\begin{aligned}(a)\quad \Delta E_{\text{vib}} &= 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} \\ &= \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.00\times10^{8}\ \text{m/s})}{(2\,300\times10^{-9}\ \text{m})(1.60\times10^{-19}\ \text{J/eV})} = 0.54\ \text{eV}.\end{aligned}$$

(b) The lowest vibrational energy has $\nu = 0$ in Eq. 5.6:

$$E = \tfrac{1}{2}2\pi\hbar f = 0.27\ \text{eV}.$$

A transition from a state with quantum numbers ν and L, to one with quantum numbers $\nu + 1$ and $L \pm 1$ (see the selection rules above), will absorb a photon of energy:

$$\begin{aligned}\Delta E &= \Delta E_{\text{vib}} + \Delta E_{\text{rot}} \\ &= \begin{cases} 2\pi\hbar f + (L+1)\dfrac{\hbar^2}{I} & \text{for } L \to L+1, \quad L = 0, 1, 2, \cdots \\ 2\pi\hbar f - L\dfrac{\hbar^2}{I} & \text{for } L \to L-1, \quad L = 1, 2, 3, \cdots, \end{cases}\end{aligned} \tag{5.9}$$

where we have used Eqs. 5.5 and 5.8. Equations 5.9 predict an absorption spectrum like that shown in Fig. 5.5, with transitions $L \to L - 1$ on the left and $L \to L + 1$ on the right.

Example 5–7. Estimate the moment of inertia of the HCl molecule using the absorption spectrum shown in Fig. 5.5. For the purpose of a rough estimate you can ignore the difference between the two isotopes.

SOLUTION. The locations of peaks in Fig. 5.5 should correspond to Eqs. 5.9. We don't know what value of L each peak corresponds to in Fig. 5.5, but we can estimate the energy difference between peaks to be about $\Delta E' = 0.0025$ eV. Then from Eqs. 5.9, the energy difference between two peaks is given by

$$\Delta E' = \Delta E_{L+1} - \Delta E_L = \frac{\hbar^2}{I}\,.$$

Then

$$I = \frac{\hbar^2}{\Delta E'} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s}/2\pi)^2}{(0.0025\ \text{eV})(1.6\times10^{-19}\ \text{J/eV})} = 2.8\times10^{-47}\ \text{kg}\cdot\text{m}^2.$$

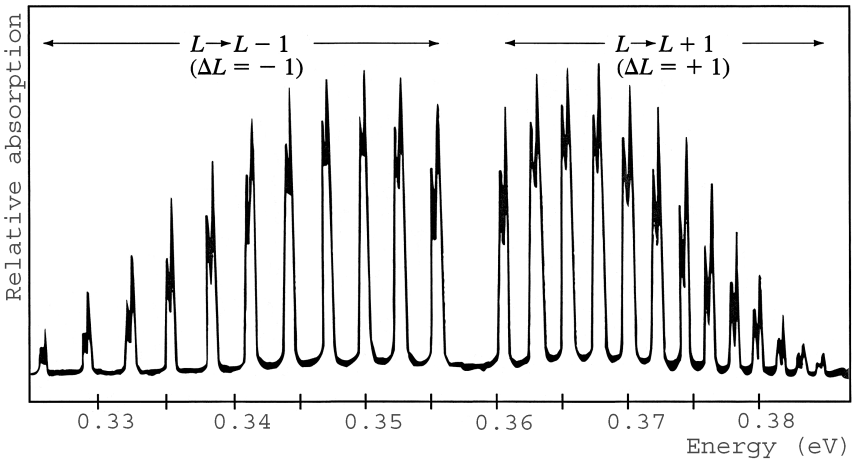


Figure 5.5. Absorption spectrum for HCl molecules. Each line has a double peak because chlorine has two isotopes of different mass and different moment of inertia.

If we use the results of Example 5–3, we can get a better idea of what this number means. We write $I = \mu r^2$ and calculate μ:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{(1.0\ \text{u})(35\ \text{u})}{36\ \text{u}}(1.66 \times 10^{-27}\ \text{kg/u}) = 1.6 \times 10^{-27}\ \text{kg}.$$

Then the bond length is given by (Eq. 5.4)

$$r = \left(\frac{I}{\mu}\right)^{1/2} = \left(\frac{2.8 \times 10^{-47}\ \text{kg} \cdot \text{m}^2}{1.6 \times 10^{-27}\ \text{kg}}\right)^{1/2} = 1.3 \times 10^{-10}\ \text{m},$$

which is the expected order of magnitude for a bond length.

Some solids are bound together by covalent and ionic bonds, just as molecules are. The potential energy of an ion in a crystal can be written

$$U = -\alpha k_e \frac{e^2}{r} + \frac{B}{r^m} \tag{5.10}$$

where r is the separation distance between neighboring ions, and α, called the *Madelung constant*, depends on the geometry of the crystal.

In metals, the electrostatic force between free electrons and positive ions helps form the metallic bond.

In the free-electron theory of metals, electrons occupy the possible energy states according to the exclusion principle. Quantity $g(E)$ is the *density of states*, such that $g(E)\,dE$ is the number of states per unit volume with energy between E and $E + dE$):

$$g(E) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}\,. \tag{5.11}$$

Example 5–8. Estimate the number of states in the range 5.0 to 5.5 eV available to electrons in a 1.0-cm cube of copper metal.

SOLUTION. Since $g(E)$ is the number of states per unit volume per unit energy interval, the number N of states is approximately (it is approximate because ΔE is not small)

$$N \approx g(E)V\,\Delta E,$$

where the volume $V = 1.0\ \text{cm}^3 = 1.0 \times 10^{-6}\ \text{m}^3$ and $\Delta E = 0.50$ eV. We evaluate $g(E)$ at 5.25 eV, and find (Eq. 5.11):

$$\begin{aligned} N &\approx g(E)V\,\Delta E \\ &= \frac{1}{2\pi^2}\left(\frac{2(9.1\times10^{-31}\ \text{kg})}{(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}\right)^{3/2} \\ &\quad \times \big((5.25\ \text{eV})(1.6\times10^{-19}\ \text{J/eV})\big)^{1/2} \\ &\quad \times (1.0\times10^{-6}\ \text{m}^3)(0.50\ \text{eV})(1.6\times10^{-19}\ \text{J/eV}) \\ &\approx 8\times10^{21} \end{aligned}$$

in 1.0 cm^3. Note that the type of metal did not enter the calculation.

At $T = 0$ K, all possible states are filled up to a maximum energy level called the *Fermi energy*, E_F, the magnitude of which is typically a few eV. To determine E_F, we integrate Eq. 5.11 from $E = 0$ to $E = E_\text{F}$:

$$\frac{N}{V} = \int_0^{E_\text{F}} g(E)\,dE, \tag{5.12}$$

where N/V is the number of conduction electrons per unit volume in the metal. Then, solving for E_{F}, the result (see Example 5–9) is

$$E_{\mathrm{F}} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}. \tag{5.13}$$

The average energy in this distribution (see Problem 5–31) is

$$\overline{E} = \tfrac{3}{5}E_{\mathrm{F}}. \tag{5.14}$$

All states above E_{F} are vacant at $T = 0$ K. At normal temperatures (300 K) the distribution of occupied states is only slightly altered and is given by the *Fermi–Dirac probability function* (or *Fermi factor*)

$$f(E) = \frac{1}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T} + 1}. \tag{5.15}$$

The *density of occupied states*, n_{O} is the product of the density of possible states, $g(E)$, and the probability that those states will be occupied, $f(E)$:

$$n_{\mathrm{O}}(E) = g(E)f(E) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\sqrt{E}}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T} + 1}. \tag{5.16}$$

Example 5–9. For the metal copper, determine (a) the Fermi energy, (b) the average energy of electrons, and (c) the speed of electrons at the Fermi level (this is called *Fermi speed*).

Solution. (a) We combine Eqs. 5.11 and 5.12:

$$\frac{N}{V} = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{E_{\mathrm{F}}} E^{1/2}\,dE = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}E_{\mathrm{F}}^{3/2}.$$

Solving for E_{F}, we obtain

$$E_{\mathrm{F}} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3},$$

and this is Eq. 5.13.

If we assume there is one free electron per atom, then the density of free electrons, N/V, would be the same as the density of Cu atoms. The atomic

mass of Cu is 63.5 u, so 63.5 g of Cu contains one mole or 6.02×10^{23} free electrons. The mass density of copper is $\rho = 8.9 \times 10^3$ kg/m^3, where $\rho = m/V$. So the number of conduction electrons per unit volume in copper is

$$\begin{aligned}\frac{N}{V} &= \frac{N}{(m/\rho)} = \frac{N(1\text{ mole})}{m(1\text{ mole})}\,\rho \\ &= \left(\frac{6.02 \times 10^{23}\text{ electrons}}{63.5 \times 10^{-3}\text{ kg}}\right)(8.9 \times 10^3\text{ kg/m}^3) \\ &= 8.4 \times 10^{28}\text{ m}^{-3}.\end{aligned}$$

Hence for copper

$$\begin{aligned}E_{\mathrm{F}} &= \frac{(1.055 \times 10^{-34}\text{ J}\cdot\text{s})^2}{2(9.1 \times 10^{-31}\text{ kg})}\left(3\pi^2(8.4 \times 10^{28}\text{ m}^{-3})\right)^{2/3} \\ &\quad \times \frac{1}{1.6 \times 10^{-19}\text{ J/eV}} \\ &= 7.0\text{ eV}.\end{aligned}$$

(b) From Eq. 5.14, $\overline{E} = \frac{3}{5}E_{\mathrm{F}} = 4.2$ eV.

(c) In our model, we took $U = 0$ inside the metal, so $E =$ kinetic energy $= \frac{1}{2}mv^2$. Therefore, at the Fermi level, the Fermi speed is

$$v = \sqrt{\frac{2E_{\mathrm{F}}}{m}} = \sqrt{\frac{2(7.0\text{ eV})(1.6 \times 10^{-19}\text{ J/eV})}{9.1 \times 10^{-31}\text{ kg}}} = 1.6 \times 10^6\text{ m/s},$$

a very high speed. The temperature of a classical gas would have to be extremely high to produce an average particle speed this large.

In a crystalline solid, the possible energy states for electrons are arranged in *bands*. Within each band the levels are very close together, but between the bands there may be forbidden energy gaps. Good conductors are characterized by the highest occupied band (the conduction band) being only partially full, so there are many accessible states available to electrons to move about and accelerate when a voltage is applied. In a good insulator, the highest occupied energy band (the valence band) is completely full, and there is a large energy gap (5 to 10 eV) to the

next highest band, the conduction band. At room temperature, molecular kinetic energy (thermal energy) available due to collisions is only about 0.04 eV so almost no electrons can jump from the valence conduction band. In a semiconductor, the gap between valence and conduction bands is much smaller, on the order of 1 eV, so a few electrons can make the transition from the essentially full valence band to the nearly empty conduction band.

Example 5–10. It is found that the conductivity of a certain semiconductor increases when light of wavelength 345 nm or shorter strikes it, suggesting that electrons are being promoted from the valence band to the conduction band. What is the energy gap, E_g, for this semiconductor?

SOLUTION. The longest wavelength, or lowest energy, photon to cause an increase in conductivity has $\lambda = 345$ nm, and it can transfer to an electron an energy

$$E_g = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{(6.63 \times 19^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^8\ \text{m/s})}{(345 \times 10^{-9}\ \text{m})(1.60 \times 10^{-19}\ \text{J/eV})} = 3.6\ \text{eV}.$$

Example 5–11. Use the Fermi-Dirac probability function, Eq. 5.15, to estimate the order of magnitude of the numbers of free electrons in the conduction band of a solid containing 10^{21} atoms, assuming the solid is at room temperature ($T = 300$ K) and is (a) a semiconductor with $E_g \approx 1.1$ eV, (b) an insulator with $E_g \approx 5$ eV. Compare to a conductor.

SOLUTION. At $T = 0$, all states above the Fermi energy E_F are empty, and all those below are full. So for semiconductors and insulators we can take E_F to be about midway between the valence and conduction bands, and it does not change significantly as we go to room temperature.

(a) For the semiconductor, the gap $E_g \approx 1.1$ eV, so $E - E_F \approx 0.55$ eV for the lowest states in the conduction band. Since at room temperature we have $k_B T \approx 0.026$ eV, then $(E - E_F)/k_B T \approx 0.55\ \text{eV}/0.026\ \text{eV} \approx 21$ and

$$f(E) = \frac{1}{e^{(E-E_F)/k_B T} + 1} \approx \frac{1}{e^{21}} \approx 10^{-9}.$$

Thus about 1 atom in 10^9 can contribute an electron to the conductivity.

(b) For the insulator with $E - E_F \approx 5.0\ \text{eV} - \frac{1}{2}(5.0\ \text{eV}) = 2.5$ eV, we get

$$f(E) = \frac{1}{e^{2.5/0.026} + 1} \approx \frac{1}{e^{96}} \approx 10^{-42}.$$

Thus in an ordinary sample containing 10^{21} atoms, there would be no free electrons in an insulator ($10^{21} \times 10^{-42} = 10^{-21}$), about 10^{12} ($10^{21} \times 10^{-9}$) free electrons in a semiconductor, and about 10^{21} free electrons in a good conductor.

Example 5–12. The energy gap for silicon is 1.14 eV at room temperature while that of zinc sulfide (ZnS) is 3.6 eV. Which one of these is opaque and which is transparent to visible light?

SOLUTION. Visible light photons span energies from roughly 1.8 eV to 3.2 eV ($E = 2\pi\hbar f = 2\pi\hbar c/\lambda$, where $\lambda = 400$ nm to 700 nm and 1 eV = 1.6×10^{-19} J). Light is absorbed by the electrons in a material. Silicon's gap is small enough to absorb these photons, thus bumping electrons well up into the conduction band, and so silicon is opaque. On the other hand, zinc sulfide's energy gap is too wide to absorb visible photons, and so the light can pass through the material; it can be transparent.

In a doped semiconductor, a small percentage of impurity atoms with five or three valence electrons replace a few of the normal silicon atoms with their four valence electrons. A five-electron impurity produces an n-type semiconductor with negative electrons as carriers of current. A three-electron impurity produces a p-type semiconductor in which positive holes carry the current. The energy level of impurity atoms lies slightly below the conduction band in an n-type semiconductor, and acts as a donor from which electrons readily pass into the conduction band. The energy level of impurity atoms in a p-type semiconductor lies slightly above the valence band and acts as an acceptor level, since electrons from the valence band easily reach it, leaving holes behind to act as charge carriers.

A semiconductor diode consists of a p-n junction and allows current to flow in one direction only. Simple theory for the current versus the bias voltage across a p-n junction yields the equation

$$I = I_s \left(e^{eV/k_B T} - 1 \right). \tag{5.17}$$

Eq. 5.17 is valid for both positive and negative values of V; note that V and I always have the same sign. In striking contrast to the symmetrical behavior of resistors that obey Ohm's law and give a straight line on an $I-V$ graph, a p-n junction conducts more readily in the direction from p to n than the reverse. The current in a typical diode varies as shown in

Fig. 5.6. As V becomes very negative, I approaches the value $-I_s$. The magnitude I_s is called the *saturation current.*

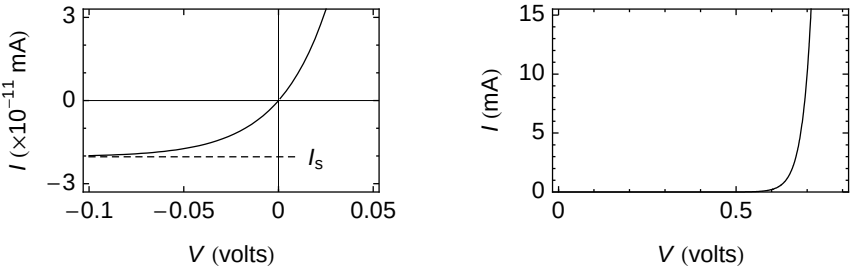


Figure 5.6. Current through a diode as a function of applied voltage. Left: reverse bias, right: forward bias. Note the different scales on the vertical axis for the forward and reverse conditions.

Example 5–13. The diode whose current-voltage characteristics are shown in Fig. 5.6 is connected in series with a 4. 0-V battery and a resistor. If a current of 10 mA is to pass through the diode, what resistance must the resistor have?

Solution. In Fig. 5.6, we see that the voltage drop across the diode is about 0. 7 V when the current is 10 mA. Therefore, the voltage drop across the resistor is $4.0\ \text{V} - 0.7\ \text{V} = 3.3\ \text{V}$, so $R = V/I = (3.3\ \text{V})/(1.0 \times 10^{-2}\ \text{A}) = 330\ \Omega$.

A *p-n* junction diode can be used as a *rectifier* to change ac to dc. A simple rectifier circuit is shown in Fig. 5.7. This *half-wave rectification* is not exactly dc, but it is unidirectional. More useful is a *full-wave rectifier* circuit, which uses four diodes as shown in Fig. 5.8. At any given instant, either one pair of diodes or the other will conduct current to the right. The output voltage is positive for all half-cycles, and the root-mean-square voltage is higher than it is for the half-wave rectifier.

A simple junction transistor consists of a crystal of one type of doped semiconductor sandwiched between two crystals of the opposite type. Both *pnp* and *npn* transistors are made. the three semiconductors are given the names *collector*, *base*, and *emitter*. Consider an *npn* transistor connected as shown in Fig. 5.9. The arrow is always placed on emitter and indicates the direction of (conventional) current flow in normal operation. A voltage

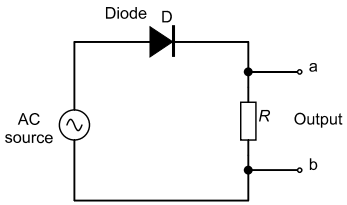


Figure 5.7. A simple (half-wave) rectifier circuit using a semiconductor diode.

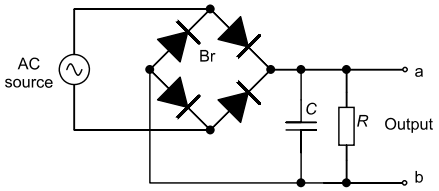


Figure 5.8. Full-wave "bridge-type" rectifier. The capacitor C helps to smooth out the current.

$V_{\rm CE}$ is maintained between the collector and emitter by the battery $\mathcal{E}_{\rm C}$. The voltage applied to the base is called the *base bias voltage*, $V_{\rm BE}$. If $V_{\rm BE}$ is positive, conduction electrons in the emitter are attracted into the base. Since the base region is very thin (perhaps 1 μm) most of these electrons flow right across into the collector, which is maintained at a positive voltage. A large current, $I_{\rm C}$, flows between collector and emitter and a much smaller current, $I_{\rm B}$, through the base. A small variation in the base voltage due to an input signal causes a large change in the collector current and therefore a large change in the voltage drop across the output resistor $R_{\rm C}$. Hence transistor can amplify electrical signals and finds many other uses.

Normally, a small ac signal is to be amplified. and when added to the base bias voltage causes the voltage and current at the collector to vary at the same rate but magnified. Thus what is important for amplification is the change in collector current for a given input change in base current. The *current gain* is defined as the ratio

$$\beta_I = \frac{i_{\rm C}}{i_{\rm B}} \tag{5.18}$$

where $i_{\rm B}$ and $i_{\rm C}$ are the ac base current (input signal) and ac collector current (output). Similarly, the *voltage gain* is

$$\beta_V = \frac{\text{output (collector) ac voltage}}{\text{input (base) ac voltage}}\,. \tag{5.19}$$

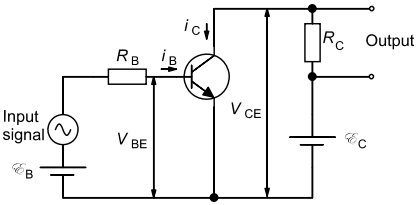


Figure 5.9. An *npn* transistor used as an amplifier.

A *pnp* transistor operates like an *npn*, except that holes move instead of electrons. The collector voltage is negative, and so is the base voltage in normal operation.

An integrated circuit consists of a tiny semiconductor crystal or "chip" on which many transistors, diodes, resistors, and other circuit elements have been constructed using careful placement of impurities.

5.2 Bonding in molecules. Potential-energy diagram. Weak (van der Waals) bonds

5–1. Estimate the binding energy of a KCl molecule by calculating the electrostatic potential energy when the K^+ and Cl^- ions are at their stable separation of 0. 28 nm. Assume each has a charge of magnitude $1.0\,e$.

Solution. With the reference level at infinity, the binding energy of the two ions is

$$\text{Binding energy} = -U = k_e\frac{e^2}{r} = \frac{2.30\times10^{-28}\ \text{J}\cdot\text{m}}{0.28\times10^{-9}\ \text{m}} = 8.21\times10^{-19}\ \text{J} = 5.1\ \text{eV}.$$

5–2. Binding energies are often measured experimentally in kcal per mole, and then the binding energy in eV per molecule is calculated from that result. What is the conversion factor in going from kcal per mole to eV per molecule? What is the binding energy of KCl (= 4. 43 eV) in kcal per mole?

SOLUTION. We convert the units:

$$\begin{aligned} 1 \text{ kcal/mol} &= \frac{(1 \text{ kcal/mol})(4\,186 \text{ J/kcal})}{(6.02 \times 10^{23} \text{ molecules/mol})(1.60 \times 10^{-19} \text{ J/eV})} \\ &= 0.0435 \text{ eV/molecule}. \end{aligned}$$

For KCl we have

$$(4.43 \text{ eV/molecule}) \frac{1 \text{ kcal/mol}}{0.0435 \text{ eV/molecule}} = 10^2 \text{ kcal/mol}.$$

5–3. The measured binding energy of KCl is 4.43 eV. From the result of Problem 5–1, estimate the contribution to the binding energy of the repelling electron clouds at the equilibrium distance $r_0 = 0.28$ nm.

SOLUTION. With the repulsion of the electron clouds, the binding energy is

$$\begin{aligned} \text{Binding energy} &= -U - U_{\text{clouds}}; \\ 4.43 \text{ eV} &= 5.1 \text{ eV} - U_{\text{clouds}}, \end{aligned}$$

which gives $U_{\text{clouds}} = 0.7$ eV.

5–4. Estimate the binding energy of the H_2 molecule, assuming the two H nuclei are 0.074 nm apart and the two electrons spend 33 percent of their time midway between them.

SOLUTION. When the electrons are midway between the protons, each electron will have a potential energy due to the two protons:

$$\begin{aligned} U_{\text{ep}} &= -k_{\text{e}} \frac{(2)(0.33)e^2}{(r/2)} = -\frac{(4)(0.33)(2.30 \times 10^{-28} \text{ J} \cdot \text{m})}{(0.074 \times 10^{-9} \text{ m})(1.60 \times 10^{-19} \text{ J/eV})} \\ &= -25.6 \text{ eV}. \end{aligned}$$

The protons have a potential energy:

$$\begin{aligned} U_{\text{pp}} &= +k_{\text{e}} \frac{e^2}{r} = +\frac{2.30 \times 10^{-28} \text{ J} \cdot \text{m}}{(0.074 \times 10^{-9} \text{ m})(1.60 \times 10^{-19} \text{ J/eV})} \\ &= +19.4 \text{ eV}. \end{aligned}$$

When the bond breaks, each hydrogen atom will be in the ground state with an energy $E_1 = -13.6$ eV. Thus the binding energy is

$$\begin{aligned} \text{Binding energy} &= 2E_1 - (2U_{\text{ep}} + U_{\text{pp}}) \\ &= 2(-13.6 \text{ eV}) - (2(-25.6 \text{ eV}) + 19.4 \text{ eV}) = 4.6 \text{ eV}. \end{aligned}$$

5–5. Apply qualitative reasoning to show why the molecule He_2 is *not* formed. Show also why the He_2^+ molecular ion *is* formed (with a binding energy of 3.1 eV at $r_0 = 0.11$ nm).

SOLUTION. The neutral He atom has two electrons in the ground state, $n = 1$, $l = 0$, $m_\ell = 0$. Thus the two electrons have opposite spins, $m_s = \pm\frac{1}{2}$. If we try to form a covalent bond for the two atoms, we see that an electron from one of the atoms will have the same quantum numbers as one of the electrons on the other atom. From the exclusion principle, this is not allowed, so the electrons cannot be shared.
We consider the He_2^+ molecular ion to be formed from a neutral He atom and an He^+ ion. If the electron on the ion has a certain spin value, it is possible for one of the electrons on the atom to have the opposite spin. Thus the electron can be in the same spatial region as the electron on the ion, so a bond can be formed.

5.3 Molecular spectra

5–6. Show that the quantity $\hbar^2/I$ has units of energy.

SOLUTION. The units of $\hbar^2/I$ are

$$\frac{(\mathrm{J}\cdot\mathrm{s})^2}{\mathrm{kg}\cdot\mathrm{m}^2} = \frac{\mathrm{J}^2}{(\mathrm{kg}\cdot\mathrm{m/s}^2)\mathrm{m}} = \frac{\mathrm{J}^2}{\mathrm{N}\cdot\mathrm{m}} = \frac{\mathrm{J}^2}{\mathrm{J}} = \mathrm{J}.$$

5–7. What is the reduced mass of the molecules (a) NaCl; (b) N_2; (c) HCl?

SOLUTION. The reduced mass of the molecule is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

(a) Using data from the periodic table, for NaCl we have

$$\mu = \frac{m_{\mathrm{Na}} m_{\mathrm{Cl}}}{m_{\mathrm{Na}} + m_{\mathrm{Cl}}} = \frac{(22.9898\ \mathrm{u})(35.4527\ \mathrm{u})}{22.9898\ \mathrm{u} + 35.4527\ \mathrm{u}} = 13.941\ \mathrm{u}.$$

(b) For N_2 we have

$$\mu = \frac{m_{\mathrm{N}} m_{\mathrm{N}}}{m_{\mathrm{N}} + m_{\mathrm{N}}} = \frac{(14.0067\ \mathrm{u})(14.0067\ \mathrm{u})}{14.0067\ \mathrm{u} + 14.0067\ \mathrm{u}} = 7.0034\ \mathrm{u}.$$

(c) For HCl we have

$$\mu = \frac{m_{\mathrm{H}} m_{\mathrm{Cl}}}{m_{\mathrm{H}} + m_{\mathrm{Cl}}} = \frac{(1.00794\ \mathrm{u})(35.4527\ \mathrm{u})}{1.00794\ \mathrm{u} + 35.4527\ \mathrm{u}} = 0.9801\ \mathrm{u}.$$

5–8. The so-called "characteristic rotational energy," $\hbar^2/2I$, for N_2 is 2.48×10^{-4} eV. Calculate the N_2 bond length.

SOLUTION. The reduced mass of the N_2 molecule is

$$\mu_{\mathrm{N}} = \frac{m_{\mathrm{N}} m_{\mathrm{N}}}{m_{\mathrm{N}} + m_{\mathrm{N}}} = \frac{(14.0067\ \mathrm{u})(14.0067\ \mathrm{u})}{14.0067\ \mathrm{u} + 14.0067\ \mathrm{u}} = 7.0034\ \mathrm{u}.$$

We find the bond length from

$$\begin{aligned} \frac{\hbar^2}{2I} &= \frac{\hbar^2}{2\mu_{\mathrm{N}} r^2}; \\ \frac{2.48 \times 10^{-4}\ \mathrm{eV}}{0.625 \times 10^{19}\ \mathrm{eV/J}} &= \frac{(1.055 \times 10^{-34}\ \mathrm{J \cdot s})^2}{2(7.0034\ \mathrm{u})(1.66 \times 10^{-27}\ \mathrm{kg/u}) r^2}, \end{aligned}$$

which gives $r = 1.10 \times 10^{-10}$ m.

5–9. The fundamental vibration frequency for the CO molecule is 6.42×10^{13} Hz. Determine (a) the reduced mass, and (b) the effective value of the constant k. Compare to k for the H_2 molecule.

SOLUTION. (a) The reduced mass of the CO molecule is

$$\mu = \frac{m_{\mathrm{C}} m_{\mathrm{O}}}{m_{\mathrm{C}} + m_{\mathrm{O}}} = \frac{(12.0\ \mathrm{u})(16.0\ \mathrm{u})}{12.0\ \mathrm{u} + 16.0\ \mathrm{u}} = 6.86\ \mathrm{u}.$$

(b) We find the effective spring constant from

$$\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}; \\ 6.42 \times 10^{13}\ \mathrm{Hz} &= \frac{1}{2\pi}\sqrt{\frac{k}{(6.86\ \mathrm{u})(1.66 \times 10^{-27}\ \mathrm{kg/u})}}, \end{aligned}$$

which gives $k = 1.85 \times 10^3$ N/m. This is 3.4 times the constant for H_2.

5–10. Explain why there is no transition for $E = 2\pi\hbar f$ in Fig. 5.5.

SOLUTION. The total energy change for the absorption or emission is

$$\Delta E = \Delta E_{\text{vib}} + \Delta E_{\text{rot}} = 2\pi\hbar f + \Delta E_{\text{rot}}.$$

If $\Delta E = 2\pi\hbar f$, we have $\Delta E_{\text{rot}} = 0$. When L changes by ΔL, we get

$$\begin{aligned} \Delta E_{\text{rot}} &= \frac{\hbar^2}{2I}\left((L+\Delta L)(L+\Delta L+1) - L(L+1)\right) \\ &= \frac{\hbar^2}{2I}\left(2L+1+\Delta L\right)\Delta L. \end{aligned}$$

Thus $\Delta E_{\text{rot}} = 0$ requires $\Delta L = 0$, which is a violation of the selection rule $\Delta L = \pm 1$.

5–11. (a) Calculate the characteristic rotational energy, $\hbar^2/2I$, for the O_2 molecule whose bond length is 0.121 nm. (b) What are the energy and wavelength of photons emitted in a $L = 2$ to $L = 1$ transition?

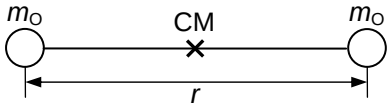


Figure 5.10. Problem 5–11.

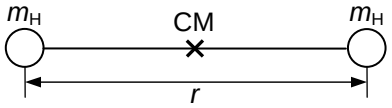


Figure 5.11. Problem 5–12.

SOLUTION. (a) The moment of inertia of O_2 about its CM (Fig. 5.10) is

$$I = 2m_O\left(\frac{r}{2}\right)^2 = \frac{1}{2}m_O r^2.$$

We find the characteristic rotational energy from

$$\begin{aligned} \frac{\hbar^2}{2I} &= \frac{\hbar^2}{m_O r^2} = \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{(16.0\ \text{u})(1.66\times10^{-27}\ \text{kg/u})(0.121\times10^{-9}\ \text{m})^2} \\ &= 2.86\times10^{-23}\ \text{J} = 1.79\times10^{-4}\ \text{eV}. \end{aligned}$$

(b) The rotational energy is

$$E_{\text{rot}} = L(L+1)\frac{\hbar^2}{2I}.$$

Thus the energy of the emitted photon from the $L = 2$ to $L = 1$ transition is

$$\begin{aligned} 2\pi\hbar f &= \Delta E_{\text{rot}} = ((2)(2+1) - (1)(1+1))\,\frac{\hbar^2}{2I} = 4\,\frac{\hbar^2}{2I} \\ &= 4(1.79 \times 10^{-4}\ \text{eV}) = 7.16 \times 10^{-4}\ \text{eV}. \end{aligned}$$

The wavelength is

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} = \frac{1.24 \times 10^3\ \text{eV} \cdot \text{nm}}{7.16 \times 10^{-4}\ \text{eV}} = 1.73 \times 10^6\ \text{nm} = 1.73\ \text{mm}.$$

5–12. The equilibrium separation of H atoms in the H_2 molecule is 0.074 nm (Fig. 5.1). Calculate the energies and wavelengths of photons for the rotational transitions (a) $L = 1$ to $L = 0$, (b) $L = 2$ to $L = 1$, and (c) $L = 3$ to $L = 2$.

SOLUTION. The moment of inertia of H_2 about its CM (Fig. 5.11) is

$$I = 2m_{\text{H}}\left(\frac{r}{2}\right)^2 = \frac{m_{\text{H}}r^2}{2}.$$

We find the characteristic rotational energy from

$$\begin{aligned} \frac{\hbar^2}{2I} &= \frac{\hbar^2}{m_{\text{H}}r^2} = \frac{(1.055 \times 10^{-34}\ \text{J} \cdot \text{s})^2}{(1.67 \times 10^{-27}\ \text{kg})(0.074 \times 10^{-9}\ \text{m})^2} \\ &= 1.22 \times 10^{-21}\ \text{J} = 7.61 \times 10^{-3}\ \text{eV}. \end{aligned}$$

The rotational energy is

$$E_{\text{rot}} = L(L+1)\frac{\hbar^2}{2I}.$$

Thus the energy of the emitted photon from the L to $L-1$ transition is

$$2\pi\hbar f = \Delta E_{\text{rot}} = ((L)(L+1) - (L-1)(L))\,\frac{\hbar^2}{2I} = 2L\frac{\hbar^2}{2I}.$$

(a) For the $L = 1$ to $L = 0$ transition, we get

$$2\pi\hbar f = 2L\frac{\hbar^2}{2I} = 2(1)(7.61 \times 10^{-3}\ \text{eV}) = 1.52 \times 10^{-2}\ \text{eV}.$$

The wavelength is

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{1.52 \times 10^{-2} \text{ eV}} = 8.16 \times 10^4 \text{ nm} = 0.082 \text{ mm}.$$

(b) For the $L = 2$ to $L = 1$ transition, we get

$$2\pi\hbar f = 2L\frac{\hbar^2}{2I} = 2(2)(7.61 \times 10^{-3} \text{ eV}) = 3.04 \times 10^{-2} \text{ eV}.$$

The wavelength is

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{3.04 \times 10^{-2} \text{ eV}} = 4.08 \times 10^4 \text{ nm} = 0.041 \text{ mm}.$$

(c) For the $L = 3$ to $L = 2$ transition, we get

$$2\pi\hbar f = 2L\frac{\hbar^2}{2I} = 2(3)(7.61 \times 10^{-3} \text{ eV}) = 4.56 \times 10^{-2} \text{ eV}.$$

The wavelength is

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{4.56 \times 10^{-2} \text{ eV}} = 2.72 \times 10^4 \text{ nm} = 0.027 \text{ mm}.$$

5–13. Calculate the bond length for the NaCl molecule given that three successive wavelengths for rotational transitions are 23.1 mm, 11.6 mm, and 7.71 mm.

Solution. We find the energies for the transitions from $\Delta E = 2\pi\hbar f = 2\pi\hbar c/\lambda = (1.24 \times 10^3 \text{ eV} \cdot \text{nm})/\lambda$:

$$\begin{aligned}
\Delta E_1 &= \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{23.1 \times 10^6 \text{ nm}} = 5.37 \times 10^{-5} \text{ eV}; \\
\Delta E_2 &= \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{11.6 \times 10^6 \text{ nm}} = 10.7 \times 10^{-5} \text{ eV}; \\
\Delta E_3 &= \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{7.71 \times 10^6 \text{ nm}} = 16.1 \times 10^{-5} \text{ eV}.
\end{aligned}$$

The rotational energy is

$$E_{\text{rot}} = L(L+1)\frac{\hbar^2}{2I}.$$

Thus the energy of the emitted photon from the L to $L-1$ transition is

$$2\pi\hbar f = \Delta E_{\text{rot}} = ((L)(L+1) - (L-1)(L))\,\frac{\hbar^2}{2I} = 2L\frac{\hbar^2}{2I}\,.$$

Because $\Delta E_3 = 3\Delta E_1$, and $\Delta E_2 = 2\Delta E_1$, the three transitions must be from the $L = 1$, 2, and 3 states.

We find the moment of inertia about the center of mass from

$$\begin{aligned}\Delta E_3 &= (16.1\times10^{-5}\text{ eV})(1.60\times10^{-19}\text{ J/eV})\\ &= \frac{2(3)(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{2I}\,,\end{aligned}$$

which gives

$$I = 1.29\times10^{-45}\text{ kg}\cdot\text{m}^2.$$

The reduced mass of the NaCl molecule is

$$\mu = \frac{m_{\text{Na}}m_{\text{Cl}}}{m_{\text{Na}}+m_{\text{Cl}}} = \frac{(22.9898\text{ u})(35.4527\text{ u})}{22.9898\text{ u}+35.4527\text{ u}} = 13.946\text{ u}.$$

We find the bond length from

$$\begin{aligned}I &= \mu r^2;\\ 1.29\times10^{-45}\text{ kg}\cdot\text{m}^2 &= (13.946\text{ u})(1.66\times10^{-27}\text{ kg/u})r^2,\end{aligned}$$

which gives $r = 2.36\times10^{-10}$ m.

5–14. Derive Eqs. 5.9.

SOLUTION. The vibrational energy levels are

$$E_{\text{vib}} = 2\pi\left(\nu+\tfrac{1}{2}\right)\hbar f.$$

The rotational energy levels are

$$E_{\text{rot}} = L(L+1)\frac{\hbar^2}{2I}\,.$$

For an absorption from level (ν, L) to ($\nu+1$, $L+1$), the absorbed

energy is

$$\begin{aligned}\Delta E &= \Delta E_{\text{vib}} + \Delta E_{\text{rot}} \\ &= \left(\left(\nu + 1 + \tfrac{1}{2}\right) - \left(\nu + \tfrac{1}{2}\right)\right) 2\pi\hbar f \\ &\quad + \left((L+1)(L+2) - (L)(L+1)\right) \frac{\hbar^2}{2I} \\ &= 2\pi\hbar f + 2(L+1)\frac{\hbar^2}{2I} = 2\pi\hbar f + (L+1)\frac{\hbar^2}{I}, \quad L = 0,\ 1,\ 2,\ \ldots\ .\end{aligned}$$

For an absorption from level $(\nu,\ L)$ to $(\nu + 1,\ L - 1)$, the absorbed energy is

$$\begin{aligned}\Delta E &= \Delta E_{\text{vib}} + \Delta E_{\text{rot}} \\ &= \left(\left(\nu + 1 + \tfrac{1}{2}\right) - \left(\nu + \tfrac{1}{2}\right)\right) 2\pi\hbar f \\ &\quad + \left((L-1)(L) - (L)(L+1)\right) \frac{\hbar^2}{2I} \\ &= 2\pi\hbar f - 2L\frac{\hbar^2}{2I} = 2\pi\hbar f - L\frac{\hbar^2}{I} \quad L = 0,\ 1,\ 2,\ \ldots\ .\end{aligned}$$

5.4 Bonding in solids

5–15. Estimate the ionic cohesive energy for NaCl taking $\alpha = 1.75$, $m = 8$, and $r_0 = 0.28$ nm.

Solution. The ionic cohesive energy is

$$\begin{aligned}U_0 &= -\frac{\alpha k_{\text{e}} e^2}{r}\left(1 - \frac{1}{m}\right) \\ &= -\frac{(1.75)(2.30 \times 10^{-28}\ \text{J}\cdot\text{m})}{(0.28 \times 10^{-9}\ \text{m})(1.60 \times 10^{-19}\ \text{J/eV})}\left(1 - \frac{1}{8}\right) \\ &= -7.9\ \text{eV}.\end{aligned}$$

5–16. The spacing between "nearest neighbor" Na and Cl ions in a NaCl crystal is 0.24 nm. What is the spacing between two nearest neighbor Na ions?

Solution. From Fig. 5.12 we see that the distance between nearest neighbor Na ions is the diagonal of the cube:

$$D = d\sqrt{2} = (0.24\ \text{nm})\sqrt{2} = 0.34\ \text{nm}.$$

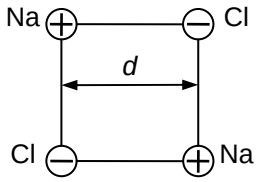


Figure 5.12. Problem 5–16.

5–17. Common salt, NaCl, has a density of $2.165\ \text{g/cm}^3$. The molecular weight of NaCl is 58.44. Estimate the distance between nearest neighbor Na and Cl ions. [*Hint*: Each ion can be considered to have one "cube" or "cell" of side s (our unknown) extending out from it.]

SOLUTION. Because each ion occupies a cell of side s, a molecule occupies two cells. Thus the density is

$$\begin{aligned} \rho &= \frac{m_{\text{NaCl}}}{2s^3}; \\ 2.165 \times 10^3\ \text{kg/m}^3 &= \frac{(58.44\ \text{u})(1.66 \times 10^{-27}\ \text{kg/u})}{2s^3}, \end{aligned}$$

which gives

$$s = 2.83 \times 10^{-10}\ \text{m} = 0.283\ \text{nm}.$$

5–18. Repeat Problem 5–17 for KCl whose density is $1.99\ \text{g/cm}^3$.

SOLUTION. Because each ion occupies a cell of side s, a molecule occupies two cells. Thus the density is

$$\begin{aligned} \rho &= \frac{m_{\text{KCl}}}{2s^3}; \\ 1.99 \times 10^3\ \text{kg/m}^3 &= \frac{(39.1\ \text{u} + 35.5\ \text{u})(1.66 \times 10^{-27}\ \text{kg/u})}{2s^3}, \end{aligned}$$

which gives

$$s = 3.15 \times 10^{-10}\ \text{m} = 0.315\ \text{nm}.$$

5–19. (a) Starting from Eq. 5.10, show that the ionic cohesive energy is given by

$$U_0 = -\frac{\alpha k_e e^2}{r_0}\left(1 - \frac{1}{m}\right).$$

Determine U_0 for (b) NaI ($r_0 = 0.33$ nm) and (c) MgO ($r_0 = 0.21$ nm). Assume $m = 10$. (d) If you used $m = 8$ instead, how far off would your answers be?

SOLUTION. (a) The potential energy is

$$U = -\alpha k_e \frac{e^2}{r} + \frac{B}{r^m}.$$

The separation at equilibrium occurs at the minimum in the energy. We can relate this distance r_0 to the value of B by setting $dU/dr = 0$:

$$\frac{dU}{dr} = \alpha k_e \frac{e^2}{r^2} - \frac{mB}{r^{m+1}} = 0$$

or

$$\frac{\alpha k_e e^2}{r_0^2} = \frac{mB}{r_0^{m+1}},$$

which gives

$$B = \frac{\alpha k_e e^2 r_0^{m-1}}{m}.$$

The ionic cohesive energy is the value of U at the equilibrium distance:

$$U_0 = -\alpha k_e \frac{e^2}{r_0} + k_e \frac{\alpha e^2 r_0^{m-1}}{m r_0^m} = -\frac{\alpha k_e e^2}{r_0}\left(1 - \frac{1}{m}\right).$$

(b) For NaI we have

$$U_0 = -\frac{(1.75)(2.30 \times 10^{-28}\ \text{J} \cdot \text{m})}{(1.60 \times 10^{-19}\ \text{J/eV})(0.33 \times 10^{-9}\ \text{m})}\left(1 - \frac{1}{10}\right) = -6.9\ \text{eV}.$$

(c) If we assume the same value for the Madelung constant, for MgO we have

$$U_0 = -\frac{(1.75)(2.30 \times 10^{-28}\ \text{J} \cdot \text{m})}{(1.60 \times 10^{-19}\ \text{J/eV})(0.21 \times 10^{-9}\ \text{m})}\left(1 - \frac{1}{10}\right) = -10.8\ \text{eV}.$$

(d) If we use a new value for m, the fractional change is

$$\frac{\Delta U_0}{U_0} = \frac{(1 - \frac{1}{8}) - (1 - \frac{1}{10})}{1 - \frac{1}{10}} = -0.028 = -3\%.$$

5–20. For a long one-dimensional chain of alternating positive and negative ions, show that the Madelung constant would be $\alpha = 2\ln 2$ [*Hint*: Use a series expansion for $\ln(1+x)$.]

SOLUTION. If we select a charge in the middle of the chain, there will be two charges of opposite sign a distance r away, two charges of the same sign a distance $2r$ away, etc. (Fig. 5.13). The potential energy of the charge is

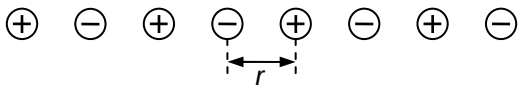


Figure 5.13. Problem 5–20.

$$\begin{aligned} U &= -k_e\frac{2e^2}{r} + k_e\frac{2e^2}{2r} - k_e\frac{2e^2}{3r} + k_e\frac{2e^2}{4r} - \dots \\ &= -k_e\frac{2e^2}{r}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right). \end{aligned}$$

If we consider the expansion of $\ln(1+x)$, which is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\ ,$$

we see that

$$\ln(1+1) = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right).$$

Thus we have

$$U = -k_e\frac{2e^2}{r}\ln(1+1),$$

so $\alpha = 2\ln 2$.

5.5 Free-electron theory of metals

5–21. Estimate the number of states available to electrons in a 1.0-cm^3 cube of copper between 6.90 and 7.00 eV.

SOLUTION. The density of energy states in a small energy interval is

$$dn = g(E)\,dE.$$

Because the energy range is small we assume a constant density at the middle of the range:

$$\begin{aligned}\Delta N &= Vg(E)\,\Delta E = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}\,\Delta E \\ &= \frac{(1.0\times10^{-6}\ \mathrm{m}^3)}{2\pi^2}\left(\frac{2(9.1\times10^{-31}\ \mathrm{kg})}{(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}\right)^{3/2} \\ &\quad\times(6.95\ \mathrm{eV})^{1/2}(0.10\ \mathrm{eV})(1.60\times10^{-19}\ \mathrm{J/eV})^{3/2} \\ &= 1.8\times10^{21}.\end{aligned}$$

5–22. What, roughly, is the ratio of the density of molecules in an ideal gas at 300 K and 1 atm (say O_2) to the density of free electrons (assume one per atom) in a metal (copper) also at 300 K?

Solution. We find the density of molecules in an ideal gas from

$$PV = nRT = Nk_{\mathrm{B}}T,$$

or

$$\frac{N}{V} = \frac{P}{k_{\mathrm{B}}T} = \frac{1.013\times10^5\ \mathrm{Pa}}{(1.38\times10^{-23}\ \mathrm{J/K})(300\ \mathrm{K})} = 2.4\times10^{25}\ \mathrm{m}^{-3}.$$

If we assume each copper atom contributes one free electron, the density of free electrons is

$$n = \frac{\rho N_A}{M} = \frac{(8.89\times10^3\ \mathrm{kg/m^3})(6.02\times10^{23}/\mathrm{mol})}{(63.5\times10^{-3}\ \mathrm{kg/mol})} = 8.4\times10^{28}\ \mathrm{m}^{-3}.$$

Thus the ratio is

$$\frac{(N/V)}{n} = \frac{2.4\times10^{25}\ \mathrm{m}^{-3}}{8.4\times10^{28}\ \mathrm{m}^{-3}} = 3\times10^{-4}.$$

5–23. Calculate the energy which has 90 percent occupancy probability for copper at (a) $T = 300$ K; (b) $T = 1\,200$ K. [For copper $E_{\mathrm{F}} = 7.0$ eV (see Example 5–9).]

Solution. (a) The value of $k_{\mathrm{B}}T$ is

$$k_{\mathrm{B}}T = \frac{(1.38\times10^{-23}\ \mathrm{J/K})(300\ \mathrm{K})}{1.60\times10^{-19}\ \mathrm{J/eV}} = 0.0259\ \mathrm{eV}.$$

We find the energy from

$$\begin{aligned} f &= \frac{1}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T}+1}; \\ 0.90 &= \frac{1}{\exp\left((E-7.0\text{ eV})/(0.0259\text{ eV})\right)+1}, \end{aligned}$$

which gives $E = 6.9$ eV.

(b) The value of $k_{\mathrm{B}}T$ is

$$k_{\mathrm{B}}T = \frac{(1.38\times10^{-23}\text{ J/K})(1\,200\text{ K})}{1.60\times10^{-19}\text{ J/eV}} = 0.1035\text{ eV}.$$

We find the energy from

$$\begin{aligned} f &= \frac{1}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T}+1}; \\ 0.90 &= \frac{1}{\exp\left((E-7.0\text{ eV})/(0.1035\text{ eV})\right)+1}, \end{aligned}$$

which gives $E = 6.8$ eV.

5–24. Calculate the energy which has 10 percent occupancy probability for copper at (a) $T = 300$ K; (b) $T = 1\,200$ K. [For copper $E_{\mathrm{F}} = 7.0$ eV (see Example 5–9).]

SOLUTION. (a) The value of $k_{\mathrm{B}}T$ is

$$k_{\mathrm{B}}T = \frac{(1.38\times10^{-23}\text{ J/K})(300\text{ K})}{1.60\times10^{-19}\text{ J/eV}} = 0.0259\text{ eV}.$$

We find the energy from

$$\begin{aligned} f &= \frac{1}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T}+1}; \\ 0.10 &= \frac{1}{\exp\left((E-7.0\text{ eV})/(0.0259\text{ eV})\right)+1}, \end{aligned}$$

which gives $E = 7.1$ eV.

(b) The value of $k_{\mathrm{B}}T$ is

$$k_{\mathrm{B}}T = \frac{(1.38\times10^{-23}\text{ J/K})(1\,200\text{ K})}{1.60\times10^{-19}\text{ J/eV}} = 0.1035\text{ eV}.$$

We find the energy from

$$\begin{aligned} f &= \frac{1}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T}+1}; \\ 0.10 &= \frac{1}{\exp\left((E-7.0\text{ eV})/(0.1035\text{ eV})\right)+1}, \end{aligned}$$

which gives $E = 7.2$ eV.

5–25. What is the occupancy probability for a conduction electron in copper at $T = 300$ K for an energy $E = 1.010\,E_{\mathrm{F}}$? [For copper $E_{\mathrm{F}} = 7.0$ eV (see Example 5–9).]

Solution. The value of $k_{\mathrm{B}}T$ is

$$k_{\mathrm{B}}T = \frac{(1.38\times10^{-23}\text{ J/K})(300\text{ K})}{1.60\times10^{-19}\text{ J/eV}} = 0.0259\text{ eV}.$$

We find the occupancy probability from

$$\begin{aligned} f &= \frac{1}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T}+1} \\ &= \frac{1}{\exp\left((1.010-1.000)(7.0\text{ eV})/(0.0259\text{ eV})\right)+1} \\ &= 0.063 = 6.3\%. \end{aligned}$$

5–26. Calculate the number of possible electron states in a 1.00-cm^3 cube of silver between $0.99E_{\mathrm{F}}$ and E_{F} ($= 5.48$ eV).

Solution. The density of energy states in a small energy interval is

$$dn = g(E)\,dE.$$

Because the energy range is small we assume a constant density at the middle of the range:

$$\begin{aligned} \Delta N &= Vg(E)\Delta E = \frac{(1.0\times10^{-6}\text{ m}^3)}{2\pi^2}\left(\frac{2(9.11\times10^{-31}\text{ kg})}{(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}\right)^{3/2} \\ &\quad\times\left((0.995)(5.48\text{ eV})\right)^{1/2} \\ &\quad\times(0.01)(5.48\text{ eV})(1.60\times10^{-19}\text{ J/eV})^{3/2} \\ &= 8.7\times10^{20}. \end{aligned}$$

5–27. Calculate the Fermi energy and Fermi speed for sodium, which has a density of 0.97×10^3 kg/m^3 and has one conduction electron per atom.

SOLUTION. Because each sodium atom contributes one conduction electron, the density of conduction electrons is

$$n = \frac{\rho N_{\mathrm{A}}}{M} = \frac{(0.97 \times 10^3 \text{ kg/m}^3)(6.02 \times 10^{23}/\text{mol})}{23.0 \times 10^{-3} \text{ kg/mol}} = 2.539 \times 10^{28} \text{ m}^{-3}.$$

We find the Fermi energy from

$$\begin{aligned} E_{\mathrm{F}} &= \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \\ &= \frac{(1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(9.11 \times 10^{-31} \text{ kg})}\left(3\pi^2(2.539 \times 10^{28} \text{ m}^{-3})\right)^{2/3} \\ &= 5.05 \times 10^{-19} \text{ J} = 3.2 \text{ eV}. \end{aligned}$$

We find the Fermi speed as the speed which gives a kinetic energy equal to the Fermi energy:

$$\begin{aligned} E_{\mathrm{F}} &= \frac{m v_{\mathrm{F}}^2}{2}; \\ 5.05 \times 10^{-18} \text{ J} &= \frac{(9.11 \times 10^{-31} \text{ kg}) v_{\mathrm{F}}^2}{2}, \end{aligned}$$

which gives $v_{\mathrm{F}} = 1.05 \times 10^6$ m/s.

5–28. The atoms in zinc metal ($\rho = 7.1 \times 10^3$ kg/m^3) have two free electrons. Calculate (a) the density of conduction electrons, (b) their Fermi energy, and (c) their Fermi speed.

SOLUTION. (a) Because each zinc atom contributes two free electrons, the density of free electrons is twice the density of atoms:

$$n = \frac{2\rho N_{\mathrm{A}}}{M} = \frac{2(7.1 \times 10^3 \text{ kg/m}^3)(6.02 \times 10^{23}/\text{mol})}{65.38 \times 10^{-3} \text{ kg/mol}} = 1.31 \times 10^{29} \text{ m}^{-3}.$$

(b) We find the Fermi energy from

$$\begin{aligned} E_{\mathrm{F}} &= \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \\ &= \frac{(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}{2(9.11\times10^{-31}\ \mathrm{kg})}\left(3\pi^2(1.31\times10^{29}\ \mathrm{m^{-3}})\right)^{2/3} \\ &= 1.50\times10^{-18}\ \mathrm{J} = 9.4\ \mathrm{eV}. \end{aligned}$$

(c) We find the Fermi speed as the speed which gives a kinetic energy equal to the Fermi energy:

$$\begin{aligned} E_{\mathrm{F}} &= \frac{m v_{\mathrm{F}}^2}{2}; \\ 1.50\times10^{-18}\ \mathrm{J} &= \frac{(9.11\times10^{-31}\ \mathrm{kg})v_{\mathrm{F}}^2}{2}, \end{aligned}$$

which gives $v_{\mathrm{F}} = 1.81\times10^6$ m/s.

5–29. Given that the Fermi energy of aluminum is 11.63 eV, (a) calculate the density of free electrons using Eq. 5.13, and (b) estimate the valence of aluminum using this model and the known density (2.70×10^3 kg/m^3) and atomic weight (27.0) of aluminum.

Solution. (a) We find the density of free electrons from

$$\begin{aligned} E_{\mathrm{F}} &= \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}; \\ (11.63\ \mathrm{eV})(1.60\times10^{-19}\ \mathrm{J/eV}) &= \frac{(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}{2(9.11\times10^{-31}\ \mathrm{kg})}\left(3\pi^2 n\right)^{2/3} \end{aligned}$$

which gives $n = 1.79\times10^{29}\ \mathrm{m^{-3}}$.

(b) If we let ν be the valence, we can relate the density of free electrons to the atomic density:

$$\begin{aligned} n &= \frac{\nu\rho N_{\mathrm{A}}}{M}; \\ 1.79\times10^{29}\ \mathrm{m^{-3}} &= \frac{\nu(2.70\times10^3\ \mathrm{kg/m^3})(6.02\times10^{23}/\mathrm{mol})}{27.0\times10^{-3}\ \mathrm{kg/mol}}, \end{aligned}$$

which gives $\nu = 3$.

5–30. The neutrons in a neutron star can be treated as a Fermi gas with neutrons in place of the electrons in our model of an electron gas. Determine the Fermi energy for a neutron star of radius 10 km and mass twice that of our Sun. Assume that the star is made entirely of neutrons and is of uniform density.

SOLUTION. We find the density of neutrons from

$$n = \frac{M_{\rm S}}{m_{\rm n} V} = \frac{2(2.0 \times 10^{30}\ {\rm kg})}{(1.67 \times 10^{-27}\ {\rm kg})\frac{4}{3}\pi(10 \times 10^{3}\ {\rm m})^3} = 5.72 \times 10^{44}\ {\rm m}^{-3}.$$

We find the Fermi energy from

$$\begin{aligned} E_{\rm F} &= \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \\ &= \frac{(1.055 \times 10^{-34}\ {\rm J \cdot s})^2}{2(1.67 \times 10^{-27}\ {\rm kg})}\left(3\pi^2(5.72 \times 10^{44}\ {\rm m}^{-3})\right)^{2/3} \\ &= 2.18 \times 10^{-11}\ {\rm J} = 1.4 \times 10^{2}\ {\rm MeV}. \end{aligned}$$

5–31. Show that the average energy of conduction electrons in a metal at $T = 0$ K is $\overline{E} = \frac{3}{5}E_{\rm F}$ (Eq. 5.14) by calculating

$$\overline{E} = \frac{\int E n_{\rm O}(E)\, dE}{\int n_{\rm O}(E)\, dE}.$$

SOLUTION. The maximum energy of an electron at $T = 0$ K is $E_{\rm F}$. All states above $E_{\rm F}$ are unoccupied, while all states with $E = E_{\rm F}$ are occupied. We find the average energy of an electron from

$$\begin{aligned} \overline{E} &= \frac{E_{\rm tot}}{N} = \frac{\displaystyle\int_0^{E_{\rm F}} E n_{\rm O}(E)\, dE}{\displaystyle\int_0^{E_{\rm F}} n_{\rm O}(E)\, dE} = \frac{\left(\sqrt{2}\,\pi^{-2} m^{3/2}\hbar^{-3}\right)\displaystyle\int_0^{E_{\rm F}} E^{3/2}\, dE}{\left(\sqrt{2}\,\pi^{-2} m^{3/2}\hbar^{-3}\right)\displaystyle\int_0^{E_{\rm F}} E^{1/2}\, dE} \\ &= \frac{\frac{2}{5}E_{\rm F}^{5/2}}{\frac{2}{3}E_{\rm F}^{3/2}} = \frac{3}{5}E_{\rm F}. \end{aligned}$$

5–32. Show that the probability for the state at the Fermi energy being occupied is exactly $\frac{1}{2}$, independent of temperature.

SOLUTION. The Fermi-Dirac distribution gives the probability that a state is occupied. If $E = E_{\mathrm{F}}$, we have

$$f = \frac{1}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T} + 1} = \frac{1}{e^0 + 1} = \frac{1}{2}\,.$$

5–33. (a) For copper at room temperature ($T = 300$ K), calculate the Fermi factor, Eq. 5.15, for an electron with energy 0.10 eV above the Fermi energy. This represents the probability that this state is occupied. Is this reasonable? (b) What is the probability that a state 0.10 eV below the Fermi energy is occupied? (c) What is the probability that the state in part (b) is unoccupied?

SOLUTION. We find the occupation probability from

$$f = \frac{1}{e^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T} + 1}.$$

The value of $k_{\mathrm{B}}T$ is

$$k_{\mathrm{B}}T = \frac{(1.38 \times 10^{-23}\ \mathrm{J/K})(300\ \mathrm{K})}{1.6 \times 10^{-19}\ \mathrm{J/eV}} = 0.0259\ \mathrm{eV}.$$

(a) For $E - E_{\mathrm{F}} = 0.10$ eV, we have

$$f = \frac{1}{\exp\left((0.10\ \mathrm{eV})/(0.0259\ \mathrm{eV})\right) + 1} = \frac{1}{\exp(3.86) + 1} = 0.021.$$

This is reasonable. For a good conductor, higher states should have low occupancy.

(b) For $E - E_{\mathrm{F}} = -0.10$ eV, we have

$$f = \frac{1}{\exp\left((-0.10\ \mathrm{eV})/(0.0259\ \mathrm{eV})\right) + 1} = \frac{1}{\exp(-3.86) + 1} = 0.979.$$

(c) Because a state must be either occupied or unoccupied, we have

$$P_{\mathrm{unoccupied}} = 1 - f = 1 - 0.979 = 0.021.$$

Note that this is the same as the occupation probability for a state 0. 10 eV above the Fermi energy.

5–34. For a one-dimensional potential well, start with Eq. 3.9 and show that the number of states per unit energy interval for an electron gas is given by

$$g_L(E) = \sqrt{\frac{2mL^2}{\pi^2\hbar^2 E}}\,.$$

Remember that there can be two electrons (spin up and spin down) for each value of n. [*Hint*: Write the quantum number n in terms of E. Then $g_L(E) = 2\ dn/dE$ where dn is the number of energy levels between E and $E + dE$.]

SOLUTION. The energy levels for a one-dimensional potential well are

$$E = \frac{\pi^2\hbar^2}{2mL^2}\,n^2, \quad n = 1,\ 2,\ 3,\ \ldots\ .$$

The quantum number n also gives the number of levels with energies between 0 and E.

$$n = \sqrt{\frac{2mL^2}{\pi^2\hbar^2}}\,\sqrt{E}\,.$$

The number of levels with energies between E and dE is

$$dn = \frac{1}{2}\sqrt{\frac{2mL^2}{\pi^2\hbar^2}}\,\frac{1}{\sqrt{E}}\,dE.$$

Because there can be two electrons with opposite spins in each state, the number of electron states per unit energy interval is

$$g_L(E) = 2\frac{dn}{dE} = \sqrt{\frac{2mL^2}{\pi^2\hbar^2 E}}\,.$$

5–35. Proceed as follows to derive the density of states, $g(E)$, the number of states per unit volume per unit energy interval, Eq. 5.11. Let the metal be a cube of side L. Extend the discussion for an infinite well to three dimensions, giving energy levels

$$E = \frac{\pi^2\hbar^2}{2mL^2}\left(n_1^2 + n_2^2 + n_3^2\right).$$

(Explain the meaning of n_1, n_2, n_3.) Each set of values for the quantum numbers n_1, n_2, n_3 corresponds to one state. Imagine a space where n_1, n_2, n_3, are the axes, and each state is represented by a point on a cubic lattice in this space, each separated by one unit along an axis. Consider the octant $n_1 > 0$, $n_2 > 0$, $n_3 > 0$. Show that the number of states N within a radius

$$R = \sqrt{n_1^2 + n_2^2 + n_3^2}$$

is

$$2\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi R^3\right).$$

Then, to get Eq. 5.11, set

$$g(E) = \frac{1}{V}\frac{dN}{dE},$$

where $V = L^3$ is the volume of the metal.

SOLUTION. If we consider the cube to be a three-dimensional infinite well, we can apply the boundary conditions separately to each dimension. Each dimension gives a quantum number which we label n_1, n_2, n_3.

Thus there is a contribution to the energy from each dimension equal to the energy from the one-dimensional well:

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2}n_1^2;\ E_2 = \frac{\pi^2\hbar^2}{2mL^2}n_2^2;\ E_3 = \frac{\pi^2\hbar^2}{2mL^2}n_3^2,$$
$$n_1,\ n_2,\ n_3 = 1,\ 2,\ 3,\ \ldots\ .$$

The energy of a state specified by the three quantum numbers is

$$E = \frac{\pi^2\hbar^2}{2mL^2}(n_1^2 + n_2^2 + n_3^2).$$

If we create a three-dimensional space with the axes labeled by n_1, n_2, n_3, as shown in Fig. 5.14, each state corresponds to a point in the lattice. When we construct a sphere of radius R, where $R^2 = n_1^2 + n_2^2 + n_3^2$, each point within the octant corresponding to positive values for n_1, n_2, n_3 represents a state with energy between 0 and $E = \pi^2\hbar^2R^2/2mL^2$. Because

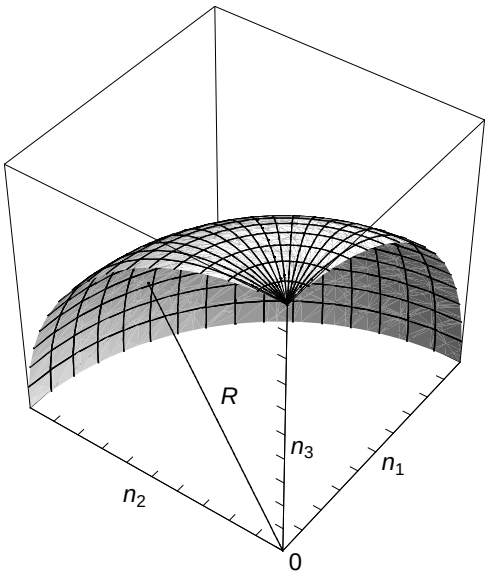


Figure 5.14. Problem 5–35.

the density of points is one and there can be two electrons in each state, the number of electrons with energy between 0 and E is

$$N = 2\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi R^3\right) = \frac{1}{3}\pi R^3 = \left(\frac{\pi}{3}\right)\left(\frac{2mL^2E}{\pi^2\hbar^2}\right)^{3/2}.$$

We find the density of states with energy E from

$$\begin{aligned} g(E) &= \frac{dn}{dE} = \frac{1}{V}\frac{dN}{dE} = \frac{3}{2}\left(\frac{1}{L^3}\right)\left(\frac{\pi}{3}\right)\left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{3/2}\sqrt{E} \\ &= \frac{\sqrt{2}\,m^{3/2}}{\pi^2\hbar^3}\sqrt{E}\,. \end{aligned}$$

5.6 Band theory of solids

5–36. A semiconductor, bombarded with light of slowly increased frequency, begins to conduct when the wavelength of light is 640 nm; estimate

the size of the energy gap E_g.

SOLUTION. The photon with the minimum frequency for conduction must have an energy equal to the energy gap:

$$E_g = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{640 \text{ nm}} = 1.94 \text{ eV}.$$

5–37. Explain on the basis of energy bands why the sodium chloride crystal is a good insulator. [*Hint*: Consider shells of Na^+ and Cl^- ions.]

SOLUTION. The partially filled shell in Na is the $3s$ shell, which has 1 electron in it. The partially filled shell in Cl is the $2p$ shell which has 5 electrons in it. In NaCl the electron from the $3s$ shell in Na is transferred to the $2p$ shell in Cl, which results in filled shells for both ions. Thus when many ions are considered, the resulting bands are either completely filled (the valence band) or completely empty (the conduction band). Thus a large energy is required to create a conduction electron by raising an electron from the valence band to the conduction band.

5–38. We saw that there are $2N$ possible electron states in the $3s$ band of Na, where N is the total number of atoms. How many possible electron states are there in the (a) $2s$ band, (b) $2p$ band, and (c) $3p$ band? (d) State a general formula for the total number of possible states in any given electron band.

SOLUTION. (a) In the $2s$ shell of an atom, $\ell = 0$, so there are two states: $m_s = \pm\frac{1}{2}$. When N atoms form bands, each atom provides 2 states, so the total number of states in the band is $2N$.

(b) In the $2p$ shell of an atom, $\ell = 1$, so there are three states from the m_ℓ values: $m_\ell = 0, \pm 1$; each of which has two states from the m_s values: $m_s = \pm\frac{1}{2}$, for a total of 6 states. When N atoms form bands, each atom provides 6 states, so the total number of states in the band is $6N$.

(c) In the $3p$ shell of an atom, $\ell = 1$, so there are three states from the m_ℓ values: $m_\ell = 0, \pm 1$; each of which has two states from the m_s values: $m_s = \pm\frac{1}{2}$, for a total of 6 states. When N atoms form bands, each atom provides 6 states, so the total number of states in the band is $6N$.

(d) In general, for a value of ℓ, there are $2\ell + 1$ states from the m_ℓ values: $m_\ell = 0,\ \pm 1,\ \ldots,\ \pm\ell$. For each of these there are two states from the m_s values: $m_s = \pm\frac{1}{2}$, for a total of $2(2\ell + 1)$ states. When N atoms form bands, each atom provides $2(2\ell + 1)$ states, so the total number of states in the band is $2N(2\ell + 1)$.

5–39. Calculate the longest-wavelength photon that can cause an electron in silicon ($E_g = 1.1$ eV) to jump from the valence band to the conduction band.

SOLUTION. The photon with the longest wavelength or minimum frequency for conduction must have an energy equal to the energy gap:

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} = \frac{2\pi\hbar c}{E_g} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{1.1 \text{ eV}} = 1.1 \times 10^3 \text{ nm}.$$

5–40. The energy gap E_g in germanium is 0.72 eV. When used as a photon detector, roughly how many electrons can be made to jump from the valence to the conduction band by the passage of a 710-keV photon that loses all its energy in this fashion?

SOLUTION. The minimum energy provided to an electron must be equal to the energy gap:

$$E_g = 0.72 \text{ eV}.$$

Thus the maximum number of electrons is

$$N = \frac{2\pi\hbar f}{E_g} = \frac{710 \times 10^3 \text{ eV}}{0.72 \text{ eV}} = 9.9 \times 10^5.$$

5.7 Semiconductors and doping

5–41. Suppose that a silicon semiconductor is doped with phosphorus so that one silicon atom in 10^6 is replaced by a phosphorus atom. Assuming that the "extra" electron in every phosphorus atom is donated to the conduction band, by what factor is the density of conduction electrons increased? The density of silicon is 2 330 kg/m^3, and the density of conduction electrons in pure silicon is about 10^{16} m^{-3} at room temperature.

SOLUTION. If we consider a mole of pure silicon (28 g or 6.02×10^{23} atoms), the number of conduction electrons is

$$N_{\text{Si}} = \frac{28 \times 10^{-3} \text{ kg}}{2\,330 \text{ kg/m}^3} (10^{16} \text{ electrons/m}^3) = 1.20 \times 10^{11}.$$

The additional conduction electrons provided by the doping is

$$N_{\text{doping}} = \frac{6.02 \times 10^{23} \text{ atoms}}{10^6} = 6.02 \times 10^{17}.$$

Thus the density of conduction electrons has increased by

$$\frac{N_{\text{doping}}}{N_{\text{Si}}} = \frac{6.02 \times 10^{17}}{1.20 \times 10^{11}} = 5 \times 10^{6}.$$

5.8 Semiconductor diodes

5–42. At what wavelength will an LED radiate if it made from a material with an energy gap $E_{\text{g}} = 1.4$ eV?

SOLUTION. The photon will have an energy equal to the energy gap:

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} = \frac{2\pi\hbar c}{E_{\text{g}}} = \frac{1.24 \times 10^{3}\ \text{eV} \cdot \text{nm}}{1.4\ \text{eV}} = 8.9 \times 10^{2}\ \text{nm}.$$

5–43. If an LED emits light of wavelength $\lambda = 650$ nm, what is the energy gap (in eV) between valence and conduction bands?

SOLUTION. The photon will have an energy equal to the energy gap:

$$E_{\text{g}} = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{1.24 \times 10^{3}\ \text{eV} \cdot \text{nm}}{650\ \text{nm}} = 1.91\ \text{eV}.$$

5–44. A silicon diode, whose current-voltage characteristics are given in Fig. 5.6, is connected in series with a battery and a 760-Ω resistor. What battery voltage is needed to produce a 12-mA current?

SOLUTION. From the current-voltage characteristic, we see that a current of 12 mA means a voltage of 0.7 V across the diode. Thus the battery voltage is

$$V_{\text{battery}} = V_{\text{diode}} + V_R = 0.7\ \text{V} + (12 \times 10^{-3}\ \text{A})(760\ \Omega) = 9.8\ \text{V}.$$

5–45. Suppose that the diode of Fig. 5.6 is connected in series to a 100-Ω resistor and a 2.0-V battery. What current flows in the circuit? [*Hint*: Draw a line on Fig. 5.6 representing the current in the resistor as a function of the voltage across the diode; the intersection of this line with the characteristic curve will give the answer.]

SOLUTION. The battery voltage is

$$\begin{aligned} V_{\text{battery}} &= V_{\text{diode}} + V_R; \\ 2.0\ \text{V} &= V(I) + I(0.100\ \text{k}\Omega), \end{aligned}$$

or

$$V(I) = 2.0\ \text{V} - I(0.100\ \text{k}\Omega).$$

This is a straight line which passes through the points (20 mA, 0 V) and (12 mA, 0.8 V), as drawn in Fig. 5.15. Because $V(I)$ is represented by both curves, the intersection will give the current, which we see is 13 mA.

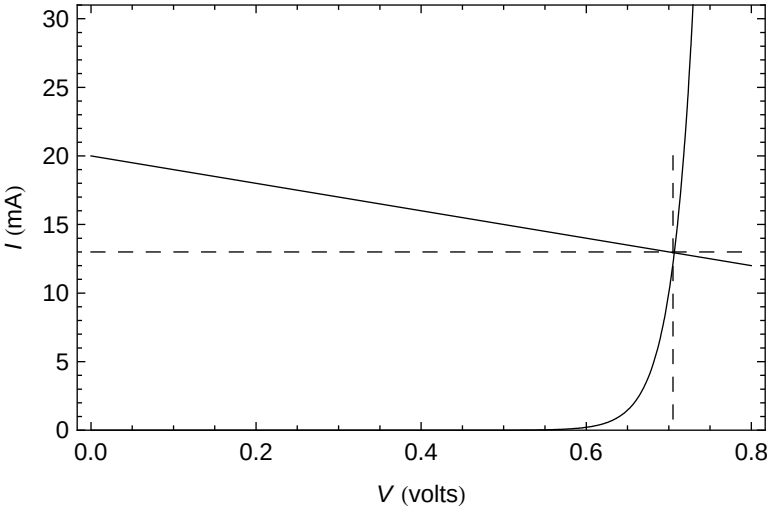


Figure 5.15. Problem 5–45.

5–46. Sketch the resistance as a function of current, for $V > 0$, for the diode shown in Fig. 5.6.

SOLUTION. See Fig. 5.16.

5–47. An ac voltage of 220 V rms is to be rectified. Estimate very roughly the average current in the output resistor R (28 kΩ) for (a) a half-wave rectifier (Fig. 5.7), and (b) a full-wave rectifier (Fig. 5.8) without capacitor.

SOLUTION. (a) For a half-wave rectifier without a capacitor, the current is zero for half the time. Thus the average current is

$$I_{\text{av}} = \frac{1}{2}\frac{V_{\text{rms}}}{R} = \frac{1}{2}\frac{220\ \text{V}}{28\ \text{k}\Omega} = 3.9\ \text{mA}.$$

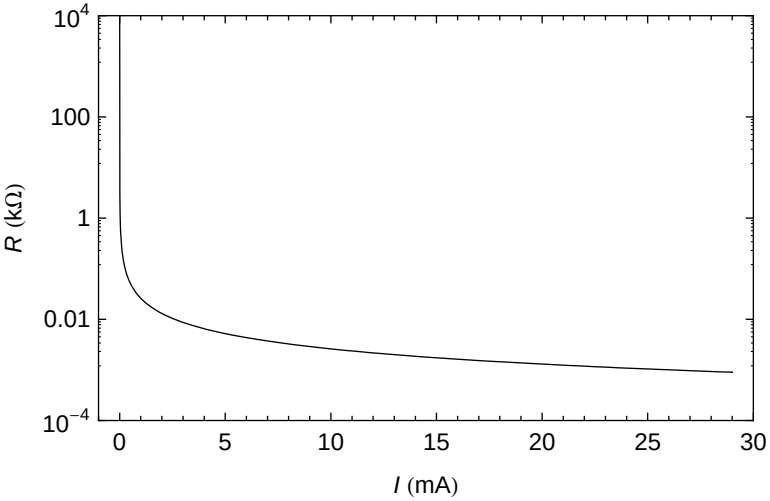


Figure 5.16. Problem 5–46.

(b) For a full-wave rectifier without a capacitor, the current is positive all the time. Thus the average current is

$$I_{\text{av}} = \frac{V_{\text{rms}}}{R} = \frac{220\ \text{V}}{28\ \text{k}\Omega} = 7.9\ \text{mA}.$$

5–48. A silicon diode passes significant current only if the forward-bias voltage exceeds about 0.6 V. Make a rough estimate of the average current in the output resistor R of (a) a half-wave rectifier (Fig. 5.7), and (b) a full-wave rectifier (Fig. 5.8) without a capacitor. Assume that $R = 150\ \Omega$ in each case and that the ac voltage is 9.0 V rms in each case.

SOLUTION. There will be a current in the resistor while the ac voltage varies from 0.6 V to 9.0 V rms. Because the 0.6 V is small, the voltage across the resistor will be almost sinusoidal, so the rms voltage across the resistor will be close to

$$9.0\ \text{V} - 0.6\ \text{V} = 8.4\ \text{V}.$$

(a) For a half-wave rectifier without a capacitor, the current is zero for half the time. If we ignore the short time it takes to reach 0.6 V, this will

also be true for the resistor. Thus the average current is

$$I_{\text{av}} = \frac{1}{2}\frac{V_{\text{rms}}}{R} = \frac{1}{2}\frac{8.4\text{ V}}{0.150\text{ k}\Omega} = 28\text{ mA}.$$

(b) For a full-wave rectifier without a capacitor, the current is positive all the time. If we ignore the short time it takes to reach 0.6 V, this will also be true for the resistor. Thus the average current is

$$I_{\text{av}} = \frac{V_{\text{rms}}}{R} = \frac{8.4\text{ V}}{0.150\text{ k}\Omega} = 56\text{ mA}.$$

5–49. A 220-V rms 50-Hz voltage is to be rectified with a full-wave rectifier as in Fig. 5.8, where $R = 18\text{ k}\Omega$, and $C = 25\ \mu\text{F}$. (a) Make a rough estimate of the average current. (b) What happens if $C = 0.10\ \mu\text{F}$?

SOLUTION. (a) The time constant for the circuit is

$$\tau_1 = RC_1 = (18 \times 10^3\ \Omega)(25 \times 10^{-6}\text{ F}) = 0.45\text{ s}.$$

Because there are two peaks per cycle, the period of the rectified voltage is

$$T = \frac{1}{2f} = \frac{1}{2(50\text{ Hz})} = 1.0 \times 10^{-2}\text{ s}.$$

Because $\tau_1 \gg T$, the voltage across the capacitor will be essentially constant during a cycle, so the voltage will be the peak voltage. Thus the average current is

$$I_{\text{av}} = \frac{V_0}{R} = \frac{\sqrt{2}\,(220\text{ V})}{18\text{ k}\Omega} = 17\text{ mA (smooth)}.$$

(b) The time constant for the circuit is

$$\tau_2 = RC_2 = (18 \times 10^3\ \Omega)(0.10 \times 10^{-6}\text{ F}) = 1.8 \times 10^{-3}\text{ s}.$$

Because $\tau_2 < T$, the voltage across the capacitor will be rippled, so the average voltage will be close to the rms voltage. Thus the average current is

$$I_{\text{av}} = \frac{V_{\text{rms}}}{R} = \frac{220\text{ V}}{18\text{ k}\Omega} = 12\text{ mA (rippled)}.$$

5.9 Transistors

5–50. Suppose that the current gain of the transistor in Fig. 5.9 is $\beta = i_C/i_B = 80$. If $R_C = 3.3$ kΩ, calculate the ac output voltage for a time-varying input current of 2.0 μA.

SOLUTION. The output voltage is the voltage across the resistor:

$$V = i_C R_C = \beta i_B R_C = (80)(2.0 \times 10^{-6}\ \text{A})(3.3 \times 10^3\ \Omega) = 0.53\ \text{V}.$$

5–51. If the current gain of the transistor amplifier in Fig. 5.9 is $\beta = i_C/i_B = 100$, what value must R_C have if a 1.0-μA ac base current is to produce an ac output voltage of 0.40 V?

SOLUTION. The output voltage is the voltage across the resistor:

$$\begin{aligned} V &= i_C R_C = \beta i_B R_C; \\ 0.40\ \text{V} &= (100)(1.0 \times 10^{-6}\ \text{A}) R_C, \end{aligned}$$

which gives $R_C = 4.0 \times 10^3\ \Omega = 4.0$ kΩ.

5–52. A transistor, whose current gain $\beta = i_C/i_B = 70$, is connected as in Fig. 5.9 with $R_B = 3.2$ kΩ and $R_C = 6.8$ kΩ. Calculate (a) the voltage gain, and (b) the power amplification.

SOLUTION. (a) The output voltage is the voltage across R_C:

$$V_C = i_C R_C,$$

while the input voltage is the voltage across R_B:

$$V_B = i_B R_B.$$

Thus the voltage gain is

$$\frac{V_C}{V_B} = \frac{i_C R_C}{i_B R_B} = \beta \frac{R_C}{R_B} = \frac{(70)(6.8\ \text{k}\Omega)}{3.2\ \text{k}\Omega} = 1.49 \times 10^2.$$

(b) The power amplification is

$$\frac{P_{\text{output}}}{P_{\text{input}}} = \frac{i_C V_C}{i_B V_B} = \beta \frac{V_C}{V_B} = (70)(1.49 \times 10^2) = 1.04 \times 10^4.$$

5–53. An amplifier has a voltage gain of 80 and a 15-kΩ load (output) resistance. What is the peak output current through the load resistor if the input voltage is an ac signal with a peak of 0.080 V?

SOLUTION. The output current is

$$i_{\mathrm{C}} = \frac{V_{\mathrm{C}}}{R_{\mathrm{C}}} = (80)\frac{V_{\mathrm{B}}}{R_{\mathrm{C}}} = \frac{(80)(0.080\ \mathrm{V})}{15\ \mathrm{k\Omega}} = 0.43\ \mathrm{mA}.$$

5.10 General problems

5–54. Estimate the binding energy of the H_2 molecule by calculating the difference in kinetic energy of the electrons between when they are in separate atoms and when they are in the molecule, using the uncertainty principle. Take Δx for the electrons in the separated atoms to be the radius of the first Bohr orbit, 0.053 nm, and for the molecule take Δx to be the separation of the nuclei, 0.074 nm.

SOLUTION. For an electron confined within Δx, we find the uncertainty in the momentum from

$$\Delta p = \frac{\hbar}{2\,\Delta x},$$

which we take to be the momentum of the particle. The kinetic energy of the electron is

$$K = \frac{p^2}{2m} = \frac{\hbar^2}{8m(\Delta x)^2}.$$

When the two electrons are in separated atoms, we get

$$\begin{aligned} K_1 &= \frac{2\hbar^2}{8m(\Delta x_1)^2} = \frac{2(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}{8(9.11\times10^{-31}\ \mathrm{kg})(0.053\times10^{-9}\ \mathrm{m})^2} \\ &= 1.09\times10^{-18}\ \mathrm{J} = 6.80\ \mathrm{eV}. \end{aligned}$$

When the electrons are in the molecule, we get

$$\begin{aligned} K_2 &= \frac{2\hbar^2}{8m(\Delta x_2)^2} = \frac{2(1.055\times10^{-34}\ \mathrm{J\cdot s})^2}{8(9.11\times10^{-31}\ \mathrm{kg})(0.074\times10^{-9}\ \mathrm{m})^2} \\ &= 5.58\times10^{-19}\ \mathrm{J} = 3.49\ \mathrm{eV}. \end{aligned}$$

Thus the binding energy is

$$K_1 - K_2 = 6.80\ \mathrm{eV} - 3.49\ \mathrm{eV} = 3.31\ \mathrm{eV}.$$

5–55. The average translational kinetic energy of an atom or molecule is about $\overline{K} = \frac{3}{2}k_{\mathrm{B}}T$, where $k_{\mathrm{B}} = 1.38\times10^{-23}$ J/K is Boltzmann's constant.

At what temperature T will $\overline{K}$ be on the order of the bond energy (and hence the bond able to be broken by thermal motion) for (a) a covalent bond (say H_2) of binding energy 4.5 eV, and (b) a "weak" hydrogen bond of binding energy 0.15 eV?

SOLUTION. (a) We find the temperature from

$$\begin{aligned} K &= \tfrac{3}{2}k_{\mathrm{B}}T; \\ (4.5\text{ eV})(1.60\times10^{-19}\text{ J/eV}) &= \tfrac{3}{2}(1.38\times10^{-23}\text{ J/K})T, \end{aligned}$$

which gives $T = 3.5\times10^4$ K.

(b) We find the temperature from

$$\begin{aligned} K &= \tfrac{3}{2}k_{\mathrm{B}}T; \\ (0.15\text{ eV})(1.60\times10^{-19}\text{ J/eV}) &= \tfrac{3}{2}(1.38\times10^{-23}\text{ J/K})T, \end{aligned}$$

which gives $T = 1.2\times10^3$ K.

5–56. In the ionic salt KF, the separation distance between ions is about 0.27 nm. (a) Estimate the electrostatic potential energy between the ions assuming them to be point charges (magnitude $1\,e$). (b) It is known that F releases 4.07 eV of energy when it "grabs" an electron, and 4.34 eV is required to ionize K. Find the binding energy of KF relative to free K and F atoms, neglecting the energy of repulsion.

SOLUTION. (a) The potential energy for the point charges is

$$\begin{aligned} U &= -k_{\mathrm{e}}\frac{e^2}{r} = -\frac{2.30\times10^{-28}\text{ J}\cdot\text{m}}{0.27\times10^{-9}\text{ m}} = -8.52\times10^{-19}\text{ J} = -5.33\text{ eV} \\ &= -5.3\text{ eV}. \end{aligned}$$

(b) Because the potential energy of the ions is negative, 5.3 eV is released when the ions are brought together. A release of energy means that energy must be provided to return the ions to the state of free atoms. Thus the total binding energy of the KF ions is

$$\text{Binding energy} = 5.33\text{ eV} + 4.07\text{ eV} - 4.34\text{ eV} = 5.1\text{ eV}.$$

5–57. The fundamental vibration frequency for the HCl molecule is 8.66×10^{13} Hz. Determine (a) the reduced mass, and (b) the effective

value of the constant k. Compare to k for the H_2 molecule (estimated in Example 5–5).

SOLUTION. (a) The reduced mass of the HCl molecule is

$$\mu = \frac{m_H m_{Cl}}{m_H + m_{Cl}} = \frac{(1.00794\ \text{u})(35.4527\ \text{u})}{1.00794\ \text{u} + 35.4527\ \text{u}} = 0.9801\ \text{u}.$$

(b) We find the effective spring constant from

$$\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}; \\ 8.66 \times 10^{13}\ \text{Hz} &= \frac{1}{2\pi}\sqrt{\frac{k}{(0.9801\ \text{u})(1.66 \times 10^{-27}\ \text{kg/u})}}, \end{aligned}$$

which gives $k = 482$ N/m. This is 84% of the constant for H_2.

5–58. Imagine the two atoms of a diatomic molecule as if they were connected by a spring, Fig. 5.17. Show that the classical frequency of vibration is given by Eq. 5.7. [*Hint*: Let x_1 and x_2 be the displacements of each mass from initial equilibrium positions; then

$$m_1\frac{d^2x_1}{dt^2} = -kx, \quad \text{and} \quad m_2\frac{d^2x_2}{dt^2} = -kx,$$

where $x = x_1 + x_2$. Find another relationship between x_1 and x_2, assuming that the center of mass of the system stays at rest, and then show that $\mu d^2x/dt^2 = -kx$.]

SOLUTION. The reduced mass of the molecule is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

The center of mass does not move. If the equilibrium separation of the atoms is L and we choose the directions indicated in Fig. 5.17, the center of mass relative to the equilibrium position of m_1 is

$$x_{CM} = \frac{m_2 L}{m_1 + m_2} = \frac{-m_1 x_1 + m_2(L + x_2)}{m_1 + m_2},$$

which gives $m_1 x_1 = m_2 x_2$.

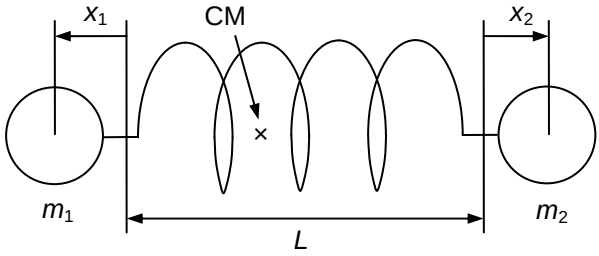


Figure 5.17. Problem 5–58.

The stretch of the spring is $x = x_1 + x_2$. Thus for the two atoms we have

$$m_1 \frac{d^2x_1}{dt^2} = -kx, \quad m_2 \frac{d^2x_2}{dt^2} = -kx.$$

This is consistent with the result from the center of mass calculation. If we rearrange and combine the equations, we get

$$\begin{aligned} \frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} &= -kx\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = -kx\frac{m_1 + m_2}{m_1 m_2}; \\ \frac{d^2x}{dt^2} &= -\frac{k}{\mu}x. \end{aligned}$$

This is the equation for simple harmonic motion with $\omega^2 = k/\mu$. Thus the frequency of vibration is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}\,.$$

5–59. Explain, using the Boltzmann factor (Eq. 4.16), why the heights of the peaks in Fig. 5.5 are different from one another. Explain also why the lines are not equally spaced. [*Hint*: Does the moment of inertia necessarily remain constant?]

Solution. From the Boltzmann factor the population of a state with energy E is proportional to $e^{-E/k_B T}$. The rotational energy of a state is

$$E = \frac{\hbar^2}{2I} L(L+1).$$

The selection rule requires $\Delta L = \pm 1$. States with higher values of L are less likely to be occupied and thus less likely to absorb a photon. For example, there is a greater probability for absorption from $L = 1$ to $L = 2$ than from $L = 2$ to $L = 3$. The molecule is not rigid and thus I will depend on L, which will affect the spacing.

5–60. Do we need to consider quantum effects for everyday rotating objects? Estimate the differences between rotational energy levels for a spinning baton compared to the energy of the baton. Assume the baton consists of a uniform 30-cm-long bar with a mass of 200 g and two small end masses, each of mass 300 g, and that it rotates at 1.6 rev/s about the bar's center.

SOLUTION. The moment of inertia of the baton of length d about its center of mass is

$$\begin{aligned} I &= \frac{Md^2}{12} + 2m\left(\frac{d}{2}\right)^2 = \frac{(0.200\ \text{kg})(0.30\ \text{m})^2}{12} + 2(0.300\ \text{kg})(0.15)^2 \\ &= 0.015\ \text{kg}\cdot\text{m}^2. \end{aligned}$$

The rotational energy of the baton is

$$E = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(0.015\ \text{kg}\cdot\text{m}^2)\left((1.6\ \text{rev/s})(2\pi\ \text{rad/rev})\right)^2 = 0.76\ \text{J}.$$

We find the rotational quantum number from

$$\begin{aligned} L^2 = (I\omega)^2 &= \ell(\ell+1)\hbar^2; \\ \big((0.015\ \text{kg}\cdot\text{m}^2)(1.6\ \text{rev/s}) & \\ \times (2\pi\ \text{rad/rev})\big)^2 &= \ell(\ell+1)(1.055\times 10^{-34}\ \text{J}\cdot\text{s})^2, \end{aligned}$$

which gives $\ell = 1.4 \times 10^{33}$.

Thus the difference in rotational energy levels is

$$\Delta E_{\text{rot}} = \frac{\ell\hbar^2}{I} = \frac{(1.4\times 10^{33})(1.055\times 10^{-34}\ \text{J}\cdot\text{s})^2}{0.015\ \text{kg}\cdot\text{m}^2} = 1.1\times 10^{-33}\ \text{J}.$$

No, because $\Delta E_{\text{rot}} \ll E$, we do not consider quantum effects for the baton.

5–61. When solid argon melts at $-189°$C, its latent heat of fusion goes directly into breaking the bonds between the atoms. Solid argon is

a weakly bound cubic lattice, with each atom connected to six neighbors, each bond having a binding energy of 3.9×10^{-3} eV. What is the latent heat of fusion for argon, in J/kg? [*Hint*: Show that in a simple cubic lattice (Fig. 5.18), there are three times as many bonds as there are atoms, when the number of atoms is large.]

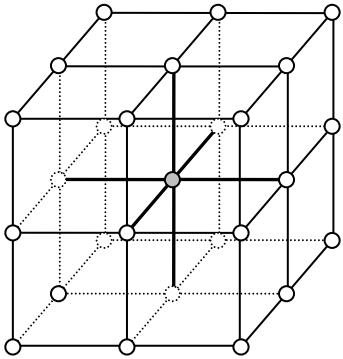


Figure 5.18. Problem 5–61.

SOLUTION. From the diagram of the cubic lattice, we see that an atom inside the cube is bonded to the six nearest neighbors (Fig. 5.18). Because each bond is shared by two atoms, the number of bonds per atom is 3. We find the heat of fusion for argon from the energy required to break the bonds:

$$\begin{aligned} L_{\text{fusion}} &= \left(\frac{\text{number of bonds}}{1 \text{ atom}}\right)\left(\frac{\text{number of atoms}}{1 \text{ kg}}\right) E_{\text{bond}} \\ &= \left(\frac{3}{\text{atom}}\right)\left(\frac{6.02 \times 10^{23} \text{ atoms/mol}}{39.95 \times 10^{-3} \text{ kg/mol}}\right) \\ &\quad \times (3.9 \times 10^{-3} \text{ eV})(1.60 \times 10^{-19} \text{ J/eV}) \\ &= 2.8 \times 10^{4} \text{ J/kg}. \end{aligned}$$

5–62. A strip of silicon 1.5 cm wide and 1.0 mm thick is immersed in a magnetic field of strength 1.6 T perpendicular to the strip (Fig. 5.19). When a current of 0.20 mA is run through the strip, there is a resulting Hall effect voltage of 18 mV across the strip. How many electrons

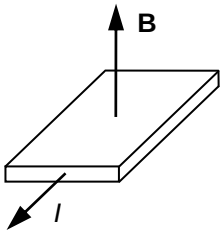


Figure 5.19. Problem 5–62.

per silicon atom are in the conduction band? The density of silicon is $2\,330$ kg/m^3.

SOLUTION. The Hall voltage is produced by the drifting electrons:

$$\begin{aligned} \mathcal{E}_{\mathrm{H}} &= v_{\mathrm{d}} BL; \\ 18 \times 10^{-3}\ \mathrm{V} &= v_{\mathrm{d}}(1.6\ \mathrm{T})(1.5 \times 10^{-2}\ \mathrm{m}), \end{aligned}$$

which gives $v_{\mathrm{d}} = 0.75$ m/s.

We find the density of drifting electrons from the current:

$$\begin{aligned} I &= neAv_{\mathrm{d}}; \\ 0.20 \times 10^{-3}\ \mathrm{A} &= n(1.60 \times 10^{-19}\ \mathrm{C})(1.5 \times 10^{-2}\ \mathrm{m}) \\ &\quad \times (1.0 \times 10^{-3}\ \mathrm{m})(0.75\ \mathrm{m/s}), \end{aligned}$$

which gives $n = 1.11 \times 10^{20}$ electrons/m^3.

The density of silicon atoms is

$$\begin{aligned} N &= \frac{2.33 \times 10^{6}\ \mathrm{g/m^3}}{28\ \mathrm{g/mol}}\,(6.02 \times 10^{23}\ \mathrm{atoms/mol}) \\ &= 5.0 \times 10^{28}\ \mathrm{atoms/m^3}. \end{aligned}$$

Thus the ratio of electrons to atoms is

$$\frac{n}{N} = \frac{1.11 \times 10^{20}\ \mathrm{electrons/m^3}}{5.0 \times 10^{28}\ \mathrm{atoms/m^3}} = 2.2 \times 10^{-9}.$$

5–63. Most of the Sun's radiation has wavelengths shorter than 1 000 nm. For a solar cell to absorb all this, what energy gap ought the material have?

SOLUTION. The photon with the longest wavelength has the minimum energy, so the energy gap must be

$$E_g = \frac{2\pi\hbar c}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{1\,000 \text{ nm}} = 1.24 \text{ eV}.$$

5–64. The energy gap between valence and conduction bands in germanium is 0.72 eV. What range of wavelengths can a photon have to excite an electron from the top of the valence band into the conduction band?

SOLUTION. The energy of the photon must be greater than or equal to the energy gap. Thus the longest wavelength that will excite an electron is

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} = \frac{2\pi\hbar c}{E_g} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{0.72 \text{ eV}} = 1.7 \times 10^3 \text{ nm}.$$

Thus the wavelength range is $\lambda \leqslant 1.7$ μm.

5–65. A TV remote control emits IR light. If the detector on the TV set is *not* to react to visible light, could it make use of silicon as a "window" with its energy gap $E_g = 1.14$ eV? What is the shortest-wavelength light that can strike silicon without causing electrons to jump from the valence band to the conduction band?

SOLUTION. To use silicon to filter the wavelengths, we want wavelengths below the IR to be able to cause the electron to be raised to the conduction band, so the photon is absorbed in the silicon. We find the shortest wavelength from

$$\lambda = \frac{c}{f} = \frac{2\pi\hbar c}{2\pi\hbar f} > \frac{2\pi\hbar c}{E_g} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{1.14 \text{ eV}} = 1.09 \times 10^3 \text{ nm}.$$

Because this is in the IR, the shorter wavelengths of visible light will excite the electron, so silicon could be used as a window.

5–66. The *Fermi temperature*, T_F is defined as that temperature at which the thermal energy $k_B T$ (without the $\frac{3}{2}$) is equal to the Fermi energy: $k_B T_F = E_F$. (a) Determine the Fermi temperature for copper. (b) Show

that for $T \gg T_F$, the Fermi factor (Eq. 5.15) approaches the Boltzmann factor. (Note: This last result is not very useful for understanding conductors. Why?)

SOLUTION. (a) For copper, $E_F = 7.0$ eV (see Example 5–9). From the definition of the Fermi temperature, we have

$$T_F = \frac{E_F}{k_B} = \frac{(7.0 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})}{1.38 \times 10^{-23} \text{ J/K}} = 8.1 \times 10^4 \text{ K}.$$

(b) We can write the Fermi distribution as

$$f = \frac{1}{e^{(E-E_F)/k_B T} + 1} = \frac{1}{e^{E/k_B T} e^{-T_F/T} + 1}$$

If $T \gg T_F$, the second exponential factor is 1. If $E/k_B T \gg 1$, we can also ignore the 1 to get

$$f \approx \frac{1}{e^{E/k_B T}} = e^{-E/k_B T}.$$

This is not useful for conductors because T_F is much higher than the melting point, so $T \ll T_F$.

5–67. For an arsenic donor atom in a doped silicon semiconductor, assume that the "extra" electron moves in a Bohr orbit about the arsenic ion. For this electron in the ground state, take into account the dielectric constant $\varkappa = 12$ of the Si lattice (which represents the weakening of the Coulomb force due to all the other atoms or ions in the lattice), and estimate (a) the binding energy, and (b) the orbit radius for this extra electron. [*Hint*: Substitute $\varepsilon = \varkappa\varepsilon_0$ in Coulomb's law.]

SOLUTION. In a dielectric, Coulomb's law becomes

$$F = \frac{e^2}{4\pi\varkappa\varepsilon_0 r^2} = \frac{k_e e^2}{\varkappa r^2}.$$

Thus where ε_0 appears in an equation, we insert $\varkappa$. If the "extra" electron is outside the arsenic ion, the effective Z will be 1, and we can use the hydrogen results.

(a) The energy of the electron is

$$E = -\frac{Z^2 k_e^2 e^4 m_e}{2\varkappa^2 \hbar^2 n^2} = -\frac{(13.6 \text{ eV}) Z^2}{\varkappa^2 n^2} = -\frac{(13.6 \text{ eV})(1)^2}{(12)^2 (1)^2} = -0.094 \text{ eV}.$$

Thus the binding energy is 0.094 eV.

(b) The radius of the electron orbit is

$$\begin{aligned} r &= \frac{\varkappa\hbar^2 n^2}{Z k_e e^2 m_e} = \frac{\varkappa n^2 r_0}{Z} = \frac{\varkappa n^2 (0.0529 \text{ nm})}{Z} = \frac{(12)(1)^2(0.0529 \text{ nm})}{1} \\ &= 0.63 \text{ nm}. \end{aligned}$$

Note that this result justifies the assumption that the electron is outside the arsenic ion.

5–68. A full-wave rectifier (Fig. 5.8) uses four diodes to rectify a 75-V rms 50 Hz ac voltage. If $R = 8.8$ kΩ and $C = 30$ μF what will be the approximate percent variation in the output voltage? The variation in output voltage is called *ripple voltage*. [*Hint*: Assume the discharge of the capacitor is approximately linear.]

SOLUTION. With a full-wave rectifier, there are two peaks for each cycle of the input voltage, so the time between peaks is

$$T = \frac{1}{2f} = \frac{1}{2(50 \text{ Hz})} = 0.01 \text{ s}.$$

The time constant of the rectifier is

$$\tau = RC = (8.8 \times 10^3 \ \Omega)(30 \times 10^{-6} \text{ F}) = 0.264 \text{ s}.$$

Because $T \ll \tau$, we assume that the exponential discharge of the capacitor voltage is linear:

$$V_C = V_0 \left(1 - \frac{t}{\tau}\right).$$

We approximate the lowest voltage of the capacitor by finding the value reached in the time from one peak to the next:

$$V = V_0 \left(1 - \frac{0.01 \text{ s}}{0.264 \text{ s}}\right) = 0.962 \ V_0.$$

Thus the ripple about the mean value is

$$\pm\frac{1}{2}\frac{V_0 - V}{V_0} = \pm\frac{1}{2}(1 - 0.962) = \pm 0.019 = \pm 1.9\%.$$

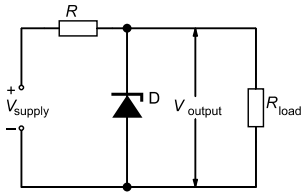


Figure 5.20. Problem 5–69.

5–69. A Zener diode voltage regulator is shown in Fig. 5.20. The voltage across the diode is connected in reverse bias. A breakdown occurs when the reverse bias becomes large enough that the top of the valence band in the p region is just higher in energy than the bottom of the conduction band in the n region. The junction region is thin enough and the probability becomes large that electrons can tunnel from the valence band of the p region to the conduction band of the n region. this process is called *Zener breakdown*. When placed over an unregulated power supply, a Zener diode can maintain the voltage at its own breakdown voltage as long as the supply voltage is always above this point. Suppose that $R = 1.80\ \text{k}\Omega$, and that the diode breakdown voltage is 130 V: the diode is rated at a maximum a 110 mA. (a) If $R_{\text{load}} = 16.0\ \text{k}\Omega$, over what range of supply voltages will the circuit maintain the voltage at 130 V? (b) If the supply voltage is 200 V, over what range of load resistance will the voltage be regulated?

SOLUTION. (a) The current through the load resistor is

$$I_{\text{load}} = \frac{V_{\text{output}}}{R_{\text{load}}} = \frac{130\ \text{V}}{16.0\ \text{k}\Omega} = 8.125\ \text{mA}.$$

At the minimum supply voltage the current through the diode will be zero, so the current through R is 8.125 mA, and the voltage across R is

$$V_{R,\,\text{min}} = I_{R,\,\text{min}} R = (8.125\ \text{mA})(1.80\ \text{k}\Omega) = 14.6\ \text{V}.$$

The minimum supply voltage is

$$V_{\text{min}} = V_{R,\,\text{min}} + V_{\text{output}} = 14.6\ \text{V} + 130\ \text{V} = 145\ \text{V}.$$

At the maximum supply voltage the current through the diode will be 110 mA, so the current through R is

$$110\ \text{mA} + 8.125\ \text{mA} = 118.1\ \text{mA},$$

and the voltage across R is

$$V_{R,\,\max} = I_{R,\,\max} R = (118.1\ \text{mA})(1.80\ \text{k}\Omega) = 213\ \text{V}.$$

The maximum supply voltage is

$$V_{\max} = V_{R,\,\max} + V_{\text{output}} = 213\ \text{V} + 130\ \text{V} = 343\ \text{V}.$$

Thus the range of supply voltages is $145\ \text{V} \leqslant V \leqslant 343\ \text{V}$.

(b) At a constant supply voltage the voltage across R is

$$200\ \text{V} - 130\ \text{V} = 70\ \text{V},$$

so the current in R is

$$IR = \frac{70\ \text{V}}{1.80\ \text{k}\Omega} = 38.9\ \text{mA}.$$

If there is no current through the diode, this current must be in the load resistor, so we have

$$R_{\text{load}} = \frac{130\ \text{V}}{38.9\ \text{mA}} = 3.34\ \text{k}\Omega.$$

If R_{load} is less than this, there will be a greater current through R, and thus the voltage across the load will drop and regulation will be lost. If R_{load} is greater than $3.34\ \text{k}\Omega$, the current through R_{load} will decrease and there will be current through the diode. The current through the diode is 38.9 mA when R_{load} is infinite, which is less than the maximum of 110 mA. Thus the range for load resistance is $3.34\ \text{k}\Omega \leqslant R_{\text{load}} < \infty$.

6 Nuclear Physics and Radioactivity

6.1 Review

Nuclear physics is the study of atomic nuclei. Nuclei contain *protons* and *neutrons*, which are collectively known as *nucleons*. The total number of nucleons, A, is the *atomic mass number*. The number of protons, Z, is the *atomic number*. The number of neutrons equals $A - Z$. *Isotopes* are nuclei with the same Z, but with different numbers of neutrons. For an element X, an isotope of given Z and A is represented by

$$^{A}_{Z}\mathrm{X}.$$

The nuclear radius increases with A according to the approximate formula

$$r \approx (1.2 \times 10^{-15}\ \mathrm{m})A^{1/3}, \tag{6.1}$$

indicating that all nuclei have about the same density.

Example 6–1. Estimate the diameter of the following nuclei: (a) $^{1}_{1}$H, (b) $^{40}_{20}$Ca, (c) $^{208}_{82}$Pb, (d) $^{235}_{92}$U.

Solution. (a) For hydrogen, $A = 1$, Eq. 6.1 gives

$$d = \text{diameter} = 2r \approx 2.4 \times 10^{-15}\ \mathrm{m}$$

since $A^{1/3} = (1)^{1/3} = 1$.

(b) For calcium

$$d = 2r \approx (2.4 \times 10^{-15}\ \mathrm{m})(40)^{1/3} = 8.2 \times 10^{-15}\ \mathrm{m}.$$

(c) For lead

$$d \approx (2.4 \times 10^{-15}\ \mathrm{m})(208)^{1/3} = 14 \times 10^{-15}\ \mathrm{m}.$$

(d) For uranium

$$d \approx (2.4 \times 10^{-15}\ \mathrm{m})(235)^{1/3} = 15 \times 10^{-15}\ \mathrm{m}.$$

The range of nuclear diameters is only from 2.4 fm to 15 fm.

A nucleus, made up of protons and neutrons, has a *nuclear spin* quantum number, I, that can be either integer or half integer. The *nuclear angular momentum* of a nucleus is given by $\hbar\sqrt{I(I+1)}$. Nuclear magnetic moments are measured in terms of the *nuclear magneton*

$$\mu_{\rm N} = \frac{e\hbar}{2m_{\rm p}}\,, \tag{6.2}$$

which is defined by analogy with the Bohr magneton for electrons ($\mu_{\rm B} = e\hbar/2m_{\rm e}$).

Nuclear masses are specified in *unified atomic mass units* (u), where the mass of ${}^{12}_{6}$C (including its 6 electrons) is defined as exactly 12.000000 u, or in terms of their energy equivalent (because $E = mc^2$), where

$$1\ {\rm u} = 931.5\ {\rm MeV}/c^2 = 1.66 \times 10^{-27}\ {\rm kg}.$$

The mass of a stable nucleus is less than the sum of the masses of its constituent nucleons, as the following Example shows.

Example 6–2. Compare the mass of a ${}^{4}_{2}$He nucleus to that of its constituent nucleons.

SOLUTION. The mass of a neutral ${}^{4}_{2}$He atom, from Appendix C, is 4.002603 u. The mass of two neutrons and two protons (including the two electrons) is

$$\begin{aligned} 2m_{\rm n} &= 2.017330\ {\rm u} \\ 2m({}^{1}_{1}{\rm H}) &= \underline{2.015650\ {\rm u}} \\ &\quad 4.032980\ {\rm u}. \end{aligned}$$

We almost always deal with masses of neutral atoms—that is, nuclei with Z electrons—since this is how masses are measured. We must therefore be sure to balance out the electrons when we compare masses, which is why we used the mass of ${}^{1}_{1}$H in this Example rather than that of the proton alone.

The difference in mass (times c^2) is the *total binding energy*. It represents the energy needed to break the nucleus into its constituent nucleons. The *binding energy per nucleon* averages about 8 MeV per nucleon, and is lowest for low mass and high mass nuclei as illustrated by Fig. 6.1.

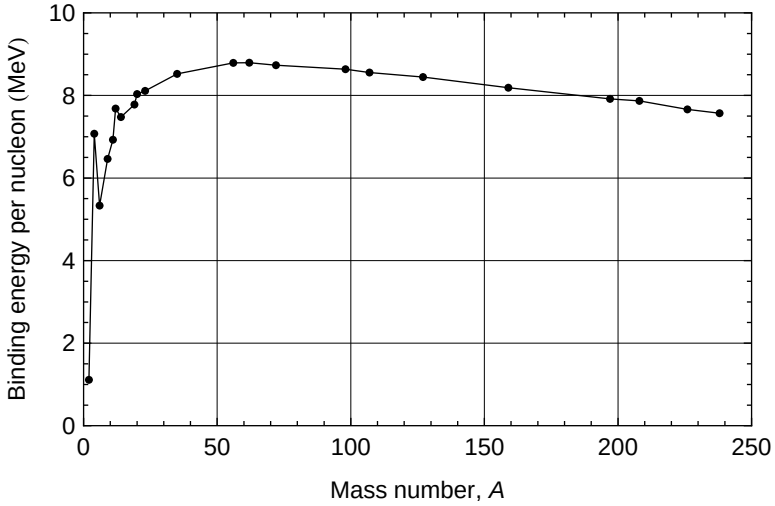


Figure 6.1. Binding energy per nucleon as a function of mass number A for stable nuclei.

Example 6–3. Calculate the total binding energy and the average binding energy per nucleon for $^{56}_{26}$Fe, the most common stable isotope of iron.

SOLUTION. $^{56}_{26}$Fe has 26 protons and 30 neutrons whose separate masses are

$$\begin{aligned}
(26)(1.007825\text{ u}) &= 26.2035\text{ u (includes electrons)} \\
(30)(1.008665\text{ u}) &= \underline{30.2600\text{ u}} \\
\text{Total} &= 56.4635\text{ u.} \\
\text{Subtract mass of } ^{56}_{26}\text{Fe}: & \quad \underline{-55.9349\text{ u}}\text{ (Appendix C)} \\
\Delta m &= 0.5286\text{ u.}
\end{aligned}$$

The total binding energy is thus

$$(0.5286\text{ u})(931.5\text{ MeV/u}) = 492.4\text{ MeV}$$

and the average binding energy per nucleon is

$$\frac{492.4 \text{ MeV}}{56 \text{ nucleons}} = 88 \text{ MeV}.$$

Example 6–4. What is the binding energy of the last neutron in ${}^{13}_{6}\text{C}$?

SOLUTION. We compare the mass of ${}^{13}_{6}\text{C}$ to that of the atom with one less neutron, ${}^{12}_{6}\text{C}$, plus a free neutron (Appendix C):

$$\begin{aligned} \text{Mass } {}^{13}_{6}\text{C} &= 12.000000 \text{ u} \\ \text{Mass } {}^{1}_{0}\text{n} &= \underline{1.008665 \text{ u}} \\ \text{Total} &= 13.008665 \text{ u}. \\ \text{Subtract mass of } {}^{13}_{6}\text{C}: & \quad \underline{-13.003355 \text{ u}} \\ \Delta m &= 0.005310 \text{ u} \end{aligned}$$

which in energy is $(931.5 \text{ MeV/u})(0.005310 \text{ u}) = 4.95 \text{ MeV}$. That is, it would require 4.95 MeV input of energy to remove one neutron from ${}^{13}_{6}\text{C}$.

Unstable nuclei undergo *radioactive decay*; they change into other nuclei with the emission of an α, β, or γ particle.

$$\begin{aligned} &\alpha \text{ decay}: \\ &\quad {}^{A}_{Z}N \rightarrow {}^{A-4}_{Z-2}N' + {}^{4}_{2}\text{He} \\ &\beta \text{ decay}: \\ &\quad {}^{A}_{Z}N \rightarrow {}_{Z+1}{}^{A}N' + \text{e}^- + \tilde{\nu} \\ &\quad {}^{A}_{Z}N \rightarrow {}_{Z-1}{}^{A}N' + \text{e}^+ + \nu \\ &{}^{A}_{Z}N + \text{e}^- \rightarrow {}_{Z-1}{}^{A}N' + \nu \qquad \text{(electron capture)} \\ &\gamma \text{ decay}: \\ &\quad {}^{A}_{Z}N^* \rightarrow {}^{A}_{Z}N + \gamma \end{aligned}$$

An α particle is a ${}^{4}_{2}\text{He}$ nucleus; a β particle is an electron or positron; and a γ ray is a high-energy photon. In β decay, a *neutrino* is also emitted. The transformation of the parent into the daughter nucleus is called *transmutation* of the elements. Radioactive decay occurs spontaneously only when the rest mass of the products is less than the mass of the parent nucleus. The loss in mass appears as kinetic energy of the products. The

total energy released is called the *disintegration energy*, Q, or the Q-value of the decay. From conservation of energy,

$$Q = M_{\mathrm{P}}c^2 - (M_{\mathrm{D}} + m_\alpha)c^2 \tag{6.3}$$

where M_{P}, M_{D}, and m_α are the masses of the parent, daughter, and α particle, respectively.

Example 6–5. Calculate the disintegration energy when ${}^{232}_{92}\mathrm{U}$ (mass = 232.037146 u) decays to ${}^{228}_{90}\mathrm{Th}$ (228.028731 u) with the emission of an α particle. (As always, masses are for neutral atoms.)

SOLUTION. Since the mass of the ${}^{4}_{2}\mathrm{He}$ is 4.002603 u (Appendix C), the total mass in the final state is

$$228.028731\ \mathrm{u} + 4.002603\ \mathrm{u} = 232.031334\ \mathrm{u}.$$

The mass lost when the ${}^{232}_{92}\mathrm{U}$ decays is

$$232.037146\ \mathrm{u} - 232.031334\ \mathrm{u} = 0.005812\ \mathrm{u}.$$

Since 1 u = 931.5 MeV, the energy Q released is

$$Q = (0.005812\ \mathrm{u})(931.5\ \mathrm{MeV/u}) \approx 5.4\ \mathrm{MeV},$$

and this energy appears as kinetic energy of the a particle and the daughter nucleus. (Using conservation of momentum, it can be shown that the α particle emitted by a ${}^{232}_{92}\mathrm{U}$ nucleus at rest has a kinetic energy of about 5.3 MeV. Thus, the daughter nucleus—which recoils in the opposite direction from the emitted α particle—has about 0.1 MeV of kinetic energy. See Problem 6–68.)

Nuclei are held together by the *strong nuclear force*. The *weak nuclear force* makes itself apparent in β decay. These two forces, plus the gravitational and electromagnetic forces, are the four known types of force. Electric charge, linear and angular momentum, mass–energy, and *nucleon number* are *conserved* in all decays.

Example 6–6. How much energy is released when ${}^{14}_{6}\mathrm{C}$ decays to ${}^{14}_{7}\mathrm{N}$ by β emission

$${}^{14}_{6}\mathrm{C} \rightarrow {}^{14}_{7}\mathrm{N} + \mathrm{e}^- + \tilde{\nu}? \tag{6.4}$$

Use Appendix C.

SOLUTION. The masses given in Appendix C are those of the neutral atom, and we have to keep track of the electrons involved. Assume the parent nucleus has six orbiting electrons so it is neutral, and its mass is 14.003242 u. The daughter, which in this decay is $^{14}_{7}$N, is not neutral since it has the same six electrons circling it but the nucleus has a charge of +7 e. However, the mass of this daughter with its six electrons, plus the mass of the emitted electron (which makes a total of seven electrons), is just the mass of a neutral nitrogen atom. That is, the total mass in the final state is

$$(\text{mass of } ^{14}_{7}\text{N nucleus} + 6 \text{ electrons}) + (\text{mass of 1 electron}),$$

and this is equal to

$$\text{mass of neutral } ^{14}_{7}\text{N (includes 7 electrons)},$$

which, from Appendix C is a mass of 14.003074 u. (Note that the neutrino doesn't contribute to either the mass or charge balance since it has $m = 0$ and $q = 0$.) Hence the mass after decay is 14.003074 u, whereas before decay, it was 14.003242 u. So the mass difference is 0.000168 u, which corresponds to 0.156 MeV or 156 keV.

Radioactive decay is a statistical process. For a given type of radioactive nucleus, the number of nuclei that decay (ΔN) in a time Δt is proportional to the number N of parent nuclei present:

$$\Delta N = -\lambda N \, \Delta t. \tag{6.5}$$

The proportionality constant, λ, is called the *decay constant* and is characteristic of the given nucleus. The infinitesimal form of Eq. 6.5 is

$$dN = -\lambda N \, dt. \tag{6.6}$$

The number N of nuclei remaining after a time t decreases exponentially:

$$N = N_0 e^{-\lambda t}, \tag{6.7}$$

where N_0 is the number of parent nuclei present at $t = 0$ and N is the number of remaining at time t.

The rate of decay, or number of decays per second, dN/dt, is called *activity* of a given sample. The activity also decreases exponentially in time:

$$\frac{dN}{dt} = -\lambda N_0 e^{-\lambda t} = \left(\frac{dN}{dt}\right)_0 e^{-\lambda t}, \tag{6.8}$$

The *half-life*, $T_{1/2}$, is the time required for half the nuclei of a radioactive sample to decay. It is related to the decay constant by

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}. \tag{6.9}$$

Example 6–7. The isotope $^{14}_{6}$C has a half-life of 5 731 yr. If at some time a sample contains 1.00×10^{22} carbon-14 nuclei, what is the activity of the sample?

SOLUTION. First we calculate the decay constant λ from Eq. 6.9, and obtain

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{(5\,730 \text{ yr})(3.156 \times 10^7 \text{ s/yr})} = 3.83 \times 10^{-12} \text{ s}^{-1},$$

since the number of seconds in a year is $(60)(60)(24)(365\frac{1}{4}) = 3.156 \times 10^7$ s. From Eq. 6.6, the magnitude of the activity or rate of decay is

$$\frac{dN}{dt} = \lambda N = (3.83 \times 10^{-12} \text{ s}^{-1})(1.00 \times 10^{22}) = 3.83 \times 10^{10} \text{ decays/s}.$$

(The unit "decays/s" is often written simply as s^{-1} since "decays" is not a unit but refers only to the number.)

Example 6–8. A laboratory has 1.4 μg of pure $^{13}_{7}$N, which has a half-life of 10.0 min (600 s). (a) How many nuclei are present initially? (b) What is the activity initially? (c) What is the activity after 1.00 h? (d) After approximately how long will the activity drop to less than one per second?

SOLUTION. (a) Since the atomic mass is 13.0, then 13.0 g will contain 6.02×10^{23} nuclei (Avogadro's number). Since we have only 1.49×10^{-6} g, the number of nuclei, N_0, that we have initially is given by the ratio

$$\frac{N_0}{1.49 \times 10^{-6} \text{ g}} = \frac{6.02 \times 10^{23}}{13.0 \text{ g}},$$

so $N_0 = 6.90 \times 10^{16}$ nuclei.

(b) From Eq. 6.9, $\lambda = (0.693)/(600\text{ s}) = 1.16 \times 10^{-3}\text{ s}^{-1}$. Then, at $t = 0$ (Eq. 6.8)

$$\left(\frac{dN}{dt}\right)_0 = \lambda N_0 = (1.16 \times 10^{-3}\text{ s}^{-1})(6.90 \times 10^{16}) = 8.00 \times 10^{13}\text{ s}^{-1}.$$

(c) After 1.00 h = 3 600 s, the activity will be (Eq. 6.8)

$$\begin{aligned}\frac{dN}{dt} &= \left(\frac{dN}{dt}\right)_0 e^{-\lambda t} \\ &= (8.00 \times 10^{13}\text{ s}^{-1}) \exp\left(-(1.16 \times 10^{-3}\text{ s}^{-1})(3\,600\text{ s})\right) \\ &= 1.23 \times 10^{12}\text{ s}^{-1}.\end{aligned}$$

This result can be obtained in another way: since 1.00 h represents six half-lives (6×10.0 min), the activity will decrease to $(\frac{1}{2})^6 = \frac{1}{64}$ of its original value, or $(8.00 \times 10^{13}\text{ s}^{-1})/64 = 1.25 \times 10^{12}\text{ s}^{-1}$. (The slight discrepancy between the two values arises because we kept only three significant figures.)

(d) We want to determine the time t when $dN/dt = 1.00\text{ s}^{-1}$. From Eq. 6.8 we have

$$e^{-\lambda t} = \frac{(dN/dt)}{(dN/dt)_0} = \frac{1.00\text{ s}^{-1}}{8.00 \times 10^{13}\text{ s}^{-1}} = 1.25 \times 10^{-14}.$$

We take the natural logarithm of both sides and divide by λ to find

$$t = -\frac{\ln(1.25 \times 10^{-14})}{\lambda} = \frac{32.0}{1.16 \times 10^{-3}\text{ s}^{-1}} = 2.76 \times 10^4\text{ s} = 7.67\text{ h}.$$

It is often the case that one radioactive isotope decays to another isotope that is also radioactive. Such successive decays are said to form a *decay series*. An important example is illustrated in Fig. 6.2. As can be seen, ${}^{238}_{92}\text{U}$ decays by α emission to ${}^{234}_{90}\text{Th}$, which in turn decays by β decay to ${}^{234}_{91}\text{Pa}$. The series continues as shown, with several possible branches near the bottom and ends at the stable lead isotope ${}^{206}_{82}\text{Pb}$.

Example 6–9. The decay chain starting with ${}^{234}\text{U}$ in Fig. 6.2 has nuclides with half-lives of 250 000 yr, 75 000 yr, 1 600 yr, and a little under

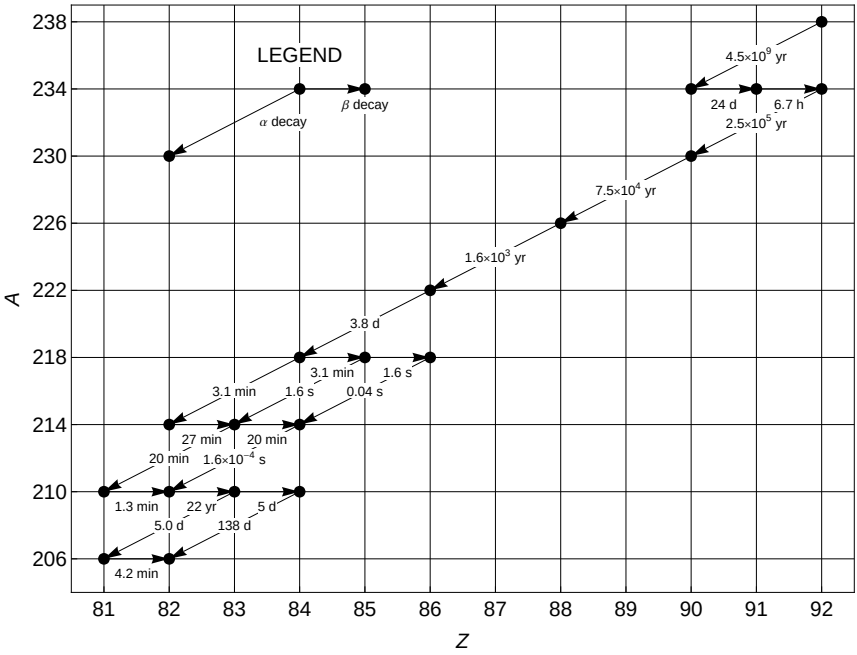


Figure 6.2. Decay series beginning with $^{238}_{92}$U. Nuclei in the series are specified by a dot representing A and Z values. Note that a horizontal arrow represents β decay (A does not change), whereas a diagonal arrow represents α decay (A changes by 4, Z changes by 2).

4 days, respectively. Each decay in the chain has an α particle of a characteristic energy, and so we can monitor the radioactive decay rate of each nuclide. Given a sample that was pure ^{234}U a million years ago, which α decay would you expect to have the highest activity rate in the sample?

Solution. The first instinct is to say that the process with the shortest half-life would show the highest activity. Surprisingly, however, the activity rates in this sample are all the same! The reason is that in each case the decay of the parent acts as a bottleneck to the decay of the daughter. Compared to the 1 600-yr half-life of ^{226}Ra, for example, its daughter, ^{222}Rn decays almost immediately, but it cannot decay until it is made.

An interesting application of radioactivity is the dating of archeological and geological specimens by measuring the concentration of radioactive isotopes. The most familiar example is *carbon dating*. The unstable isotope ^{14}C, produced during nuclear reactions in the atmosphere that result from cosmic-ray bombardment, gives a small proportion of ^{14}C in the CO_2 in the atmosphere. Plants that obtain their carbon from this source contain the same proportion of ^{14}C as the atmosphere. When a plant dies, it stops taking in carbon, and its ^{14}C β^- decays to ^{14}N with a half-life of 5 730 years. By measuring the proportion of ^{14}C in the remains, we can determine how long ago the organism died.

Example 6–10. An animal bone fragment found in an archeological site has a carbon mass of 200 g. It registers an activity of 16 decays/s. What is the age of the bone?

SOLUTION. When the animal was alive, the ratio of $^{14}_{6}$C to $^{12}_{6}$C in the 200-g piece of bone was 1.3×10^{-12}. The number of $^{14}_{6}$C nuclei at that time was

$$N_0 = \left(\frac{6.02 \times 10^{23} \text{ atoms}}{12 \text{ g}}\right)(200 \text{ g})(1.3 \times 10^{-12}) = 1.3 \times 10^{13}.$$

From Eq. 6.8, considering only the magnitude,

$$\left(\frac{dN}{dt}\right)_0 = \lambda N_0,$$

where $\lambda = 3.83 \times 10^{-12}$ s^{-1} (Example 6–7). So the original activity was

$$\left(\frac{dN}{dt}\right)_0 = (3.83 \times 10^{-12} \text{ s}^{-1})(1.3 \times 10^{13}) = 50 \text{ s}^{-1},$$

From Eq. 6.8,

$$\frac{dN}{dt} = \left(\frac{dN}{dt}\right)_0 e^{-\lambda t},$$

and we rewrite this as

$$e^{\lambda t} = \frac{(dN/dt)_0}{(dN/dt)}.$$

Now we take the natural logarithm of both sides to get

$$\begin{aligned} t &= \frac{1}{\lambda}\ln\left(\frac{(dN/dt)_0}{(dN/dt)}\right) = \frac{1}{3.83\times10^{-12}\ \mathrm{s}^{-1}}\ln\left(\frac{50\ \mathrm{s}^{-1}}{16\ \mathrm{s}^{-1}}\right) \\ &= 2.98\times10^{11}\ \mathrm{s} = 9\,400\ \mathrm{yr}, \end{aligned}$$

which is the time elapsed since the death of the animal.

6.2 Structure and properties of the nucleus

6–1. What is the rest mass of an α particle in MeV/c^2?

SOLUTION. To find the rest mass of an α particle, we subtract the rest mass of the two electrons from the rest mass of a helium atom:

$$\begin{aligned} m_\alpha &= M_{\mathrm{He}} - 2m_{\mathrm{e}} \\ &= (4.002603\ \mathrm{u})(931.5\ \mathrm{MeV/uc^2}) - 2(0.511\ \mathrm{MeV}/c^2) \\ &= 3\,727\ \mathrm{MeV}/c^2. \end{aligned}$$

6–2. A π meson has a mass of 139 MeV/c^2. What is this in atomic mass units?

SOLUTION. We convert the units:

$$m = \frac{139\ \mathrm{MeV}/c^2}{931.5\ \mathrm{MeV/uc^2}} = 0.149\ \mathrm{u}.$$

6–3. What is the approximate radius of an α particle ($^4_2$He)?

SOLUTION. The α particle is a helium nucleus:

$$\begin{aligned} r &= (1.2\times10^{-15}\ \mathrm{m})\sqrt[3]{A} = (1.2\times10^{-15}\ \mathrm{m})\sqrt[3]{4} \\ &= 1.9\times10^{-15}\ \mathrm{m} = 1.9\ \mathrm{fm}. \end{aligned}$$

6–4. By what percentage is the radius of the isotope $^{14}_6$C greater than that of its sister $^{12}_6$C?

SOLUTION. The radius of a nucleus is

$$r = (1.2\times10^{-15}\ \mathrm{m})\sqrt[3]{A}\,.$$

If we form the ratio for the two isotopes, we get

$$\frac{r_{14}}{r_{12}} = \sqrt[3]{\frac{14}{12}} = 1.053.$$

Thus the radius of ^{14}C is 5.3% greater than that for ^{12}C.

6–5. (a) Determine the density of nuclear matter in kg/m^3, and show that it is essentially the same for all nuclei. (b) What would be the radius of the Earth if it had its actual mass but had the density of nuclei? (c) What would be the radius of a $^{238}_{92}$U nucleus if it had the density of the Earth?

SOLUTION. (a) The mass of a nucleus with mass number A is A u and its radius is

$$r = (1.2 \times 10^{-15}\ \text{m})\sqrt[3]{A}\,.$$

Thus the density is

$$\begin{aligned}\rho &= \frac{m}{V} = \frac{A(1.66 \times 10^{-27}\ \text{kg/u})}{\frac{4}{3}\pi r^3} = \frac{A(1.66 \times 10^{-27}\ \text{kg/u})}{\frac{4}{3}\pi(1.2 \times 10^{-15}\ \text{m})^3 A} \\ &= 2.3 \times 10^{17}\ \text{kg/m}^3,\end{aligned}$$

independent of A.

(b) We find the radius from

$$\begin{aligned}M &= \rho V; \\ 5.98 \times 10^{24}\ \text{kg} &= (2.3 \times 10^{17}\ \text{kg/m}^3)\frac{4}{3}\pi R^3,\end{aligned}$$

which gives $R = 184$ m.

(c) For equal densities, we have

$$\begin{aligned}\rho &= \frac{M_{\text{Earth}}}{\frac{4}{3}\pi R_{\text{Earth}}^3} = \frac{m_{\text{U}}}{\frac{4}{3}\pi r_{\text{U}}^3}; \\ \frac{5.98 \times 10^{24}\ \text{kg}}{(6.38 \times 10^{6}\ \text{m})^3} &= \frac{(238\ \text{u})(1.66 \times 10^{-27}\ \text{kg/u})}{r_{\text{U}}^3},\end{aligned}$$

which gives $r_{\text{U}} = 2.6 \times 10^{-10}$ m.

6–6. (a) What is the approximate radius of a $^{64}_{29}$Cu nucleus? (b) Approximately what is the value of A for a nucleus whose radius is 3.9×10^{-15} m?

SOLUTION. (a) The radius of ^{64}Cu is

$$\begin{aligned}r &= (1.2 \times 10^{-15}\ \text{m})\sqrt[3]{A} = (1.2 \times 10^{-15}\ \text{m})\sqrt[3]{64} = 4.8 \times 10^{-15}\ \text{m} \\ &= 4.8\ \text{fm}.\end{aligned}$$

(b) We find the value of A from

$$\begin{aligned} r &= (1.2\times10^{-15}\text{ m})\sqrt[3]{A}\,; \\ 3.9\times10^{-15}\text{ m} &= (1.2\times10^{-15}\text{ m})\sqrt[3]{A}\,, \end{aligned}$$

which gives $A = 34$.

6–7. How much energy must an α particle have to just "touch" the surface of a $^{238}_{92}$U nucleus?

SOLUTION. We find the radii of the two nuclei from

$$\begin{aligned} r &= r_0\sqrt[3]{A}\,; \\ r_\alpha &= (1.2\text{ fm})\sqrt[3]{4} = 1.9\text{ fm}; \\ r_\text{U} &= (1.2\text{ fm})\sqrt[3]{238} = 7.4\text{ fm}. \end{aligned}$$

If the two nuclei are just touching, the Coulomb potential energy must be the initial kinetic energy of the α particle:

$$\begin{aligned} K &= U = k_\text{e}\frac{Z_\alpha Z_\text{U} e^2}{r_\alpha + r_\text{U}} = \frac{(2)(92)(1.44\text{ MeV}\cdot\text{fm})}{1.9\text{ fm} + 7.4\text{ fm}} \\ &= 28\text{ MeV}. \end{aligned}$$

6–8. If an α particle were released from rest near the surface of a $^{243}_{95}$Am nucleus, what would its kinetic energy be when far away?

SOLUTION. We find the radii of the two nuclei from

$$\begin{aligned} r &= r_0\sqrt[3]{A}\,; \\ r_\alpha &= (1.2\text{ fm})\sqrt[3]{4} = 1.9\text{ fm}; \\ r_\text{Am} &= (1.2\text{ fm})\sqrt[3]{243} = 7.5\text{ fm}. \end{aligned}$$

We assume that the nucleus is so much heavier than the α particle that we can ignore the recoil of the nucleus. We find the kinetic energy of the α particle from the conservation of energy:

$$\begin{aligned} K_\text{i} + U_\text{i} &= K_\text{f} + U_\text{f}; \\ 0 + k_\text{e}\frac{Z_\alpha Z_\text{Am} e^2}{r_\alpha + r_\text{Am}} &= K_\text{f} + 0; \\ \frac{(2)(95)(1.44\text{ MeV}\cdot\text{fm})}{1.9\text{ fm} + 7.5\text{ fm}} &= K_\text{f}, \end{aligned}$$

which gives $K_{\mathrm{f}} = 29$ MeV.

6–9. What stable nucleus has approximately half the radius of a uranium nucleus? [*Hint*: Find A and use Appendix C to get Z.]

SOLUTION. The radius of a nucleus is

$$r = (1.2 \times 10^{-15}\ \mathrm{m})\sqrt[3]{A}\,.$$

If we form the ratio for the two nuclei, we get

$$\begin{aligned} \frac{r_{\mathrm{X}}}{r_{\mathrm{U}}} &= \sqrt[3]{\frac{A_{\mathrm{X}}}{A_{\mathrm{U}}}}\,; \\ \frac{1}{2} &= \sqrt[3]{\frac{A_{\mathrm{X}}}{238}}\,, \end{aligned}$$

which gives $A_{\mathrm{X}} = 30$.

From the Appendix C, we see that the stable nucleus could be ${}^{31}_{15}\mathrm{P}$.

6.3 Binding energy and nuclear forces

6–10. Estimate the total binding energy for ${}^{40}_{20}\mathrm{Ca}$, using Fig. 6.1.

SOLUTION. From Fig. 6.1, we see that the average binding energy per nucleon at $A = 40$ is 8.6 MeV. Thus the total binding energy for ${}^{40}\mathrm{Ca}$ is

$$(40)(8.6\ \mathrm{MeV}) = 340\ \mathrm{MeV}.$$

6–11. Use Fig. 6.1 to estimate the total binding energy of (a) ${}^{238}_{92}\mathrm{U}$, and (b) ${}^{84}_{36}\mathrm{Kr}$.

SOLUTION. (a) From Fig. 6.1, we see that the average binding energy per nucleon at $A = 238$ is 7.5 MeV. Thus the total binding energy for ${}^{238}\mathrm{U}$ is

$$(238)(7.5\ \mathrm{MeV}) = 1.8 \times 10^{3}\ \mathrm{MeV}.$$

(b) From Fig. 6.1, we see that the average binding energy per nucleon at $A = 84$ is 8.7 MeV. Thus the total binding energy for ${}^{84}\mathrm{Kr}$ is

$$(84)(8.7\ \mathrm{MeV}) = 7.3 \times 10^{2}\ \mathrm{MeV}.$$

6–12. Use Appendix C to calculate the binding energy of ${}^{2}_{1}\mathrm{H}$ (deuterium).

SOLUTION. Deuterium consists of one proton and one neutron. We find the binding energy from the masses:

$$\begin{aligned}\text{Binding energy} &= \left(M(^1\text{H}) + m(^1\text{n}) - M(^2\text{H})\right)c^2 \\ &= ((1.007825\text{ u}) + (1.008665\text{ u}) - (2.014102\text{ u})) \\ &\quad \times c^2(931.5\text{ MeV/u}c^2) \\ &= 2.22\text{ MeV}.\end{aligned}$$

6–13. Calculate the binding energy per nucleon for a $^{14}_{7}\text{N}$ nucleus.

SOLUTION. ^{14}N consists of seven protons and seven neutrons. We find the binding energy from the masses:

$$\begin{aligned}\text{Binding energy} &= \left(7M(^1\text{H}) + 7m(^1\text{n}) - M(^{14}\text{N})\right)c^2 \\ &= (7(1.007825\text{ u}) + 7(1.008665\text{ u}) - (14.003074\text{ u})) \\ &\quad \times c^2(931.5\text{ MeV/u}c^2) \\ &= 104.7\text{ MeV}.\end{aligned}$$

Thus the binding energy per nucleon is

$$\frac{104.7\text{ MeV}}{14} = 7.48\text{ MeV}.$$

6–14. Determine the binding energy of the last neutron in a $^{40}_{19}\text{K}$ nucleus.

SOLUTION. We find the binding energy of the last neutron from the masses:

$$\begin{aligned}\text{Binding energy} &= \left(M(^{39}\text{K}) + m(^1\text{n}) - M(^{40}\text{K})\right)c^2 \\ &= ((38.963707\text{ u}) + (1.008665\text{ u}) - (39.963999\text{ u})) \\ &\quad \times c^2(931.5\text{ MeV/u}c^2) \\ &= 7.80\text{ MeV}.\end{aligned}$$

6–15. Calculate the total binding energy, and the binding energy per nucleon, for (a) $^{6}_{3}\text{Li}$, (b) $^{208}_{82}\text{Pb}$. Use Appendix C.

SOLUTION. (a) ^{6}Li consists of three protons and three neutrons. We find the binding energy from the masses:

$$\begin{aligned}\text{Binding energy} &= \left(3M(^1\text{H}) + 3m(^1\text{n}) - M(^6\text{Li})\right)c^2 \\ &= (3(1.007825\text{ u}) + 3(1.008665\text{ u}) - (6.015122\text{ u})) \\ &\quad \times c^2(931.5\text{ MeV/u}c^2) \\ &= 32.0\text{ MeV}.\end{aligned}$$

Thus the binding energy per nucleon is

$$\frac{(32.0\text{ MeV})}{6} = 5.33\text{ MeV}.$$

(b) ^{208}Pb consists of 82 protons and 126 neutrons. We find the binding energy from the masses:

$$\begin{aligned}\text{Binding energy} &= \left(82M(^1\text{H}) + 126m(^1\text{n}) - M(^{208}\text{Pb})\right)c^2 \\ &= (82(1.007825\text{ u}) + 126(1.008665\text{ u}) \\ &\quad - (207.976635\text{ u}))\,c^2(931.5\text{ MeV/u}c^2) \\ &= 1\,636\text{ MeV}.\end{aligned}$$

Thus the binding energy per nucleon is

$$\frac{(1\,636\text{ MeV})}{208} = 7.87\text{ MeV}.$$

6–16. Calculate the binding energy of (a) the last proton, and (b) the last neutron, in a $^{12}_{6}$C nucleus. Use Appendix C.

SOLUTION. (a) ^{12}C less a proton becomes ^{11}B. We find the binding energy of the last proton from the masses:

$$\begin{aligned}\text{Binding energy} &= \left(M(^{11}\text{B}) + M(^1\text{H}) - M(^{12}\text{C})\right)c^2 \\ &= ((11.009305\text{ u}) + (1.007825\text{ u}) - (12.000000\text{ u})) \\ &\quad \times c^2(931.5\text{ MeV/u}c^2) \\ &= 16.0\text{ MeV}.\end{aligned}$$

(b) ^{12}C less a neutron becomes ^{11}C. We find the binding energy of the last neutron from the masses:

$$\begin{aligned}\text{Binding energy} &= \left(M(^{11}\text{C}) + m(^{1}\text{n}) - M(^{12}\text{C})\right)c^2 \\ &= ((11.011434\ \text{u}) + (1.008665\ \text{u}) - (12.000000\ \text{u})) \\ &\quad \times c^2(931.5\ \text{MeV/u}c^2) \\ &= 18.7\ \text{MeV}.\end{aligned}$$

6–17. Compare the binding energy of a neutron in $^{23}_{11}$Na to that in $^{24}_{11}$Na.

SOLUTION. We find the binding energy of the last neutron from the masses:

$$\begin{aligned}\text{Binding energy}(^{23}\text{Na}) &= \left(M(^{22}\text{Na}) + m(^{1}\text{n}) - M(^{23}\text{Na})\right)c^2 \\ &= ((21.994437\ \text{u}) + (1.008665\ \text{u}) \\ &\quad - (22.989770\ \text{u}))\,c^2(931.5\ \text{MeV/u}c^2) \\ &= 12.4\ \text{MeV}. \\ \text{Binding energy}(^{24}\text{Na}) &= \left(M(^{23}\text{Na}) + m(^{1}\text{n}) - M(^{24}\text{Na})\right)c^2 \\ &= ((22.989770\ \text{u}) + (1.008665\ \text{u}) \\ &\quad - (23.990963\ \text{u}))\,c^2(931.5\ \text{MeV/u}c^2) \\ &= 7.0\ \text{MeV}.\end{aligned}$$

Thus the neutron is more closely bound in ^{23}Na.

6–18. How much energy is required to remove (a) a proton, (b) a neutron, from $^{16}_{8}$O? Explain the difference in your answers.

SOLUTION. We find the required energy for separation from the masses.

(a) Removal of a proton creates an isotope of nitrogen:

$$\begin{aligned}\text{Energy(p)} &= \left(M(^{15}\text{N}) + M(^{1}\text{H}) - M(^{16}\text{O})\right)c^2 \\ &= ((15.000108\ \text{u}) + (1.007825\ \text{u}) - (15.994915\ \text{u})) \\ &\quad \times c^2(931.5\ \text{MeV/u}c^2) \\ &= 12.1\ \text{MeV}.\end{aligned}$$

(b) Removal of a neutron creates another isotope of oxygen:

$$\begin{aligned}\text{Energy(n)} &= \left(M(^{15}\text{O}) + m(^{1}\text{n}) - M(^{16}\text{O})\right)c^2 \\ &= ((15.003065\ \text{u}) + (1.008665\ \text{u}) - (15.994915\ \text{u})) \\ &\quad \times c^2(931.5\ \text{MeV/u}c^2) \\ &= 15.7\ \text{MeV}.\end{aligned}$$

The nucleons are held by the attractive strong nuclear force. It takes less energy to remove the proton because there is also the repulsive electric force from the other protons.

6–19. (a) Show that the nucleus ${}^{6}_{4}\text{Be}$ (mass = 8.005305 u) is unstable to decay into two α particles, (b) Is ${}^{12}_{6}\text{C}$ stable against decay into three α particles? Show why or why not.

SOLUTION. (a) We find the binding energy from the masses:

$$\begin{aligned}E_{\text{binding}} &= \left(2M(^{4}\text{He}) - M(^{8}\text{Be})\right)c^2 \\ &= (2(4.002603\ \text{u}) - (8.005305\ \text{u}))\,c^2(931.5\ \text{MeV/u}c^2) \\ &= -0.092\ \text{MeV}.\end{aligned}$$

Because the binding energy is negative, the nucleus is unstable.

(b) We find the binding energy from the masses:

$$\begin{aligned}E_{\text{binding}} &= \left(3M(^{4}\text{He}) - M(^{12}\text{C})\right)c^2 \\ &= (3(4.002603\ \text{u}) - (12.000000\ \text{u}))\,c^2(931.5\ \text{MeV/u}c^2) \\ &= +7.3\ \text{MeV}.\end{aligned}$$

Because the binding energy is positive, the nucleus is stable.

6.4 Radioactivity. Decays. Conservation laws

6–20. How much energy is released when tritium, ${}^{3}_{1}\text{H}$, decays by β^- emission?

SOLUTION. The decay is

$${}^{3}_{1}\text{H} \rightarrow {}^{3}_{2}\text{He} + {}^{0}_{-1}\text{e} + \tilde{\nu}.$$

When we add an electron to both sides to use atomic masses, we see that the mass of the emitted β particle is included in the atomic mass of ${}^{3}\text{He}$.

Thus the energy released is

$$\begin{aligned} Q &= \left(M(^3\mathrm{H}) - M(^3\mathrm{He})\right) c^2 \\ &= ((3.016049\ \mathrm{u}) - (3.016029\ \mathrm{u}))\, c^2 (931.5\ \mathrm{MeV/uc^2}) \\ &= 0.0186\ \mathrm{MeV} = 18.6\ \mathrm{keV}. \end{aligned}$$

6–21. What is the maximum kinetic energy of an electron emitted in the β decay of a free neutron?

SOLUTION. The decay is

$$^1_0\mathrm{n} \rightarrow {}^1_1\mathrm{p} + {}^0_{-1}\mathrm{e} + \tilde{\nu}.$$

We take the electron mass to use the atomic mass of ^{1}H. The kinetic energy of the electron will be maximum if no neutrino is emitted. If we ignore the recoil of the proton, the maximum kinetic energy is

$$\begin{aligned} K &= \left(m(^1\mathrm{n}) - M(^1\mathrm{H})\right) c^2 \\ &= ((1.008665\ \mathrm{u}) - (1.007825\ \mathrm{u}))\, c^2 (931.5\ \mathrm{MeV/uc^2}) \\ &= 0.782\ \mathrm{MeV}. \end{aligned}$$

6–22. Show that the decay $^{11}_6\mathrm{C} \rightarrow {}^{10}_5\mathrm{B} + \mathrm{p}$ is not possible because energy would not be conserved.

SOLUTION. For the decay

$$^{11}_6\mathrm{C} \rightarrow {}^{10}_5\mathrm{B} + {}^1_1\mathrm{p},$$

we find the difference of the initial and the final masses:

$$\begin{aligned} \Delta m &= M(^{11}\mathrm{C}) - M(^{10}\mathrm{B}) - M(^1\mathrm{H}) \\ &= (11.011434\ \mathrm{u}) - (10.012937\ \mathrm{u}) - (1.007825\ \mathrm{u}) \\ &= -0.009328\ \mathrm{u}. \end{aligned}$$

Thus some additional energy would have to be added.

6–23. $^{22}_{11}\mathrm{Na}$ is radioactive. Is it a β^- or β^+ emitter? Write down the decay reaction, and estimate the maximum energy of the emitted β.

SOLUTION. If $^{22}_{11}\mathrm{Na}$ were a β^- emitter, the resulting nucleus would be $^{22}_{12}\mathrm{Mg}$, which has too few neutrons relative to the number of protons to be stable. Thus we have a β^+ emitter.

For the reaction

$$^{22}_{11}\mathrm{Na} \rightarrow {}^{22}_{10}\mathrm{Ne} + \beta^+ + \nu,$$

if we add 11 electrons to both sides in order to use atomic masses, we see that we have two extra electron masses on the right. The kinetic energy of the β^+ will be maximum if no neutrino is emitted. If we ignore the recoil of the neon, the maximum kinetic energy is

$$\begin{aligned} K &= \left(M(^{22}\mathrm{Na}) - M(^{22}\mathrm{Ne}) - 2m(\mathrm{e})\right) c^2 \\ &= ((21.994437\ \mathrm{u}) - (21.991386\ \mathrm{u}) - 2(0.00054858\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= 1.82\ \mathrm{MeV}. \end{aligned}$$

6–24. Give the result of a calculation that shows whether or not the following decays are possible: (a) $^{236}_{92}\mathrm{U} \rightarrow {}^{235}_{92}\mathrm{U} + \mathrm{n}$; (b) $^{16}_{8}\mathrm{O} \rightarrow {}^{15}_{8}\mathrm{O} + \mathrm{n}$; (c) $^{23}_{11}\mathrm{Na} \rightarrow {}^{22}_{11}\mathrm{Na} + \mathrm{n}$.

Solution. For each decay, we find the difference of the initial and the final masses:

$$\begin{aligned} \text{(a)} \qquad \Delta m &= M(^{236}\mathrm{U}) - M(^{235}\mathrm{U}) - m(^{1}\mathrm{n}) \\ &= (236.045561\ \mathrm{u}) - (235.043923\ \mathrm{u}) - (1.008665\ \mathrm{u}) \\ &= -0.00703\ \mathrm{u}. \end{aligned}$$

Because an increase in mass is required, the decay is not possible.

$$\begin{aligned} \text{(b)} \qquad \Delta m &= M(^{16}\mathrm{O}) - M(^{15}\mathrm{O}) - m(^{1}\mathrm{n}) \\ &= (15.994915\ \mathrm{u}) - (15.003065\ \mathrm{u}) - (1.008665\ \mathrm{u}) \\ &= -0.0168\ \mathrm{u}. \end{aligned}$$

Because an increase in mass is required, the decay is not possible.

$$\begin{aligned} \text{(c)} \qquad \Delta m &= M(^{23}\mathrm{Na}) - M(^{22}\mathrm{Na}) - m(^{1}\mathrm{n}) \\ &= (22.989770\ \mathrm{u}) - (21.994437\ \mathrm{u}) - (1.008665\ \mathrm{u}) \\ &= -0.0133\ \mathrm{u}. \end{aligned}$$

Because an increase in mass is required, the decay is not possible.

6–25. A $^{232}_{92}$U nucleus emits an α particle with kinetic energy = 5.32 MeV. What is the daughter nucleus and what is the approximate atomic mass (in u) of the daughter atom? Ignore recoil of the daughter nucleus.

SOLUTION. We find the daughter nucleus by balancing the mass and charge numbers:

$$\begin{aligned} Z(\mathrm{X}) &= Z(\mathrm{U}) - Z(\mathrm{He}) = 92 - 2 = 90; \\ A(\mathrm{X}) &= A(\mathrm{U}) - A(\mathrm{He}) = 232 - 4 = 228, \end{aligned}$$

so the daughter nucleus is $^{228}_{90}$Th.

If we ignore the recoil of the thorium, the kinetic energy of the α particle is

$$\begin{aligned} K &= \left(M(^{232}\mathrm{U}) - M(^{228}\mathrm{Th}) - M(^{4}\mathrm{He})\right) c^2; \\ 5.32\ \mathrm{MeV} &= \left((232.037146\ \mathrm{u}) - M(^{228}\mathrm{Th}) - (4.002603\ \mathrm{u})\right) \\ &\quad \times c^2(931.5\ \mathrm{MeV/uc^2}), \end{aligned}$$

which gives $M(^{228}\mathrm{Th}) = 228.02883$ u.

6–26. When $^{23}_{10}$Ne (mass = 22.9945 u) decays to $^{23}_{11}$Na (mass = 22.9898 u), what is the maximum kinetic energy the emitted electron? What is its minimum energy? What is the energy of the neutrino in each case?

SOLUTION. The kinetic energy of the electron will be maximum if no neutrino is emitted. If we ignore the recoil of the sodium, the maximum kinetic energy of the electron is

$$\begin{aligned} K &= \left(M(^{23}\mathrm{Ne}) - M(^{23}\mathrm{Na})\right) c^2 \\ &= \left((22.9945\ \mathrm{u}) - (22.9898\ \mathrm{u})\right) c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= 4.4\ \mathrm{MeV}. \end{aligned}$$

When the neutrino has all of the kinetic energy, the minimum kinetic energy of the electron is 0. The sum of the kinetic energy of the electron and the energy of the neutrino must be from the mass difference, so the energy range of the neutrino will be $0 \leqslant E_\nu \leqslant 4.4$ MeV.

6–27. The nuclide $^{32}_{15}$P decays by emitting an electron whose maximum kinetic energy can be 1.71 MeV. (a) What is the daughter nucleus? (b) What is its atomic mass (in u)?

SOLUTION. (a) We find the final nucleus by balancing the mass and charge numbers:

$$\begin{aligned} Z(\mathrm{X}) &= Z(\mathrm{P}) - Z(\mathrm{e}) = 15 - (-1) = 16; \\ A(\mathrm{X}) &= A(\mathrm{P}) - A(\mathrm{e}) = 32 - 0 = 32, \end{aligned}$$

so the final nucleus is ${}^{32}_{16}\mathrm{S}$.

(b) If we ignore the recoil of the sulfur, the maximum kinetic energy of the electron is

$$\begin{aligned} K &= \left(M({}^{32}\mathrm{P}) - M({}^{32}\mathrm{S})\right) c^2; \\ 1.71\ \mathrm{MeV} &= \left((31.973907\ \mathrm{u}) - M({}^{32}\mathrm{S})\right) c^2 (931.5\ \mathrm{MeV/u}c^2), \end{aligned}$$

which gives $M({}^{32}\mathrm{S}) = 31.97207$ u.

6–28. The isotope ${}^{218}_{84}\mathrm{Po}$ can decay by either α or β^- emission. What is the energy release in each case? The mass of ${}^{218}_{84}\mathrm{Po}$ is 218.008965 u.

SOLUTION. For alpha decay we have

$${}^{218}_{84}\mathrm{Po} \rightarrow {}^{214}_{82}\mathrm{Pb} + {}^{4}_{2}\mathrm{He}.$$

The Q value is

$$\begin{aligned} Q &= \left(M({}^{218}\mathrm{Po}) - M({}^{214}\mathrm{Pb}) - M({}^{4}\mathrm{He})\right) c^2 \\ &= ((218.008965\ \mathrm{u}) - (213.999798\ \mathrm{u}) - (4.002603\ \mathrm{u})) \\ &\quad \times c^2 (931.5\ \mathrm{MeV/u}c^2) \\ &= 6.11\ \mathrm{MeV}. \end{aligned}$$

For beta decay we have

$${}^{218}_{84}\mathrm{Po} \rightarrow {}^{218}_{85}\mathrm{At} + {}^{0}_{-1}\mathrm{e}.$$

The Q value is

$$\begin{aligned} Q &= \left(M({}^{218}\mathrm{Po}) - M({}^{218}\mathrm{At})\right) c^2 \\ &= ((218.008965\ \mathrm{u}) - (218.00868\ \mathrm{u}))\, c^2 (931.5\ \mathrm{MeV/u}c^2) \\ &= 0.26\ \mathrm{MeV}. \end{aligned}$$

6–29. How much energy is released in electron capture by beryllium: ${}^{7}_{4}\mathrm{Be} + {}^{0}_{-1}\mathrm{e} \rightarrow {}^{7}_{3}\mathrm{Li} + \nu$?

SOLUTION. For the electron capture

$$ {}^{7}_{4}\mathrm{Be} + {}^{0}_{-1}\mathrm{e} \rightarrow {}^{7}_{3}\mathrm{Li} + \nu, $$

we see that if we add three electron masses to both sides to use the atomic mass for Li, we use the captured electron for the atomic mass of Be.

We find the Q value from

$$ \begin{aligned} Q &= \left(M({}^{7}\mathrm{Be}) - M({}^{7}\mathrm{Li})\right) c^2 \\ &= \left((7.016929\ \mathrm{u}) - (7.016004\ \mathrm{u})\right) c^2 (931.5\ \mathrm{MeV/uc^2}) \\ &= 0.862\ \mathrm{MeV}. \end{aligned} $$

6–30. Decay series, as that shown in Fig. 6.2, can be classified into four families, depending on whether the mass numbers have the form $4n$, $4n+1$, $4n+2$, or $4n+3$ where n is an integer. Justify this statement and show that for a nuclide in any family, all its daughters will be in the same family.

SOLUTION. The mass number changes only with an α decay for which the change is -4. If the mass number is $4n$, then the new number is

$$ 4n - 4 = 4(n-1) = 4n'. $$

Thus for each family, we have

$$ \begin{array}{lcll} 4n & \rightarrow & 4n-4 & \rightarrow \quad 4n'; \\ 4n+1 & \rightarrow & 4n-4+1 & \rightarrow \quad 4n'+1; \\ 4n+2 & \rightarrow & 4n-4+2 & \rightarrow \quad 4n'+2; \\ 4n+3 & \rightarrow & 4n-4+3 & \rightarrow \quad 4n'+3. \end{array} $$

Thus the daughter nuclides are always in the same family.

6–31. What is the energy of the α particle emitted in the decay ${}^{210}_{84}\mathrm{Po} \rightarrow {}^{206}_{82}\mathrm{Pb} + \alpha$? Take into account the recoil of the daughter nucleus.

SOLUTION. The total kinetic energy of the daughter and the α particle

is

$$\begin{aligned} K_\alpha + K_{\mathrm{Pb}} &= Q \\ &= \left(M(^{210}\mathrm{Po}) - M(^{206}\mathrm{Pb}) - M(^{4}\mathrm{He})\right) c^2 \\ &= ((209.982857\ \mathrm{u}) - (205.974449\ \mathrm{u}) - (4.002603\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= 5.41\ \mathrm{MeV}. \end{aligned}$$

If the polonium nucleus is at rest when it decays, for momentum conservation we have

$$p_\alpha = p_{\mathrm{Pb}}.$$

The kinetic energy of the lead nucleus is

$$K_{\mathrm{Pb}} = \frac{p_{\mathrm{Pb}}^2}{2m_{\mathrm{Pb}}} = \frac{p_\alpha^2}{2m_{\mathrm{Pb}}} = \frac{m_\alpha}{m_{\mathrm{Pb}}} K_\alpha.$$

Thus we have

$$K_\alpha + \frac{m_\alpha}{m_{\mathrm{Pb}}} K_\alpha = \left(1 + \frac{4\ \mathrm{u}}{206\ \mathrm{u}}\right) K_\alpha = 5.41\ \mathrm{MeV},$$

which gives $K_\alpha = 5.31$ MeV.

6–32. The α particle emitted when $^{238}_{92}\mathrm{U}$ decays has 4.20 MeV of kinetic energy. Calculate the recoil kinetic energy of the daughter nucleus and the Q-value of the decay.

SOLUTION. The decay is

$$^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}.$$

If the uranium nucleus is at rest when it decays, for momentum conservation we have

$$p_\alpha = p_{\mathrm{Th}}.$$

Thus the kinetic energy of the thorium nucleus is

$$K_{\mathrm{Th}} = \frac{p_{\mathrm{Th}}^2}{2m_{\mathrm{Th}}} = \frac{p_\alpha^2}{2m_{\mathrm{Th}}} = \frac{m_\alpha}{m_{\mathrm{Th}}} K_\alpha = \frac{4\ \mathrm{u}}{234\ \mathrm{u}} (4.20\ \mathrm{MeV}) = 0.0718\ \mathrm{MeV}.$$

The Q value is the total kinetic energy produced:

$$Q = K_\alpha + K_{\mathrm{Th}} = 4.20 \text{ MeV} + 0.0718 \text{ MeV} = 4.27 \text{ MeV}.$$

6–33. (a) Show that when a nucleus decays by β^+ decay, the total energy released is equal to

$$(M_{\mathrm{P}} - M_{\mathrm{D}} - 2m_{\mathrm{e}})c^2,$$

where M_{P} and M_{D} are the masses of the parent and daughter atoms (neutral), and m_{e} is the mass of an electron or positron. (b) Determine the maximum kinetic energy of β^+ particles released when ${}^{11}_{6}\mathrm{C}$ decays to ${}^{11}_{5}\mathrm{B}$. What is the maximum energy the neutrino can have? What is its minimum energy?

Solution. (a) For the positron-emission process

$${}_{Z+1}{}^{A}\mathrm{X} \rightarrow {}_{Z}{}^{A}\mathrm{X}' + \mathrm{e}^+ + \nu,$$

we need to add $Z + 1$ electrons to the nuclear mass of X to be able to use the atomic mass. On the right-hand side we use Z electrons to be able to use the atomic mass of X′. Thus we have 1 electron mass and the β particle mass, which means that we must include 2 electron masses on the right-hand side. The Q value will be

$$Q = (M_{\mathrm{P}} - (M_{\mathrm{D}} + 2m_{\mathrm{e}}))\, c^2 = (M_{\mathrm{P}} - M_{\mathrm{D}} - 2m_{\mathrm{e}})c^2.$$

(b) The kinetic energy of the β^+ particle will be maximum if no neutrino is emitted. If we ignore the recoil of the boron, the maximum kinetic energy is

$$\begin{aligned} K &= Q = \left(M({}^{11}\mathrm{C}) - M({}^{11}\mathrm{B}) - 2m_{\mathrm{e}}\right) c^2 \\ &= ((11.011434 \text{ u}) - (11.009305 \text{ u}) - 2(0.00054858 \text{ u})) \\ &\quad \times c^2(931.5 \text{ MeV/u}c^2) \\ &= 0.961 \text{ MeV}. \end{aligned}$$

The sum of the kinetic energy of the β^+ particle and the energy of the neutrino must be from the mass difference, so the kinetic energy of the neutrino will range from 0.961 MeV to 0.

6–34. (a) Calculate the kinetic energy of the α particle emitted when $^{236}_{92}$U decays, (b) Use Eq. 6.1 to estimate the radius of an α particle and $^{232}_{90}$Th nucleus. Use this to estimate (c) the maximum height of the Coulomb barrier, and (d) its width AB in Fig. 3.11.

SOLUTION. (a) The decay is

$$^{236}_{92}\mathrm{U} \rightarrow {}^{232}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}.$$

The Q value is

$$\begin{aligned} Q &= \left(M(^{236}\mathrm{U}) - M(^{232}\mathrm{Th}) - M(^{4}\mathrm{He})\right) c^2 \\ &= ((236.045561\ \mathrm{u}) - (232.038050\ \mathrm{u}) - (4.002603\ \mathrm{u})) \\ &\quad \times c^2 (931.5\ \mathrm{MeV/u}c^2) \\ &= 4.572\ \mathrm{MeV}. \end{aligned}$$

If the uranium nucleus is at rest when it decays, for momentum conservation we have

$$p_\alpha = p_{\mathrm{Th}}.$$

Thus the kinetic energy of the thorium nucleus is

$$K_{\mathrm{Th}} = \frac{p_{\mathrm{Th}}^2}{2m_{\mathrm{Th}}} = \frac{p_\alpha^2}{2m_{\mathrm{Th}}} = \frac{m_\alpha}{m_{\mathrm{Th}}} K_\alpha.$$

Thus we have

$$Q = K_\alpha + \frac{m_\alpha}{m_{\mathrm{Th}}} K_\alpha = \left(1 + \frac{4\ \mathrm{u}}{232\ \mathrm{u}}\right) K_\alpha = 4.572\ \mathrm{MeV},$$

which gives $K_\alpha = 4.49$ MeV.

(b) We find the radii of the two nuclei from

$$\begin{aligned} R &= r_0 \sqrt[3]{A}\,; \\ R_\alpha &= (1.2\ \mathrm{fm})\sqrt[3]{4} = 1.9\ \mathrm{fm}; \\ R_{\mathrm{Th}} &= (1.2\ \mathrm{fm})\sqrt[3]{232} = 7.4\ \mathrm{fm}. \end{aligned}$$

(c) We assume the potential energy of the α particle (charge $= 2e$) is produced by the electric field of the remaining nuclear charge of $90e$. Thus the potential energy when the two nuclei are just touching is

$$U_{\mathrm{C}} = \frac{k_{\mathrm{e}} q Q}{R_\alpha + R_{\mathrm{Th}}} = \frac{(2)(90)(1.44\ \mathrm{MeV \cdot fm})}{1.9\ \mathrm{fm} + 7.4\ \mathrm{fm}} = 28\ \mathrm{MeV}.$$

(d) The radius when the α particle leaves the Coulomb barrier is where the Coulomb energy is equal to the final kinetic energy:

$$\begin{aligned} U_{\mathrm{B}} = \frac{k_{\mathrm{e}} q Q}{R_{\mathrm{B}}} &= K_{\alpha}; \\ \frac{(2)(90)(1.44\ \mathrm{MeV}\cdot\mathrm{fm})}{R_{\mathrm{B}}} &= 4.49\ \mathrm{MeV}, \end{aligned}$$

which gives $R_{\mathrm{B}} = 57.7$ fm.

Thus the width of the barrier is

$$R_{\mathrm{B}} - R_{\mathrm{Th}} = 57.7\ \mathrm{fm} - 7.4\ \mathrm{fm} \approx 50\ \mathrm{fm}.$$

6.5 Half-life and rate of decay. Decay series. Radioactive dating

6–35. A radioactive material produces 1 280 decays per minute at one time, and 6 h later produces 320 decays per minute. What is its half-life?

SOLUTION. We find the decay constant from

$$\begin{aligned} \lambda N &= \lambda N_0 e^{-\lambda t}; \\ (320\ \text{decays/min}) &= (1\,280\ \text{decays/min}) \exp\left(-\lambda(6\ \mathrm{h})\right), \end{aligned}$$

which gives $\lambda = 0.231\ \mathrm{h}^{-1}$. Thus the half-life is

$$T_{1/2} = \frac{0.693}{\lambda} = \frac{0.693}{0.231\ \mathrm{h}^{-1}} = 3.0\ \mathrm{h}.$$

Note that in 6.0 h the decay rate was reduced to $\frac{1}{4}$ the original rate. This means the elapsed time was 2 half-lives.

6–36. (a) What is the decay constant of $^{238}_{92}\mathrm{U}$ whose half-life is 4.5×10^9 yr? (b) The decay constant of a given nucleus is $8.2 \times 10^{-5}\ \mathrm{s}^{-1}$. What is its half-life?

SOLUTION. (a) We find the decay constant from

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{(4.5 \times 10^9\ \mathrm{yr})(3.16 \times 10^7\ \mathrm{s/yr})} = 4.9 \times 10^{-18}\ \mathrm{s}^{-1}.$$

(b) We find the half-life from

$$\begin{aligned} \lambda &= \frac{0.693}{T_{1/2}}; \\ (8.2 \times 10^{-5}\ \text{s}^{-1}) &= \frac{0.693}{T_{1/2}}, \end{aligned}$$

which gives

$$T_{1/2} = 8.45 \times 10^3\ \text{s} = 2.3\ \text{h}.$$

6–37. What is the activity of a sample of $^{14}_{6}$C that contains 3.1×10^{20} nuclei?

Solution. The activity of the sample is

$$\begin{aligned} \frac{\Delta N}{\Delta t} &= \lambda N = \frac{0.693}{T_{1/2}} N = \frac{0.693}{(5\,730\ \text{yr})(3.16 \times 10^7\ \text{s/yr})}(3.1 \times 10^{20}) \\ &= 1.2 \times 10^9\ \text{decays/s}. \end{aligned}$$

6–38. What fraction of a sample of $^{68}_{32}$Ge, whose half-life is about 9 months, will remain after 3.0 yr?

Solution. The number of half-lives that elapses is

$$n = \frac{(3.0\ \text{yr})(12\ \text{mo/yr})}{9\ \text{mo}} = 4.$$

We find the fraction remaining from

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^4 = 0.0625.$$

6–39. How many nuclei of $^{238}_{92}$U remain in a rock if the activity registers 875 decays per second?

Solution. We find the number of nuclei from the activity of the sample:

$$\begin{aligned} \frac{\Delta N}{\Delta t} &= \lambda N; \\ (875\ \text{decays/s}) &= \frac{0.693}{(4.468 \times 109\ \text{yr})(3.16 \times 10^7\ \text{s/yr})}\, N, \end{aligned}$$

which gives $N = 1.78 \times 10^{20}$ nuclei.

6–40. What fraction of a sample is left after (a) exactly 4 half-lives, (b) exactly $4\frac{1}{2}$ half-lives?

SOLUTION. (a) The fraction left is

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^4 = 0.0625.$$

(b) The fraction left is

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{4.5} = 0.0442.$$

6–41. In a series of decays, the nuclide ${}^{235}_{92}\text{U}$ becomes ${}^{207}_{82}\text{Pb}$. How many α and β^- particles are emitted in this series?

SOLUTION. Because only α particle decay changes the mass number (by 4), we have

$$N_\alpha = \frac{235 - 207}{4} = 7 \text{ } \alpha \text{ particles.}$$

An α particle decreases the atomic number by 2, while a β^- particle increases the atomic number by 1, so we have

$$N_\beta = \frac{92 - 82 - 7(2)}{-1} = 4 \text{ } \beta^- \text{ particles.}$$

6–42. The iodine isotope ${}^{131}_{53}\text{I}$ ($T_{1/2} = 8.0207$ days) is used in hospitals for diagnosis of thyroid function. If 632 μg are ingested by a patient, determine the activity (a) immediately, (b) 1.0 h later when the thyroid is being tested, and (c) 6 months later.

SOLUTION. The decay constant is

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{(8.0207 \text{ days})(24 \text{ h/day})(3\,600 \text{ s/h})} = 1.000 \times 10^{-6} \text{ s}^{-1}.$$

The initial number of nuclei is

$$N_0 = \frac{632 \times 10^{-6} \text{ g}}{131 \text{ g/mol}}(6.02 \times 10^{23} \text{ atoms/mol}) = 2.904 \times 10^{18} \text{ nuclei.}$$

(a) When $t = 0$, we get

$$\begin{aligned} \lambda N &= \lambda N_0 e^{-\lambda t} = (1.000 \times 10^{-6}\ \text{s}^{-1})(2.904 \times 10^{18}) e^0 \\ &= 2.90 \times 10^{12}\ \text{decays/s}. \end{aligned}$$

(b) When $t = 1.0$ h, the exponent is

$$\lambda t = (1.000 \times 10^{-6}\ \text{s}^{-1})(1.0\ \text{h})(3\,600\ \text{s/h}) = 3.600 \times 10^{-3},$$

so we get

$$\begin{aligned} \lambda N &= \lambda N_0 e^{-\lambda t} = (2.90 \times 10^{12}\ \text{decays/s}) e^{-0.003600} \\ &= 2.89 \times 10^{12}\ \text{decays/s}. \end{aligned}$$

(c) When $t = 6$ months, the exponent is

$$\begin{aligned} \lambda t &= (1.000 \times 10^{-6}\ \text{s}^{-1})(6\ \text{mo})(30\ \text{days/mo})(24\ \text{h/day})(3\,600\ \text{s/h}) \\ &= 15.55, \end{aligned}$$

so we get

$$\begin{aligned} \lambda N &= \lambda N_0 e^{-\lambda t} = (2.90 \times 10^{12}\ \text{decays/s}) e^{-15.55} \\ &= 5.11 \times 10^{5}\ \text{decays/s}. \end{aligned}$$

6–43. $^{124}_{55}$Cs has a half-life of 30.8 s. (a) If we have 9.8 μg initially, how many Cs nuclei are present? (b) How many are present 2.0 min later? (c) What is the activity at this time? (d) After how much time will the activity drop to less than about 1 per second?

SOLUTION. The decay constant is

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{30.8\ \text{s}} = 0.0225\ \text{s}^{-1}.$$

(a) The initial number of nuclei is

$$N_0 = \frac{9.8 \times 10^{-6}\ \text{g}}{124\ \text{g/mol}} (6.02 \times 10^{23}\ \text{atoms/mol}) = 4.8 \times 10^{16}\ \text{nuclei}.$$

(b) When $t = 2.0$ min, the exponent is

$$\lambda t = (0.0225\ \text{s}^{-1})(2.0\ \text{min})(60\ \text{s/min}) = 2.7,$$

so we get

$$N = N_0 e^{-\lambda t} = (4.8 \times 10^{16}) e^{-2.7} = 3.2 \times 10^{15} \text{ nuclei.}$$

(c) The activity is

$$\lambda N = (0.0225 \text{ s}^{-1})(3.2 \times 10^{15}) = 7.2 \times 10^{13} \text{ decays/s.}$$

(d) We find the time from

$$\begin{aligned} \lambda N &= \lambda N_0 e^{-\lambda t}; \\ 1 \text{ decay/s} &= (0.0225 \text{ s}^{-1})(4.8 \times 10^{16}) \exp\left(-(0.0225 \text{ s}^{-1})t\right), \end{aligned}$$

which gives

$$t = 1.54 \times 10^3 \text{ s} = 26 \text{ min.}$$

6–44. Calculate the activity of a pure 7.7-μg sample of $^{32}_{15}$P ($T_{1/2} = 1.23 \times 10^6$ s).

SOLUTION. The number of nuclei is

$$N = \frac{7.7 \times 10^{-6} \text{ g}}{32 \text{ g/mol}} (6.02 \times 10^{23} \text{ atoms/mol}) = 1.45 \times 10^{17} \text{ nuclei.}$$

The activity is

$$\lambda N = \frac{0.693}{1.23 \times 10^6 \text{ s}} (1.45 \times 10^{17}) = 8.2 \times 10^{10} \text{ decays/s.}$$

6–45. The activity of a sample of $^{35}_{16}$S ($T_{1/2} = 7.56 \times 10^6$ s) is 2.65×10^5 decays per second. What is the mass of the sample?

SOLUTION. We find the number of nuclei from

$$\begin{aligned} \text{Activity} &= \lambda N; \\ (2.65 \times 10^5 \text{ decays/s}) &= \frac{0.693}{(7.56 \times 10^6 \text{ s})} N, \end{aligned}$$

which gives $N = 2.89 \times 10^{12}$ nuclei. The mass is

$$m = \frac{(2.89 \times 10^{12} \text{ nuclei})}{(6.02 \times 10^{23} \text{ atoms/mol})} (35 \text{ g/mol}) = 1.68 \times 10^{-10} \text{ g.}$$

6–46. A sample of $^{233}_{92}$U ($T_{1/2} = 1.59 \times 10^5$ yr) contains 7.50×10^{19} nuclei, (a) What is the decay constant? (b) Approximately how many disintegrations will occur per minute?

SOLUTION. (a) The decay constant is

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{(1.59 \times 10^5 \text{ yr})(3.16 \times 10^7 \text{ s/yr})} = 1.38 \times 10^{-13} \text{ s}^{-1}.$$

(b) The activity is

$$\begin{aligned} \lambda N &= (1.38 \times 10^{-13} \text{ s}^{-1})(7.50 \times 10^{19}) = 1.035 \times 10^7 \text{ decays/s} \\ &= 6.21 \times 10^8 \text{ decays/min}. \end{aligned}$$

6–47. The activity of a sample drops by a factor of 10 in 8.6 minutes. What is its half-life?

SOLUTION. We find the number of half-lives from

$$\begin{aligned} \frac{\Delta N/\Delta t}{(\Delta N/\Delta t)_0} &= \left(\frac{1}{2}\right)^n; \\ \frac{1}{10} &= \left(\frac{1}{2}\right)^n, \\ \text{or} \quad n \lg 2 &= \lg 10, \end{aligned}$$

which gives $n = 3.32$. Thus the half-life is

$$T_{1/2} = \frac{t}{n} = \frac{8.6 \text{ min}}{3.32} = 2.6 \text{ min}.$$

6–48. A 185-g sample of pure carbon contains 1.3 parts in 10^{12} (atoms) of $^{14}_{6}$C. How many disintegrations occur per second?

SOLUTION. Because the fraction of atoms that are ^{14}C is so small, we use the atomic weight of ^{12}C to find the number of carbon atoms in 185 g:

$$N = \frac{(185 \text{ g})}{(12 \text{ g/mol})}(6.02 \times 10^{23} \text{ atoms/mol}) = 9.28 \times 10^{24} \text{ atoms}.$$

The number of ^{14}C nuclei is

$$N_{14} = \frac{1.3}{10^{12}}(9.28 \times 10^{24}) = 1.207 \times 10^{13} \text{ nuclei}.$$

The activity is

$$\lambda N = \frac{0.693}{(5\,730 \text{ yr})(3.16 \times 10^7 \text{ s/yr})}(1.207 \times 10^{13}) = 46 \text{ decays/s}.$$

6–49. A sample of $^{40}_{19}$K is decaying at a rate of 8.70×10^2 decays/s. What is the mass of the sample?

SOLUTION. We find the number of nuclei from

$$\begin{aligned} \text{Activity} &= \lambda N; \\ (8.70 \times 10^2 \text{ decays/s}) &= \frac{0.693}{(1.28 \times 10^9 \text{ yr})(3.16 \times 10^7 \text{ s/yr})} N, \end{aligned}$$

which gives $N = 5.08 \times 10^{19}$ nuclei. The mass is

$$m = \frac{(5.08 \times 10^{19} \text{ nuclei})}{(6.02 \times 10^{23} \text{ atoms/mol})}(40 \text{ g/mol}) = 3.4 \times 10^{-3} \text{ g} = 3.4 \text{ mg}.$$

6–50. The rubidium isotope $^{87}_{37}$Rb, a β emitter with a half-life of 4.75×10^{10} yr, is used to determine the age of rocks and fossils. Rocks containing fossils of early animals contain a ratio of $^{87}_{38}$Sr to $^{87}_{37}$Rb of 0.0160. Assuming that there was no $^{87}_{38}$Sr present when the rocks were formed, calculate the age of these fossils.

SOLUTION. We assume that the elapsed time is much smaller than the half-life, so we can use a constant decay rate. Because ^{87}Sr is stable, and there was none present when the rocks were formed, every atom of ^{87}Rb that decayed is now an atom of ^{87}Sr. Thus we have

$$N_{\text{Sr}} = -\Delta N_{\text{Rb}} = \lambda N_{\text{Rb}} \Delta t,$$

or

$$\begin{aligned} \frac{N_{\text{Sr}}}{N_{\text{Rb}}} &= \frac{0.693}{T_{1/2}} \Delta t; \\ 0.0160 &= \frac{0.693}{(4.75 \times 10^{10} \text{ yr})} \Delta t, \end{aligned}$$

which gives $\Delta t = 1.1 \times 10^9$ yr. This is $\approx 2\%$ of the half-life, so our original assumption is valid.

6–51. Use Fig. 6.2 and calculate the relative decay rates for α decay of ${}^{218}_{84}$Po and ${}^{214}_{84}$Po.

SOLUTION. The decay rate is

$$\frac{\Delta N}{\Delta t} = \lambda N.$$

If we assume equal numbers of nuclei decaying by α emission, we have

$$\frac{(\Delta N/\Delta t)_{218}}{(\Delta N/\Delta t)_{214}} = \frac{\lambda_{218}}{\lambda_{214}} = \frac{T_{1/2,\ 214}}{T_{1/2,\ 218}} = \frac{1.6 \times 10^{-4}\ \text{s}}{(3.1\ \text{min})(60\ \text{s/min})} = 8.6 \times 10^{-7}.$$

6–52. ${}^{7}_{4}$Be decays with a half-life of about 53 days. It is produced in the upper atmosphere, and filters down onto the Earth's surface. If a plant leaf is detected to have 250 decays/s of ${}^{7}_{4}$Be, how long do we have to wait for the decay rate to drop to 10 s^{-1}? Estimate the initial mass of ${}^{7}_{4}$Be on the leaf.

SOLUTION. The decay constant is

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{53\ \text{days}} = 0.0131\ \text{day}^{-1} = 1.52 \times 10^{-7}\ \text{s}^{-1}.$$

We find the number of half-lives from

$$\begin{aligned} \frac{(\Delta N/\Delta t)}{(\Delta N/\Delta t)_0} &= \left(\frac{1}{2}\right)^n; \\ \frac{(10\ \text{decays/s})}{(250\ \text{decays/s})} &= \left(\frac{1}{2}\right)^n, \\ \text{or} \quad n \lg 2 &= \lg 25, \end{aligned}$$

which gives $n = 4.64$. Thus the elapsed time is

$$\Delta t = nT_{1/2} = (4.64)(53\ \text{days}) = 2.5 \times 10^2\ \text{days} \approx 8\ \text{months}.$$

We find the number of nuclei from

$$\begin{aligned} \text{Activity} &= \lambda N; \\ (250\ \text{decays/s}) &= (1.52 \times 10^{-7}\ \text{s}^{-1})N, \end{aligned}$$

which gives $N = 1.64 \times 10^9$ nuclei.

The mass is

$$\begin{aligned} m &= \frac{(1.64 \times 10^9 \text{ nuclei})}{(6.02 \times 10^{23} \text{ atoms/mol})}(7 \text{ g/mol}) = 1.9 \times 10^{-14} \text{ g} \\ &= 1.9 \times 10^{-17} \text{ kg}. \end{aligned}$$

6–53. The radioactive isotopes of lead have the following half-lives:

Mass number A	Decay mode	Half-life
210	β^-, γ, α	22.3 yr
211	β^-, γ	36.1 min
212	β^-, γ	10.64 h
214	β^-, γ	26.8 min

Which radioactive isotope of lead is being produced in a reaction where the measured activity of a sample drops to 1.050 percent of its original activity in 4.00 h?

SOLUTION. We find the number of half-lives from

$$\begin{aligned} \frac{(\Delta N/\Delta t)}{(\Delta N/\Delta t)_0} &= \left(\frac{1}{2}\right)^n; \\ 1.050 \times 10^{-2} &= 2^{-n}, \\ \text{or} \quad n \lg 2 &= \lg \frac{1}{1.050 \times 10^{-2}}, \end{aligned}$$

which gives $n = 6.57$. Thus the half-life is

$$T_{1/2} = \frac{t}{n} = \frac{(4.00 \text{ h})}{6.57} = 0.609 \text{ h} = 36.5 \text{ min}.$$

From the Appendix C we see that the isotope is $^{211}_{82}$Pb.

6–54. An ancient club is found that contains 190 g of carbon and has an activity of 5.0 decays per second. Determine its age assuming that in living trees the ratio of ^{14}C/^{12}C atoms is about 1.3×10^{-12}.

SOLUTION. The decay constant is

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{(5\,730 \text{ yr})} = 1.209 \times 10^{-4} \text{ yr}^{-1}.$$

Because the fraction of atoms that are ^{14}C is so small, we use the atomic weight of ^{12}C to find the number of carbon atoms in 190 g:

$$N = \frac{(190\text{ g})}{(12\text{ g/mol})}(6.02 \times 10^{23}\text{ atoms/mol}) = 9.53 \times 10^{24}\text{ atoms},$$

so the number of ^{14}C nuclei in a sample from a living tree is

$$N_{14} = (1.3 \times 10^{-12})(9.53 \times 10^{24}) = 1.24 \times 10^{13}\text{ nuclei}.$$

Because the carbon is being replenished in living trees, we assume that this number produced the activity when the club was made. We determine its age from

$$\begin{aligned} \lambda N &= \lambda N_{14} e^{-\lambda t}; \\ 5.0\text{ decays/s} &= \frac{(1.209 \times 10^{-4}\text{ yr}^{-1})}{(3.16 \times 10^{7}\text{ s/yr})}(1.24 \times 10^{13}\text{ nuclei}) \\ &\quad \times \exp\left(-(1.209 \times 10^{-4}\text{ yr}^{-1})t\right), \end{aligned}$$

which gives $t = 1.9 \times 10^4$ yr.

6–55. At $t = 0$, a pure sample of radioactive nuclei contains N_0 nuclei whose decay constant is λ. Determine a formula for the number of daughter nuclei, N_D, as a function of time. Assume the daughter is stable and that $N_D = 0$ at $t = 0$.

SOLUTION. The number of radioactive nuclei decreases exponentially:

$$N = N_0 e^{-\lambda t}.$$

Every radioactive nucleus that decays becomes a stable daughter nucleus, so we have

$$N_D = N_0 - N = N_0(1 - e^{-\lambda t}).$$

6–56. At $t = 0$, a pure sample of a radioactive nuclide (the parent) contains N_{P0} nuclei whose half-life is T_P. The daughter nuclide is also radioactive, with half-life T_D. (a) Determine the number of daughter nuclei, N_D, as a function of time, assuming that $N_D = 0$ at $t = 0$. (b) Plot N_D versus t for the cases $T_P = T_D$, $T_P = 3T_D$, $T_P = \frac{1}{3}T_D$. [*Hint*: Eq. 6.6 must be modified.]

SOLUTION. (a) We find the number of parent nuclei from the rate at which parent nuclei are decaying:

$$\frac{dN_{\rm P}}{dt} = -\lambda_{\rm P} N_{\rm P},$$

which gives

$$N_{\rm P} = N_{\rm P0} e^{-\lambda_{\rm P} t}.$$

Each decay of a parent nucleus produces a daughter nucleus. Because the daughter nuclei are also decaying, the rate at which daughter nuclei are increasing is

$$\frac{dN_{\rm D}}{dt} = \lambda_{\rm P} N_{\rm P} - \lambda_{\rm D} N_{\rm D} = \lambda_{\rm P} N_{\rm P0} e^{-\lambda_{\rm P} t} - \lambda_{\rm D} N_{\rm D},$$

or

$$\frac{dN_{\rm D}}{dt} + \lambda_{\rm D} N_{\rm D} = \lambda_{\rm P} N_{\rm P0} e^{-\lambda_{\rm P} t}.$$

If we multiply both sides by the integrating factor $e^{\lambda_{\rm D} t}$, we have

$$e^{\lambda_{\rm D} t}\left(\frac{dN_{\rm D}}{dt} + \lambda_{\rm D} N_{\rm D}\right) = \frac{d}{dt}\left(N_{\rm D} e^{\lambda_{\rm D} t}\right) = \lambda_{\rm P} N_{\rm P0} e^{-\lambda_{\rm P} t} e^{\lambda_{\rm D} t},$$

or

$$d\left(N_{\rm D} e^{\lambda_{\rm D} t}\right) = \lambda_{\rm P} N_{\rm P0} e^{-(\lambda_{\rm P} - \lambda_{\rm D}) t}\, dt.$$

When we integrate this from $t = 0$ (when $N_{\rm D} = 0$) to $t = t$, we get

$$\begin{aligned} N_{\rm D} e^{\lambda_{\rm D} t}\Big|_0^t &= \frac{-\lambda_{\rm P} N_{\rm P0}}{\lambda_{\rm P} - \lambda_{\rm D}}\, e^{-(\lambda_{\rm P} - \lambda_{\rm D}) t}\Big|_0^t; \\ N_{\rm D} e^{\lambda_{\rm D} t} &= \frac{\lambda_{\rm P} N_{\rm P0}}{\lambda_{\rm P} - \lambda_{\rm D}}\left(1 - e^{-(\lambda_{\rm P} - \lambda_{\rm D}) t}\right), \end{aligned}$$

or

$$N_{\rm D} = \frac{\lambda_{\rm P} N_{\rm P0}}{\lambda_{\rm P} - \lambda_{\rm D}}\left(e^{-\lambda_{\rm D} t} - e^{-\lambda_{\rm P} t}\right).$$

In terms of the half-lives this is

$$N_{\rm D} = \frac{T_{\rm D} N_{\rm P0}}{T_{\rm D} - T_{\rm P}}\left(\exp\left(-0.693\, t/T_{\rm D}\right) - \exp\left(-0.693\, t/T_{\rm P}\right)\right).$$

(b) We write our result as

$$\frac{N_{\mathrm{D}}}{N_{\mathrm{P0}}} = \frac{\left(\exp\left(-0.693\,t/T_{\mathrm{D}}\right) - \exp\left(-0.693\,t/T_{\mathrm{P}}\right)\right)}{T_{\mathrm{D}} - T_{\mathrm{P}}}\, T_{\mathrm{D}}.$$

When $T_{\mathrm{P}} = T_{\mathrm{D}}$, both numerator and denominator are zero, so we must use l'Hôpital's rule:

$$\begin{aligned}
&\frac{\partial_{T_{\mathrm{P}}}\left(\exp\left(-0.693\,t/T_{\mathrm{D}}\right) - \exp\left(-0.693\,t/T_{\mathrm{P}}\right)\right)|_{T_{\mathrm{P}}=T_{\mathrm{D}}}}{\partial_{T_{\mathrm{P}}}(T_{\mathrm{D}} - T_{\mathrm{P}})|_{T_{\mathrm{P}}=T_{\mathrm{D}}}} \\
&= \frac{\left(-0.693\,t/T_{\mathrm{D}}^2\right)\exp\left(-0.693\,t/T_{\mathrm{D}}\right)}{-1} \\
&= \frac{0.693\,t}{T_{\mathrm{D}}^2}\exp\left(-0.693\,t/T_{\mathrm{D}}\right).
\end{aligned}$$

Thus we have

$$\frac{N_{\mathrm{D}}}{N_{\mathrm{P0}}} = \frac{0.693\,t}{T_{\mathrm{D}}^2}\exp\left(-\frac{0.693\,t}{T_{\mathrm{D}}}\right).$$

When $T_{\mathrm{P}} = 3T_{\mathrm{D}}$, we have

$$\frac{N_{\mathrm{D}}}{N_{\mathrm{P0}}} = \frac{1}{2}\left(\exp\left(-\frac{0.693\,t}{3T_{\mathrm{D}}}\right) - \exp\left(-\frac{0.693\,t}{T_{\mathrm{D}}}\right)\right).$$

When $T_{\mathrm{P}} = \frac{1}{3}T_{\mathrm{D}}$, we have

$$\frac{N_{\mathrm{D}}}{N_{\mathrm{P0}}} = \frac{3}{2}\left(\exp\left(-\frac{0.693\,t}{T_{\mathrm{D}}}\right) - \exp\left(-\frac{3(0.693)t}{T_{\mathrm{D}}}\right)\right).$$

The results are graphed on Fig. 6.3.

6.6 General problems

6–57. (a) What is the fraction of the hydrogen atom's mass that is in the nucleus? (b) What is the fraction of the hydrogen atom's volume that is occupied by the nucleus? (c) What is the density of nuclear matter? Compare this with water.

Solution. (a) The fraction of mass is

$$\frac{m_{\mathrm{p}}}{m_{\mathrm{p}} + m_{\mathrm{e}}} = \frac{1.6726 \times 10^{-27}\ \mathrm{kg}}{1.6726 \times 10^{-27}\ \mathrm{kg} + 9.11 \times 10^{-31}\ \mathrm{kg}} = 0.99946.$$

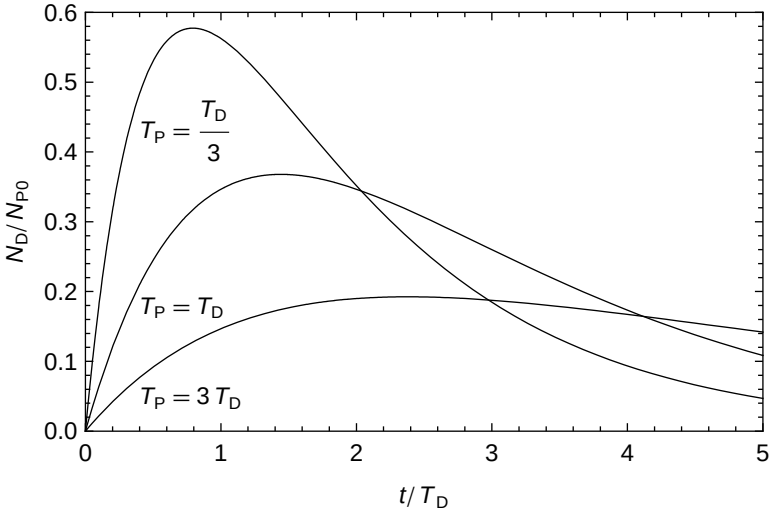


Figure 6.3. Problem 6–56.

(b) The fraction of volume is

$$\left(\frac{r_{\text{nucleus}}}{r_{\text{atom}}}\right)^3 = \left(\frac{1.2 \times 10^{-15}\ \text{m}}{0.53 \times 10^{-10}\ \text{m}}\right)^3 = 1.2 \times 10^{-14}.$$

(c) If we take the density of the hydrogen nucleus as the density of nuclear matter, we get

$$\rho = \frac{m_{\text{p}}}{\frac{4}{3}\pi r^3} = \frac{1.67 \times 10^{-27}\ \text{kg}}{\frac{4}{3}\pi(1.2 \times 10^{-15}\ \text{m})^3} = 2.3 \times 10^{17}\ \text{kg/m}^3.$$

The density of water is 1 000 kg/m^3, so nuclear matter is 1 014× greater.

6–58. Using the uncertainty principle and the radius of a nucleus, estimate the minimum possible kinetic energy of a nucleon in, say, iron. Ignore relativistic corrections. [*Hint*: A particle can have an energy at least as large as its uncertainty.]

Solution. The radius of the iron nucleus is

$$r = (1.2 \times 10^{-15}\ \text{m})\sqrt[3]{A} = (1.2 \times 10^{-15}\ \text{m})\sqrt[3]{56} = 4.6 \times 10^{-15}\ \text{m}.$$

We use the radius as the uncertainty in position for the nucleon. We find the uncertainty in the momentum from

$$\Delta p \geqslant \frac{\hbar}{2\,\Delta x} = \frac{1.055 \times 10^{-34}\ \mathrm{J \cdot s}}{2(4.6 \times 10^{-15}\ \mathrm{m})} = 1.15 \times 10^{-20}\ \mathrm{kg \cdot m/s}.$$

If we assume that the lowest value for the momentum is the least uncertainty, we estimate the minimum possible kinetic energy as

$$K = \frac{(\Delta p)^2}{2m} = \frac{(1.15 \times 10^{-20}\ \mathrm{kg \cdot m/s})^2}{2(1.66 \times 10^{-27}\ \mathrm{kg})} = 0.396 \times 10^{-13}\ \mathrm{J} = 248\ \mathrm{keV}.$$

6–59. How much recoil energy does a $^{40}_{19}\mathrm{K}$ nucleus get when it emits a 1.46 MeV gamma ray?

SOLUTION. If the $^{40}\mathrm{K}$ nucleus in the excited state is at rest, the gamma ray and the nucleus must have equal and opposite momenta:

$$p_{\mathrm{K}} = p_\gamma = \frac{E_\gamma}{c},$$

or

$$p_{\mathrm{K}} c = E_\gamma = 1.46\ \mathrm{MeV}.$$

The kinetic energy of the nucleus is

$$\begin{aligned} K &= \frac{p_{\mathrm{K}}^2}{2m} = \frac{(p_{\mathrm{K}} c)^2}{2mc^2} = \frac{(1.46\ \mathrm{MeV})^2}{2(40\ \mathrm{u})(931.5\ \mathrm{MeV/uc^2})c^2} \\ &= 2.86 \times 10^{-5}\ \mathrm{MeV} = 28.6\ \mathrm{eV}. \end{aligned}$$

6–60. An old wooden tool is found to contain only 9.0 percent of $^{14}_{6}\mathrm{C}$ that a sample of fresh wood would. How old is the tool?

SOLUTION. Because the carbon is being replenished in living trees, we assume that the amount of $^{14}\mathrm{C}$ is constant until the wood is cut, and then it decays. We find the number of half-lives from

$$\begin{aligned} \frac{N}{N_0} &= \left(\frac{1}{2}\right)^n; \\ 0.090 &= \left(\frac{1}{2}\right)^n, \\ \text{or} \quad n \lg 2 &= \lg\left(\frac{1}{0.090}\right), \end{aligned}$$

which gives $n = 3.47$. Thus the time is

$$t = nT_{1/2} = (3.47)(5\,730 \text{ yr}) = 2.0 \times 10^4 \text{ yr}.$$

6–61. A neutron star consists of neutrons at approximately nuclear density. Estimate, for a 10-km-diameter neutron star, (a) its mass number, (b) its mass (kg), and (c) the acceleration of gravity at its surface.

SOLUTION. (a) We find the mass number from the radius:

$$\begin{aligned} r &= (1.2 \times 10^{-15} \text{ m})\sqrt[3]{A}\,; \\ 5.0 \times 10^3 \text{ m} &= (1.2 \times 10^{-15} \text{ m})\sqrt[3]{A}\,, \end{aligned}$$

which gives $A = 7.2 \times 10^{55}$.

(b) The mass of the neutron star is

$$\begin{aligned} m &= A(1.66 \times 10^{-27} \text{ kg/u}) = (7.2 \times 10^{55} \text{ u})(1.66 \times 10^{-27} \text{ kg/u}) \\ &= 1.2 \times 10^{29} \text{ kg}. \end{aligned}$$

Note that this is about 6% of the mass of the Sun.

(c) The acceleration of gravity on the surface of the neutron star is

$$\begin{aligned} g &= \frac{Gm}{r^2} = \frac{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(1.2 \times 10^{29} \text{ kg})}{(5.0 \times 10^3 \text{ m})^2} \\ &= 3.2 \times 10^{11} \text{ m/s}^2. \end{aligned}$$

6–62. The ${}^3_1\text{H}$ isotope of hydrogen, which is called tritium (because it contains three nucleons), has a half-life of 12.33 yr. It can be used to measure the age of objects up to about 100 yr. It is produced in the upper atmosphere by cosmic rays and brought to Earth by rain. As an application, determine approximately the age of a bottle of wine whose ${}^3_1\text{H}$ radiation is about $\frac{1}{10}$ that present in new wine.

SOLUTION. Because the tritium in water is being replenished, we assume that the amount is constant until the wine is made, and then it decays. We find the number of half-lives from

$$\begin{aligned} \frac{N}{N_0} &= \left(\frac{1}{2}\right)^n; \\ 0.10 &= \left(\frac{1}{2}\right)^n, \\ \text{or} \quad n \lg 2 &= \lg 10, \end{aligned}$$

which gives $n = 3.32$. Thus the time is

$$t = nT_{1/2} = (3.32)(12.33 \text{ yr}) = 41 \text{ yr}.$$

6–63. Some elementary particle theories suggest that the proton may be unstable, with a half-life $\geqslant 10^{32}$ yr. How long would you expect to wait for one proton in your body to decay (consider that your body is all water)?

SOLUTION. If we assume a body has 70 kg of water, the number of water molecules is

$$N_{\text{water}} = \frac{70 \times 10^3 \text{ g}}{18 \text{ g/mol}}(6.02 \times 10^{23} \text{ atoms/mol}) = 2.34 \times 10^{27} \text{ molecules}.$$

The number of protons in a water molecule (H_2O) is $2 + 8 = 10$, so the number of protons is

$$N_0 = 2.34 \times 10^{28} \text{ protons}.$$

If we assume that the time is much less than the half-life, the rate of decay is constant, so we have

$$\begin{aligned} \frac{\Delta N}{\Delta t} &= \lambda N = \frac{0.693}{T_{1/2}} N; \\ \frac{1 \text{ proton}}{\Delta t} &= \frac{0.693}{1\,032 \text{ yr}}(2.34 \times 10^{28} \text{ protons}), \end{aligned}$$

which gives $\Delta t = 6 \times 10^3$ yr.

6–64. When water is placed near an intense neutron source, the neutrons can be slowed down by collisions with the water molecules and eventually captured by a hydrogen nucleus to form the stable isotope called deuterium, ${}^2_1\text{H}$, giving off a gamma ray. What is the energy of the gamma ray?

SOLUTION. The capture is

$${}^1_1\text{H} + {}^1_0\text{n} \rightarrow {}^2_1\text{H} + \gamma.$$

Because the kinetic energies of the particles are small, the gamma energy is the energy released:

$$\begin{aligned} Q &= \left(M(^1\mathrm{H}) + m(^1\mathrm{n}) - M(^2\mathrm{H})\right) c^2 \\ &= ((1.007825\ \mathrm{u}) + (1.008665\ \mathrm{u}) - (2.014102\ \mathrm{u})) \\ &\quad \times c^2 (931.5\ \mathrm{MeV/u}c^2) \\ &= 2.22\ \mathrm{MeV}. \end{aligned}$$

6–65. How long must you wait (in half-lives) for a radioactive sample to drop to 1.00 percent of its original activity?

SOLUTION. We find the number of half-lives from

$$\begin{aligned} \frac{\Delta N/\Delta t}{(\Delta N/\Delta t)_0} &= \left(\frac{1}{2}\right)^n; \\ 1.00 \times 10^{-2} &= \left(\frac{1}{2}\right)^n, \\ \text{or} \quad n \lg 2 &= \lg 100, \end{aligned}$$

which gives $n = 6.64$, so the time is $6.64\, T_{1/2}$.

6–66. If the potassium isotope $^{40}_{19}\mathrm{K}$ gives 60 decays/s in a liter of milk, estimate how much $^{40}_{19}\mathrm{K}$ and regular $^{39}_{19}\mathrm{K}$ are in a liter of milk. The respective natural abundances of $^{39}_{19}\mathrm{K}$ and $^{40}_{19}\mathrm{K}$ are 93.2581% and 0.0117%.

SOLUTION. We find the number of $^{40}\mathrm{K}$ nuclei from

$$\begin{aligned} \text{Activity} &= \lambda N_{40}; \\ 60\ \text{decays/s} &= \frac{0.693}{(1.28 \times 10^9\ \mathrm{yr})(3.16 \times 10^7\ \mathrm{s/yr})} N_{40}, \end{aligned}$$

which gives $N_{40} = 3.5 \times 10^{18}$ nuclei.

The mass of $^{40}\mathrm{K}$ is

$$\begin{aligned} m_{40} &= (3.5 \times 10^{18})(40\ \mathrm{u})(1.66 \times 10^{-27}\ \mathrm{kg/u}) = 2.32 \times 10^{-7}\ \mathrm{kg} \\ &= 0.23\ \mathrm{mg}. \end{aligned}$$

We have

$$N_{40} = (0.0117\,\%)N, \quad \text{and} \quad N_{39} = (93.2581\,\%)N.$$

Thus the number of ^{39}K nuclei is

$$N_{39} = \frac{93.2581\,\%}{0.0117\,\%}(3.5 \times 10^{18} \text{ nuclei}) = 2.8 \times 10^{22} \text{ nuclei}.$$

The mass of ^{39}K is

$$\begin{aligned} m_{39} &= (2.8 \times 10^{22})(39 \text{ u})(1.66 \times 10^{-27} \text{ kg/u}) = 1.81 \times 10^{-3} \text{ kg} \\ &= 1.8 \text{ g}. \end{aligned}$$

6–67. Show, using the decay given by Eq. 6.4, that the neutrino must have spin $\frac{1}{2}$.

SOLUTION. If we use

$$^{14}_{6}\text{C} \rightarrow {}^{14}_{7}\text{N} + \text{e}^- + \tilde{\nu}$$

as the decay, we see that ^{14}C and ^{14}N each have an even number of nucleons with spin $\frac{1}{2}$, so their total spin must be an integer. Because e^- has spin $\frac{1}{2}$, $\tilde{\nu}$ must have spin $\frac{1}{2}$ to conserve angular momentum.

6–68. In a decay of, say, a $^{226}_{88}$Ra nucleus, show that the nucleus carries away a fraction $1/(1 + \frac{1}{4}A_\text{D})$ of the total energy available, where A_D is the mass number of the daughter nucleus. [*Hint*: Use conservation of momentum as well as conservation of energy.] Approximately what percentage of the energy available is thus carried off by the α particle in the case cited?

SOLUTION. If the initial nucleus is at rest when it decays, for momentum conservation we have

$$p_\alpha = p_\text{D}.$$

Thus the kinetic energy of the daughter is

$$K_\text{D} = \frac{p_\text{D}^2}{2m_\text{D}} = \frac{p_\alpha^2}{2m_\text{D}} = \frac{m_\alpha}{m_\text{D}}K_\alpha = \frac{A_\alpha}{A_\text{D}}K_\alpha = \frac{4}{A_\text{D}}K_\alpha.$$

Thus the fraction carried away by the daughter is

$$\frac{K_\text{D}}{K_\alpha + K_\text{D}} = \frac{4K_\alpha/A_\text{D}}{K_\alpha + 4K_\alpha/A_\text{D}} = \frac{1}{1 + \frac{1}{4}A_\text{D}}.$$

For the decay of ^{226}Ra, the daughter has $A_{\rm D} = 222$, so we get

$$\text{fraction}_{\rm D} = \frac{1}{1 + \frac{1}{4}(222)} = 0.018.$$

Thus the α particle carries away $1 - 0.018 = 0.982 = 98.2\,\%$.

6–69. Strontium-90 is produced as a nuclear fission product of uranium in both reactors and atomic bombs. Look at its location in the periodic table (Fig. C.1) to see what other elements it might be similar to chemically, and tell why you think it might be dangerous to ingest. It has too many neutrons, and it decays with a half-life of about 29 yr. How long will we have to wait for the amount of $^{90}_{38}$Sr on the Earth's surface to reach 1 percent of its current level, assuming no new material is scattered about? Write down the decay reaction, including the daughter nucleus. Is the daughter radioactive? If so, write down its decay. Finish the decay scheme until you reach a stable nucleus.

SOLUTION. We see from the periodic chart (Fig. C.1) that Sr is in the same column as calcium. If strontium is ingested, the body will treat it chemically as if it were calcium, which means it will be stored by the body in bones, where its radioactivity could affect the bone marrow.
We find the number of half-lives to reach a 1% level from

$$\begin{aligned} \frac{N}{N_0} &= \left(\frac{1}{2}\right)^n; \\ 0.01 &= \left(\frac{1}{2}\right)^n, \\ \text{or} \quad n \lg 2 &= \lg 100, \end{aligned}$$

which gives $n = 6.64$. Thus the time is

$$t = nT_{1/2} = (6.64)(29\ \text{yr}) = 193\ \text{yr}.$$

The decay reactions are

$$\begin{aligned} {}^{90}_{38}\text{Sr} &\rightarrow {}^{90}_{39}\text{Y} + {}^{0}_{-1}\text{e} + \tilde{\nu}, \qquad {}^{90}_{39}\text{Y} \text{ is radioactive;} \\ {}^{90}_{39}\text{Y} &\rightarrow {}^{90}_{40}\text{Zr} + {}^{0}_{-1}\text{e} + \tilde{\nu}, \qquad {}^{90}_{40}\text{Zr} \text{ is stable.} \end{aligned}$$

6–70. The nuclide $^{191}_{76}$Os decays with β^- energy of 0.14 MeV accompanied by γ rays of energy 0.042 MeV and 0.129 MeV. (a) What is the

daughter nucleus? (b) Draw an energy-level diagram showing the ground states of the parent and daughter and excited states of the daughter. To which of the daughter states does β^- decay of $^{191}_{76}$Os occur?

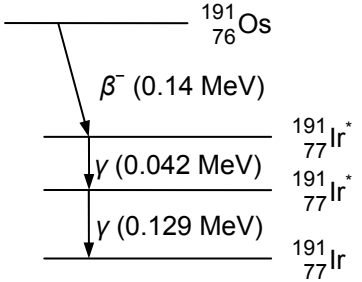


Figure 6.4. Problem 6–70.

Solution. (a) We find the daughter nucleus by balancing the mass and charge numbers:

$$\begin{aligned} Z(\mathrm{X}) &= Z(\mathrm{Os}) - Z(\mathrm{e}^-) = 76 - (-1) = 77; \\ A(\mathrm{X}) &= A(\mathrm{Os}) - A(\mathrm{e}^-) = 191 - 0 = 191, \end{aligned}$$

so the daughter nucleus is $^{191}_{77}$Ir (Fig. 6.4).

(b) Because there is only one β energy, the β decay must be to the higher excited state.

6–71. Use the uncertainty principle to argue why electrons are unlikely to be found in the nucleus. Use relativity. [*Hint*: A particle can have an energy at least as large as its uncertainty.]

Solution. We use an average nuclear radius of 5×10^{-15} m. We use the radius as the uncertainty in position for the electron to find the uncertainty in the momentum from

$$\Delta p \geqslant \frac{\hbar}{2\Delta x} = \frac{1.055 \times 10^{-34}\ \mathrm{J \cdot s}}{2(5 \times 10^{-15}\ \mathrm{m})} = 1.06 \times 10^{-20}\ \mathrm{kg \cdot m/s}.$$

If we assume that the lowest value for the momentum is the least uncer-

tainty, we estimate the minimum possible energy as

$$\begin{aligned} E &= K + mc^2 = \sqrt{p^2c^2 + m^2c^4} = \sqrt{(\Delta p)^2c^2 + m^2c^4} \\ &= \big((1.06 \times 10^{-20}\ \text{kg} \cdot \text{m/s})^2(3.00 \times 10^8\ \text{m/s})^2 \\ &\quad + (9.11 \times 10^{-31}\ \text{kg})^2(3.00 \times 10^8\ \text{m/s})^4\big)^{1/2} \\ &= 3.17 \times 10^{-12}\ \text{J} = 19.8\ \text{MeV}. \end{aligned}$$

Because this is $\gg mc^2$, the electron is unlikely to be found in the nucleus.

6–72. Estimate the total binding energy for copper and then estimate the energy, in joules, needed to break a 3.0-g copper coin into its constituent nucleons.

SOLUTION. From Fig. 6.1, we see that the average binding energy per nucleon at $A = 29$ is 8.6 MeV.
If we use the average atomic weight as the average number of nucleons for the two stable isotopes of copper, the total binding energy is

$$(63.5)(8.6\ \text{MeV}) = 550\ \text{MeV}.$$

The number of atoms in a coin is

$$N = \frac{3.0\ \text{g}}{63.5\ \text{g/mol}}(6.02 \times 10^{23}\ \text{atoms/mol}) = 2.84 \times 10^{22}\ \text{atoms}.$$

Thus the total energy needed is

$$(2.84 \times 10^{22})(550\ \text{MeV}) = 1.57 \times 10^{25}\ \text{MeV} = 2.5 \times 10^{12}\ \text{J}.$$

6–73. Instead of giving atomic masses for nuclides as in Appendix C, some tables give the *mass excess*, Δ, defined as $\Delta = M - A$, where A is the atomic number and M is the mass in u. Determine the mass excess, in u and in MeV/c^2, for: (a) ${}^4_2\text{He}$; (b) ${}^{12}_{6}\text{C}$; (c) ${}^{107}_{47}\text{Ag}$; (d) ${}^{235}_{92}\text{U}$. (e) From a glance at Appendix C, can you make a generalization about the sign of Δ as a function of Z or A?

SOLUTION.

$$\begin{aligned} \text{(a)}\quad \Delta({}^4\text{He}) &= M({}^4\text{He}) - A({}^4\text{He}) = 4.002603\ \text{u} - 4\ \text{u} \\ &= 0.002603\ \text{u} = (0.002603\ \text{u})(931.5\ \text{MeV/u}c^2) \\ &= 2.425\ \text{MeV}/c^2. \end{aligned}$$

(b) $\Delta(^{12}\mathrm{C}) = M(^{12}\mathrm{C}) - A(^{12}\mathrm{C}) = 12.000000\ \mathrm{u} - 12\ \mathrm{u} = 0.$

(c) $$\begin{aligned}\Delta(^{107}\mathrm{Ag}) &= M(^{107}\mathrm{Ag}) - A(^{107}\mathrm{Ag}) = 106.905093\ \mathrm{u} - 107\ \mathrm{u} \\ &= -0.094907\ \mathrm{u} = (-0.094907\ \mathrm{u})(931.5\ \mathrm{MeV/u}c^2) \\ &= -88.41\ \mathrm{MeV}/c^2.\end{aligned}$$

(d) $$\begin{aligned}\Delta(^{235}\mathrm{U}) &= M(^{235}\mathrm{U}) - A(^{235}\mathrm{U}) = 235.043923\ \mathrm{u} - 235\ \mathrm{u} \\ &= 0.043923\ \mathrm{u} = (0.043923\ \mathrm{u})(931.5\ \mathrm{MeV/u}c^2) \\ &= 40.92\ \mathrm{MeV}/c^2.\end{aligned}$$

(e) From Appendix C we see that

$$\begin{aligned}&\Delta \geqslant 0 \quad \text{for} \quad 1 \leqslant Z \leqslant 8 \qquad \text{and} \quad Z \geqslant 85; \\ &\Delta < 0 \quad \text{for} \quad 9 \leqslant Z \leqslant 84.\end{aligned}$$

6–74. (a) A 100-gram sample of natural carbon contains the usual fraction of $^{14}_{6}\mathrm{C}$. How long will it take on average before there is only one $^{14}_{6}\mathrm{C}$ nucleus left? (b) How does the answer in (a) change if the sample is 200 grams? What does this tell you about the limits of carbon dating?

SOLUTION. (a) The usual fraction of $^{14}\mathrm{C}$ is 1.3×10^{-12}. Because the fraction of atoms that are $^{14}\mathrm{C}$ is so small, we use the atomic weight of $^{12}\mathrm{C}$ to find the number of carbon atoms in 100 g:

$$N = \frac{100\ \mathrm{g}}{12\ \mathrm{g/mol}}(6.02 \times 10^{23}\ \mathrm{atoms/mol}) = 5.02 \times 10^{24}\ \mathrm{atoms}.$$

The number of $^{14}\mathrm{C}$ nuclei in the sample is

$$N_{14} = (1.3 \times 10^{-12})(5.02 \times 10^{24}) = 6.53 \times 10^{12}\ \mathrm{nuclei}.$$

We find the number of half-lives from

$$\begin{aligned}\frac{N}{N_0} &= \left(\frac{1}{2}\right)^n; \\ \frac{1}{6.53 \times 10^{12}} &= \left(\frac{1}{2}\right)^n, \\ \text{or} \quad n \lg 2 &= \lg(6.53 \times 10^{12}),\end{aligned}$$

which gives $n = 42.6$. Thus the time is

$$t = nT_{1/2} = (42.6)(5\,730 \text{ yr}) = 2.4 \times 10^5 \text{ yr}.$$

(b) It would take one half-life for the activity of 200 g to decay to the activity of 100 g. This is so much smaller than the time in (a) that there is no change. Thus carbon dating cannot be used for times much greater than the half-life.

6–75. If the mass of the proton were just a little closer to the mass of the neutron, the following reaction would be possible even at low collision energies:

$$\mathrm{e}^- + \mathrm{p} \rightarrow \mathrm{n} + \nu.$$

Why would this situation be catastrophic? By what percentage would the proton's mass have to be increased to make this reaction possible?

SOLUTION. Because there are so many low-energy electrons available, the reaction would turn most of the protons into neutrons, which would eliminate chemistry, and thus life.
The Q-value of the reaction is

$$\begin{aligned} Q &= \left(M(^1\mathrm{H}) - m(^1\mathrm{n})\right) c^2 \\ &= \left((1.007825 \text{ u}) - (1.008665 \text{ u})\right) c^2 (931.5 \text{ MeV/u}c^2) \\ &= -0.782 \text{ MeV}. \end{aligned}$$

The percentage increase in the proton's mass to make the Q-value = 0 is

$$\frac{\Delta m}{m}(100) = \frac{0.782 \text{ MeV}/c^2}{938.3 \text{ MeV}/c^2}(100) = 0.083\,\%.$$

6–76. (a) Show that the *mean life*, or *average lifetime* of a radioactive nuclide, defined as

$$\tau = \frac{\displaystyle\int_0^\infty N(t) t \, dt}{\displaystyle\int_0^\infty N(t) \, dt}$$

is $\tau = 1/\lambda$. (b) What fraction of the original number of nuclei remains after one mean life?

SOLUTION. (a) From the definition of the mean life we have

$$\tau = \frac{\displaystyle\int_0^\infty N(t)t\,dt}{\displaystyle\int_0^\infty N(t)\,dt} = \frac{\displaystyle\int_0^\infty N_0 e^{-\lambda t}t\,dt}{\displaystyle\int_0^\infty N_0 e^{-\lambda t}\,dt} = \frac{1}{\lambda}\frac{\displaystyle\int_0^\infty e^{-u}u\,du}{\displaystyle\int_0^\infty e^{-u}\,du},$$

where we have changed variable to $u = \lambda t$. We integrate the numerator by parts:

$$\begin{aligned}\int_0^\infty e^{-u}u\,du &= \int_0^\infty -u\,d(e^{-u}) = -u(e^{-u})\Big|_0^\infty + \int_0^\infty e^{-u}\,du \\ &= 0 + \int_0^\infty e^{-u}\,du.\end{aligned}$$

Thus we get

$$\tau = \frac{1}{\lambda}\frac{\displaystyle\int_0^\infty e^{-u}\,du}{\displaystyle\int_0^\infty e^{-u}\,du} = \frac{1}{\lambda}.$$

(b) We find the fraction remaining after one mean life from

$$\frac{N}{N_0} = e^{-\lambda t} = e^{-1} = 0.368.$$

6–77. Two of the naturally occurring radioactive decay sequences start with ${}^{232}_{90}\mathrm{Th}$, and ${}^{235}_{92}\mathrm{U}$. The first five decays of these two sequences are:

$$\alpha,\ \beta,\ \beta,\ \alpha,\ \alpha \quad \text{and} \quad \alpha,\ \beta,\ \alpha,\ \beta,\ \alpha.$$

With this information determine the resulting intermediate daughter nuclei in each case.

SOLUTION. For the $\alpha,\ \beta,\ \beta,\ \alpha,\ \alpha$ sequence we have

$$\begin{aligned}{}^{232}_{90}\mathrm{Th} &\rightarrow {}^{4}_{2}\mathrm{He} + {}^{228}_{88}\mathrm{Ra}, \\ {}^{228}_{88}\mathrm{Ra} &\rightarrow {}^{0}_{-1}\mathrm{e} + {}^{228}_{89}\mathrm{Ac}, \\ {}^{228}_{89}\mathrm{Ac} &\rightarrow {}^{0}_{-1}\mathrm{e} + {}^{228}_{90}\mathrm{Th}, \\ {}^{228}_{90}\mathrm{Th} &\rightarrow {}^{4}_{2}\mathrm{He} + {}^{224}_{88}\mathrm{Ra}, \\ {}^{224}_{88}\mathrm{Ra} &\rightarrow {}^{4}_{2}\mathrm{He} + {}^{220}_{86}\mathrm{Rn}.\end{aligned}$$

Thus the daughter nuclei are ${}^{228}_{88}\mathrm{Ra}$, ${}^{228}_{89}\mathrm{Ac}$, ${}^{228}_{90}\mathrm{Th}$, ${}^{224}_{88}\mathrm{Ra}$, ${}^{220}_{86}\mathrm{Rn}$.
For the $\alpha,\ \beta,\ \alpha,\ \beta,\ \alpha$ sequence we have

$$\begin{aligned}
{}^{235}_{92}\mathrm{U} &\rightarrow {}^{4}_{2}\mathrm{He} + {}^{231}_{90}\mathrm{Th},\\
{}^{231}_{90}\mathrm{Th} &\rightarrow {}^{0}_{-1}\mathrm{e} + {}^{231}_{91}\mathrm{Pa},\\
{}^{231}_{91}\mathrm{Pa} &\rightarrow {}^{4}_{2}\mathrm{He} + {}^{227}_{89}\mathrm{Ac},\\
{}^{227}_{89}\mathrm{Ac} &\rightarrow {}^{0}_{-1}\mathrm{e} + {}^{227}_{90}\mathrm{Th},\\
{}^{227}_{90}\mathrm{Th} &\rightarrow {}^{4}_{2}\mathrm{He} + {}^{223}_{88}\mathrm{Ra}.
\end{aligned}$$

Thus the daughter nuclei are ${}^{231}_{90}\mathrm{Th}$, ${}^{231}_{91}\mathrm{Pa}$, ${}^{227}_{89}\mathrm{Ac}$, ${}^{227}_{90}\mathrm{Th}$, ${}^{223}_{88}\mathrm{Ra}$.

6–78. What is the ratio of the kinetic energies for an α particle and a β particle if both make tracks with the same radius of curvature in a magnetic field, oriented perpendicular to the paths of the particles.

Solution. For the circular motion in the magnetic field, the magnetic force provides the radial acceleration:

$$qvB = \frac{mv^2}{R}.$$

The kinetic energy is

$$K = \frac{mv^2}{2} = \frac{q^2B^2R^2}{2m}.$$

When we form the ratio for the two particles with the same radii, we get

$$\frac{K_\alpha}{K_\beta} = \left(\frac{q_\alpha}{q_\beta}\right)^2 \frac{m_\beta}{m_\alpha} = \left(\frac{2e}{1e}\right)^2 \frac{9.11\times10^{-31}\ \mathrm{kg}}{4(1.67\times10^{-27}\ \mathrm{kg})} = 5.5\times10^{-4}.$$

Note that the particles will have opposite curvatures.

6–79. In a rough theoretical model of α decay, the α particle moves back and forth within the parent nucleus at speed v_{in}, making a collision with the Coulomb barrier (Fig. 3.11) every $2R_0/v_{\mathrm{in}}$ seconds, where R_0 is the radius of the nucleus. (a) Show that the decay constant λ in this model can be written roughly (using Eq. 3.10) as

$$\begin{aligned}
\lambda &= \frac{v_{\mathrm{in}}}{2R_0}\, e^{-2GL},\\
G &= \frac{\sqrt{2m(U_0 - E)}}{\hbar}.
\end{aligned}$$

where L is the "average" barrier thickness (the barrier is not square). (b) Use this result, plus estimates of the barrier height and width (as in Problem 6–34) to estimate the halflife of ${}^{236}_{92}\mathrm{U}$. For the barrier thickness (Fig. 3.11) try first $L = \frac{1}{2}(R_\mathrm{B} - R_0)$ and then (c) set $L = \frac{1}{3}(R_\mathrm{B} - R_0)$ as an average thickness since the barrier is not square. Choose the potential U_in inside the well to be zero. (d) What barrier width would provide, in this simple model, the measured half-life of 2×10^7 yr?

SOLUTION. (a) If the speed of the α particle within the nucleus is v_in, the time to traverse the nucleus is $2R_0/v_\mathrm{in}$. The frequency of a collision with the Coulomb barrier is $v_\mathrm{in}/(2R_0)$. According to Eq. 3.10, the probability of passing through the barrier each time is

$$T = Ce^{-2GL},$$

where L is the thickness of a square barrier, and

$$C = \frac{16E(U_0 - E)}{U_0^2}, \quad G = \frac{\sqrt{2m(U_0 - E)}}{\hbar}.$$

The decay constant is the probability of the α particle escaping from the nucleus:

$$\lambda = \frac{v_\mathrm{in}}{2R_0}\, T = \frac{v_\mathrm{in}}{R_0}\, Ce^{-2GL}.$$

The Coulomb barrier decreases as r^{-1}. For a square barrier that approximates this we will use the maximum height found in Problem 6–34 for the height of the barrier. The constant C is

$$C = \frac{16E(U_0 - E)}{U_0^2} = \frac{16(4.49\ \mathrm{MeV})(27.9\ \mathrm{MeV} - 4.49\ \mathrm{MeV})}{(27.9\ \mathrm{MeV})^2} = 2.16.$$

We find the value of G from

$$\begin{aligned} G^2 &= \frac{2m(U_0 - E)}{\hbar^2} \\ &= \frac{2(4)(1.67 \times 10^{-27}\ \mathrm{kg})(27.9\ \mathrm{MeV} - 4.49\ \mathrm{MeV})}{(1.055 \times 10^{-34}\ \mathrm{J \cdot s})^2} \\ &\quad \times (1.60 \times 10^{-13}\ \mathrm{J/MeV}), \end{aligned}$$

which gives $G = 2.1 \times 10^{15}\ \mathrm{m}^{-1} = 2.1\ \mathrm{fm}^{-1}$.

If we choose the potential inside the well to be zero, the kinetic energy of the α particle inside is the same as the kinetic energy of the α particle after the decay. We find the speed inside the well from

$$\begin{aligned} K &= \tfrac{1}{2}mv^2; \\ (4.49\text{ MeV})(1.60\times10^{-13}\text{ J/MeV}) &= \tfrac{1}{2}(4)(1.67\times10^{-27}\text{ kg})v_{\text{in}}^2, \end{aligned}$$

which gives $v_{\text{in}} = 1.47\times10^7$ m/s.

The frequency of striking the barrier is

$$\frac{v_{\text{in}}}{2R_0} = \frac{1.47\times10^7\text{ m/s}}{2(7.37\times10^{-15}\text{ m})} = 9.95\times10^{20}\text{ s}^{-1}.$$

To account for the decrease with r of the Coulomb barrier, we use a square barrier with a thickness less than the width found in Problem 6–34.

(b) If we use a thickness of $\frac{1}{2}$ the width of 50.3 fm, we have

$$\begin{aligned} \lambda &= \frac{v_{\text{in}}}{2R_0}Ce^{-2GL} \\ &= (9.95\times10^{20}\text{ s}^{-1})(2.16)\exp\left(-2(2.1\text{ fm}^{-1})(25.2\text{ fm})\right) \\ &= 2.3\times10^{-25}\text{ s}^{-1}. \end{aligned}$$

Thus the half-life is

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{2.3\times10^{-25}\text{ s}^{-1}} = 2.98\times10^{24}\text{ s} = 9.4\times10^{16}\text{ yr}.$$

(c) If we use a thickness of $\frac{1}{3}$ the width of 50.3 fm, we have

$$\begin{aligned} \lambda &= \frac{v_{\text{in}}}{2R_0}Ce^{-2GL} \\ &= (9.95\times10^{20}\text{ s}^{-1})(2.16)\exp\left(-2(2.1\text{ fm}^{-1})(16.8\text{ fm})\right) \\ &= 4.9\times10^{-10}\text{ s}^{-1}. \end{aligned}$$

Thus the half-life is

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{4.9\times10^{-10}\text{ s}^{-1}} = 1.4\times10^9\text{ s} = 45\text{ yr}.$$

Note that the result is very sensitive to the value used for the barrier thickness.

(d) To find the width that gives us the known half-life, we have

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{(\ln 2) 2R_0 e^{2GL}}{v_{\text{in}} C} ,$$

or

$$\begin{aligned} e^{2GL} &= \frac{v_{\text{in}}}{2R_0} \frac{C}{\ln 2} T_{1/2} \\ &= \frac{(9.95 \times 10^{20}\ \text{s}^{-1})(2.16)}{\ln 2} (2 \times 10^7\ \text{yr})(3.16 \times 10^7\ \text{s/yr}); \end{aligned}$$

or

$$\begin{aligned} 2GL &= 83.6; \\ L &= \frac{83.6}{2\left(2.1\ \text{fm}^{-1}\right)} = 19.9\ \text{fm}, \end{aligned}$$

which is 0.4 of the width found in Problem 6–34.

7 Nuclear Energy. Effects and Uses of Radiation

7.1 Review

A nuclear reaction occurs when two nuclei collide and two or more other nuclei (or particles) are produced. In this process, as in radioactivity, *transmutation* (change) of elements occurs. Nuclear reactions are sometimes written in a shortened form: for example, the reaction

$$\mathrm{n} + {}^{14}_{7}\mathrm{N} \rightarrow {}^{14}_{6}\mathrm{C} + \mathrm{p}$$

is written

$${}^{14}_{7}\mathrm{N}\,(\mathrm{n},\,\mathrm{p})\,{}^{14}_{6}\mathrm{C}.$$

The symbols outside the parentheses on the left and right represent the initial and final nuclei, respectively. The symbols inside the parentheses represent the bombarding particle (first) and the emitted small particle (second).

Example 7–1. A neutron is observed to strike an ${}^{16}_{8}\mathrm{O}$ nucleus and a deuteron is given off. (A *deuteron* is the nucleus of *deuterium*, the isotope of hydrogen containing one proton and one neutron, ${}^{2}_{1}\mathrm{H}$; it is sometimes given the symbol d or D.) What is the nucleus that results?

SOLUTION. We have the reaction

$$\mathrm{n} + {}^{16}_{8}\mathrm{O} \rightarrow ? + {}^{2}_{1}\mathrm{H}.$$

The total number of nucleons initially is $1 + 16 = 17$, and the total charge is $0 + 8 = 8$. The same totals apply to the right side of the reaction. Hence the product nucleus must have $Z = 7$ and $A = 15$. From the periodic table, we find that it is nitrogen that has $Z = 7$, so the nucleus produced is ${}^{15}_{7}\mathrm{N}$. The reaction can be written ${}^{16}_{8}\mathrm{O}\,(\mathrm{n},\,\mathrm{d})\,{}^{15}_{7}\mathrm{N}$, where d represents deuterium, ${}^{2}_{1}\mathrm{H}$.

The *reaction energy* or Q-value of a reaction $\mathrm{a} + \mathrm{X} \rightarrow \mathrm{Y} + \mathrm{b}$, in terms of the masses involved, is

$$Q = (M_\mathrm{a} + M_\mathrm{X} - M_\mathrm{b} - M_\mathrm{Y})c^2. \tag{7.1}$$

Since energy is conserved, Q is equal to the change in kinetic energy:

$$Q = K_\mathrm{b} + K_\mathrm{Y} - K_\mathrm{a} - K_\mathrm{X}. \tag{7.2}$$

Example 7–2. The nuclear reaction

$$\mathrm{n} + {}^{10}_{5}\mathrm{B} \rightarrow {}^{7}_{3}\mathrm{Li} + {}^{4}_{2}\mathrm{He}$$

is observed to occur even when very slow-moving neutrons ($M_\mathrm{n} = 1.0087$ u) strike a boron atom at rest. For a particular reaction in which $K_\mathrm{n} \approx 0$, the helium ($M_\mathrm{He} = 4.0026$ u) is observed to have a speed of 9.30×10^6 m/s. Determine (a) the kinetic energy of the lithium ($M_\mathrm{Li} = 7.0160$ u), and (b) the Q-value of the reaction.

SOLUTION. (a) Since the neutron and boron are both essentially at rest, the total momentum before the reaction is zero, and afterward is also zero. Therefore,

$$M_\mathrm{Li} v_\mathrm{Li} = M_\mathrm{He} v_\mathrm{He}.$$

We solve this for v_Li and substitute it into the equation for kinetic energy. We can use classical kinetic energy with little error, rather than relativistic formulas, because $v_\mathrm{He} = 9.30 \times 10^6$ m/s is not close to the speed of light c, and v_Li will be even less since $M_\mathrm{Li} > M_\mathrm{He}$. Thus we can write:

$$K_\mathrm{Li} = \frac{1}{2} M_\mathrm{Li} v_\mathrm{Li}^2 = \frac{1}{2} M_\mathrm{Li} \left(\frac{M_\mathrm{He} v_\mathrm{He}}{M_\mathrm{Li}} \right)^2 = \frac{M_\mathrm{He}^2 v_\mathrm{He}^2}{2 M_\mathrm{Li}}.$$

We put in numbers, changing the mass in u to kg and recalling that 1.60×10^{-13} J = 1 MeV:

$$\begin{aligned} K_\mathrm{Li} &= \frac{(4.0026 \text{ u})^2 (1.66 \times 10^{-27} \text{ kg/u})^2 (9.30 \times 10^6 \text{ m/s})^2}{2(7.0160 \text{ u})(1.66 \times 10^{-27} \text{ kg/u})} \\ &= 1.64 \times 10^{-13} \text{ J} = 1.02 \text{ MeV}. \end{aligned}$$

(b) We are given the data $K_\mathrm{a} = K_\mathrm{X} = 0$ in Eq. 7.2, so $Q = K_\mathrm{Li} + K_\mathrm{He}$, where

$$\begin{aligned} K_\mathrm{He} &= \tfrac{1}{2} M_\mathrm{He} v_\mathrm{He}^2 \\ &= \tfrac{1}{2} (4.0026 \text{ u})(1.66 \times 10^{-27} \text{ kg/u})(9.30 \times 10^6 \text{ m/s})^2 \\ &= 2.87 \times 10^{-13} \text{ J} = 1.80 \text{ MeV}. \end{aligned}$$

Hence, $Q = 1.02 \text{ MeV} + 1.80 \text{ MeV} = 2.82 \text{ MeV}$.

Example 7–3. Can the reaction ${}^{13}_{6}\mathrm{C}\,(\mathrm{p},\,\mathrm{n})\,{}^{13}_{7}\mathrm{N}$ occur when ${}^{13}_{6}\mathrm{C}$ is bombarded by 2.0-MeV protons?

SOLUTION. We use Eq. 7.1, looking up the masses of the nuclei in Appendix C. The total masses before and after the reaction are:

Before			After		
$M({}^{13}_{6}\mathrm{C})$	=	13.003355	$M({}^{13}_{7}\mathrm{N})$	=	13.005739
$M({}^{1}_{1}\mathrm{H})$	=	1.007825	$M(\mathrm{n})$	=	1.008665
		14.011180			14.014404

(We must use the mass of the ${}^{1}_{1}\mathrm{H}$ atom rather than that of the bare proton because the masses of ${}^{13}_{6}\mathrm{C}$ and ${}^{13}_{7}\mathrm{N}$ include the electrons, and we must include an equal number of electrons on each side of the equation since none are created or destroyed.) The products have an excess mass of $(14.014404 - 14.011180)\,\mathrm{u} = 0.003224\ \mathrm{u} \times 931.5\ \mathrm{MeV/u} = 3.00\ \mathrm{MeV}$. Thus $Q = -3.00$ MeV, and the reaction is endoergic. This reaction requires energy, and the 2.0 MeV protons do not have enough to make it go.

The *effective cross section* for a reaction is a measure of the reaction probability per target nucleus:

$$\sigma = \frac{R}{R_0 n t}, \tag{7.3}$$

where R is the number of projectile particles that are deflected per second, R_0 is the rate at which the particles strike the target (numbers/second), n is the number of nuclei per unit volume, and t is the thickness of a target. Nuclear cross sections are measured in barns: 1 barn (bn) $= 10^{-28}\ \mathrm{m}^2$.

Fig. 7.1 shows the cross section for neutron capture in cadmium ($\mathrm{n} + {}^{113}_{48}\mathrm{Cd} \rightarrow {}^{114}_{48}\mathrm{Cd} + \gamma$) as a function of neutron kinetic energy. Neutron cross sections for most materials are greater at low energies, as in Fig. 7.1.

Example 7–4. The reaction

$$\mathrm{p} + {}^{56}_{26}\mathrm{Fe} \rightarrow {}^{56}_{27}\mathrm{Co} + \mathrm{n}$$

has a cross section of 0.65 bn for a particular incident proton energy. Suppose the iron target has an area of $1.5\ \mathrm{cm}^2$, and is $2.0\ \mu\mathrm{m}$ thick. The density of iron is $7.8 \times 10^3\ \mathrm{kg/m}^3$. If the protons are incident at a rate of 2.0×10^{13} particles/s, calculate the rate at which neutrons are produced.

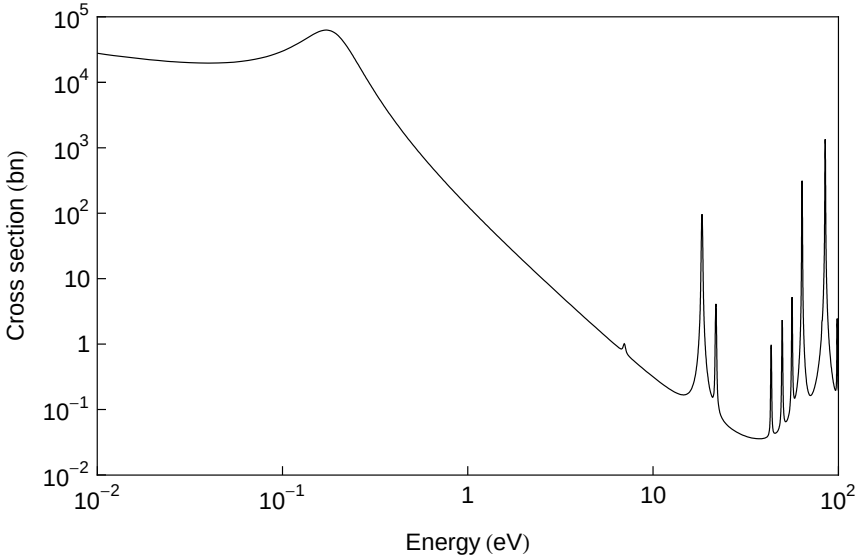


Figure 7.1. The neutron cross section for cadmium-113, which is extraordinary large for thermal neutrons. Note that both scales are logarithmic.

Solution. We use Eq. 7.3 in the form:

$$R = R_0 n t \sigma.$$

Here $R_0 = 2.0 \times 10^{13}$ particles/s, $t = 2.0 \times 10^{-6}$ m, and $\sigma = 0.65$ bn. Recalling that one mole (mass = 56 g for iron) contains 6.02×10^{23} molecules, then the number of iron atoms per unit volume is

$$\begin{aligned} n &= \frac{(7.8 \times 10^3 \text{ kg/m}^3)(6.02 \times 10^{23} \text{ atoms/mole})}{56 \times 10^{-3} \text{ kg/mole}} \\ &= 8.4 \times 10^{28} \text{ atoms/m}^3. \end{aligned}$$

Then we find that the rate at which neutrons are produced is

$$\begin{aligned} R &= (2.0 \times 10^{13} \text{ particles/s})(8.4 \times 10^{28} \text{ atoms/m}^3)(2.0 \times 10^{-6} \text{ m}) \\ &\quad \times (0.65 \times 10^{-28} \text{ m}^2) \\ &= 2.2 \times 10^{8} \text{ particles/s}. \end{aligned}$$

In *fission* a heavy nucleus such as uranium splits into two intermediate-sized nuclei after being struck by a neutron:

$$\text{n} + {}^{235}_{92}\text{U} \rightarrow {}^{236}_{92}\text{U} \rightarrow \text{N}_1 + \text{N}_2 + \text{neutrons}. \tag{7.4}$$

A typical fission is

$$\text{n} + {}^{235}_{92}\text{U} \rightarrow {}^{141}_{56}\text{Ba} + {}^{92}_{36}\text{Kr} + 3\text{n}, \tag{7.5}$$

although many others also occur.

${}^{235}_{92}\text{U}$ is fissionable by slow neutrons, whereas some fissionable nuclei require fast neutrons. Much energy is released in fission because the binding energy per nucleon is lower for heavy nuclei than it is for intermediate-sized nuclei, so the mass of a heavy nucleus is greater than the total mass of its fission products.

The fission process releases neutrons, so that a *chain reaction* is possible. The *critical mass* is the minimum mass of fuel needed to sustain a chain reaction. In a *nuclear reactor* or nuclear bomb, a moderator is needed to slow down the released neutrons.

Example 7–5. Estimate the minimum amount of ${}^{235}_{92}\text{U}$ that needs to undergo fission in order to run a 1 000-MW power reactor per year of continuous operation. Assume an efficiency of about 33 percent.

SOLUTION. For 1 000 MW output, the total power generation is 3 000 MW, of which 2 000 MW is dumped as "waste" heat. Thus the total energy release in 1 yr (3×10^7 s) from fission is about

$$(3 \times 10^9 \text{ J/s})(3 \times 10^7 \text{ s}) \approx 10^{17} \text{ J}.$$

If each fission releases 200 MeV of energy, the number of fissions required is

$$\frac{10^{17} \text{ J}}{(2 \times 10^8 \text{ eV/fission})(1.6 \times 10^{-19} \text{ J/eV})} \approx 3 \times 10^{27} \text{ fissions}.$$

The mass of a single uranium atom is about (235 u)$(1.66\times10^{-27}$ kg/u) $\approx 4\times10^{-25}$ kg, so the total mass needed is$(4\times10^{-25}$ kg$)(3\times10^{27}$ fissions) $\approx 1\,000$ kg or about a ton. Since ${}^{235}_{92}$U is only a fraction of normal uranium, and even when enriched it is never more than 10 percent of the total, the yearly requirement for uranium is on the order of tens of tons. This is orders of magnitude less than coal, both in mass and volume.

The *fusion* process, in which small nuclei combine to form larger ones, also releases energy.

Example 7–6. One of the simplest fusion reactions involves the production of deuterium, ${}^{2}_{1}$H, from a neutron and a proton: ${}^{1}_{1}\mathrm{H}+\mathrm{n}\rightarrow{}^{2}_{1}\mathrm{H}+\gamma$. How much energy is released in this reaction?

Solution. From Appendix C, the initial rest mass is

$$1.007825\text{ u}+1.008665\text{ u}=2.016490\text{ u},$$

and after the reaction the mass is that of the ${}^{2}_{1}$H, namely 2.014102 u. The mass difference is 0.002388 u, so the energy released is

$$(0.002388\text{ u})(931.5\text{ MeV/u})=2.22\text{ MeV},$$

and it is carried off by the ${}^{2}_{1}$H nucleus and the γ ray.

The energy from our Sun is believed to originate in the fusion reactions

$$\begin{array}{lllll}
\text{(a)} & {}^{1}_{1}\mathrm{H}+{}^{1}_{1}\mathrm{H} & \rightarrow & {}^{2}_{1}\mathrm{H}+\mathrm{e}^{+}+\nu & (0.42\text{ MeV}) \\
\text{(b)} & {}^{1}_{1}\mathrm{H}+{}^{2}_{1}\mathrm{H} & \rightarrow & {}^{3}_{2}\mathrm{He}+\gamma & (5.49\text{ MeV}) \\
\text{(c)} & {}^{3}_{2}\mathrm{He}+{}^{3}_{2}\mathrm{He} & \rightarrow & {}^{4}_{2}\mathrm{He}+{}^{1}_{1}\mathrm{H}+{}^{1}_{1}\mathrm{H} & (12.86\text{ MeV})
\end{array} \tag{7.6}$$

known as the *proton-proton cycle* in which four protons fuse to form a ${}^{4}_{2}$He nucleus:

$$4\,{}^{1}_{1}\mathrm{H}\rightarrow{}^{4}_{2}\mathrm{He}+2\mathrm{e}^{+}+2\nu+2\gamma. \tag{7.7}$$

Note that it takes two of each of the first two reactions (Eqs. 7.6a and b) to produce the two ${}^{3}_{2}$He for the third reaction, so the total energy release for the net reaction, Eq. 7.7, is

$$2\times0.42\text{ MeV}+2\times5.49\text{ MeV}+12.86\text{ MeV}=24.7\text{ MeV}.$$

However, each of the two e^{+} (formed in reaction Eq. 7.6a) quickly annihilates with an electron to produce $2m_{\mathrm{e}}c^{2}=1.02$ MeV, so the total energy released is $(24.7\text{ MeV}+2\times1.02\text{ MeV})=26.7$ MeV.

In stars hotter than the Sun, it is more likely that the energy output comes principally from the *carbon cycle*, which comprises the following sequence of reactions (see Problem 7–38):

$$\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{13}_{7}\mathrm{N} + \gamma \\
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + \mathrm{e}^{+} + \nu \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + \gamma \\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + \gamma \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + \mathrm{e}^{+} + \nu \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}.
\end{aligned} \tag{7.8}$$

Example 7–7. What is the heaviest element likely to be produced in fusion processes in stars?

SOLUTION. Fusion is possible as long as the final product has more binding energy (less mass) than the reactants, for then there is net release of energy. Since the binding energy curve in Fig. 6.1 peaks near $A \approx 56$ to 58, which corresponds to iron or nickel, it would not be energetically favorable to produce elements heavier than that. Nevertheless, in the center of massive stars or in supernova explosions, there is enough energy available to drive endothermic reactions that produce heavier elements, as well.

The fusion reactions most likely to succeed in a reactor involve the isotopes of hydrogen, ${}^{2}_{1}\mathrm{H}$ (deuterium) and ${}^{3}_{1}\mathrm{H}$ (tritium), and are as follows, with the energy released given in parentheses:

$$\begin{aligned}
&\text{(a)} \quad {}^{2}_{1}\mathrm{H} + {}^{2}_{1}\mathrm{H} \rightarrow {}^{3}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \quad (4.03\ \mathrm{MeV}) \\
&\text{(b)} \quad {}^{2}_{1}\mathrm{H} + {}^{2}_{1}\mathrm{H} \rightarrow {}^{3}_{2}\mathrm{He} + \mathrm{n} \quad (3.27\ \mathrm{MeV}) \\
&\text{(c)} \quad {}^{2}_{1}\mathrm{H} + {}^{3}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + \mathrm{n} \quad (17.59\ \mathrm{MeV})
\end{aligned} \tag{7.9}$$

Comparing these energy yields with that for the fission of ${}^{235}_{92}\mathrm{U}$, we can see that the energy released in fusion reactions can be greater for a given mass of fuel than in fusion.

All nuclei have a positive charge and thus repel each other. However, if they can be brought close enough together so that the short-page attractive nuclear force can come into play, the latter can pull the nuclei together and fusion will occur. Thus, in order for the nuclei to get close

enough together, they must have rather large kinetic energy to overcome the electric repulsion. High kinetic energies are easily attainable with particle accelerators, but the number of particles involved is too small. To produce realistic amounts of energy, we must deal with matter in bulk, for which high kinetic energy means higher temperatures. Indeed, very high temperatures required for fusion to occur, and fusion devices are often referred to as *thermonuclear devices*.

Example 7–8. Estimate the temperature required for deuterium-tritium fusion (d-t) to occur.

SOLUTION. We assume the nuclei approach head-on, each with kinetic energy K, and that the nuclear force comes into play when the distance between their centers equals the sum of their radii as given by Eq. 6.1, namely $r_\mathrm{d} \approx 1.5$ fm and $r_\mathrm{t} \approx 1.7$ fm. The electrostatic potential energy of the two particles at this distance must equal the total kinetic energy of the two particles when far apart:

$$\begin{aligned} 2K &\approx k_\mathrm{e}\frac{e^2}{r_\mathrm{d}+r_\mathrm{t}} \approx \frac{(9.0\times10^9\ \mathrm{N\cdot m^2/C^2})(1.6\times10^{-19}\ \mathrm{C})^2}{(3.2\times10^{-15}\ \mathrm{m})(1.6\times10^{-19}\ \mathrm{J/eV})} \\ &\approx 0.45\ \mathrm{MeV}. \end{aligned}$$

Thus, $K \approx 0.22$ MeV, and if we ask that the average kinetic energy be this high, then from equation $\frac{3}{2}k_\mathrm{B}T = \overline{K}$, we have

$$T = \frac{2K}{3k_\mathrm{B}} = \frac{2(0.22\ \mathrm{MeV})(1.6\times10^{-13}\ \mathrm{J/MeV})}{3(1.38\times10^{-23}\ \mathrm{J/K})} \approx 2\times10^9\ \mathrm{K}.$$

A working fusion reactor for power generation has not yet proved possible because of the difficulty in containing the fuel (e.g., deuterium) long enough at the high temperature required. Nonetheless, great progress has been made in confining the collection of charged ions known as a *plasma*. The two main methods are *magnetic confinement*, using a magnetic field in a device such as the toroidal-shaped *tokamak*, and *inertial confinement*, in which intense laser beams compress a fuel pellet of deuterium and tritium.

In 1957, J. D. Lawson showed that the product of ion density n and confinement time T must satisfy, at a minimum, approximately

$$n\tau \geqslant 3\times10^{20}\ \mathrm{s/m^3}. \tag{7.10}$$

This Lawson criterion must be reached to produce *ignition*, by which we mean a self-sustaining thermonuclear "burn" that continues after all external heating is turned off.

Radiation can cause damage to materials, including biological tissue. Quantifying amounts of radiation is the subject of *dosimetry*. The *curie* (Ci) and the *becquerel* (Bq) are units that measure the source activity or rate of decay of a sample:

$$1 \text{ Ci} = 3.70 \times 10^{10} \text{ disintegrations/s},$$

whereas

$$1 \text{ Bq} = 1 \text{ disintegration/s}.$$

Example 7–9. In a certain experiment, 0.016 μCi of $^{32}_{15}$P is injected into a medium containing a culture of bacteria. After 1 h, the cells are washed and a detector that is 70 percent efficient (counts 70 percent of emitted β particles) records 720 counts per minute from all the cells. What percentage of the original $^{32}_{15}$P was taken up by the cells?

SOLUTION. The halflife of $^{32}_{15}$P being about 14 days, the loss of activity over the 1 hour can be ignored. The total number of disintegrations per second originally was $(0.016 \times 10^{-6})(3.7 \times 10^{10}) = 590$. The counter could be expected to count 70 percent of this, or 410 per second. Since it counted $720/60 = 12$ per second, then $12/410 = 0.029$ or 2.9 percent was incorporated into the cells.

The *absorbed dose*, often specified in *rads*, measures the amount of energy deposited per unit mass of absorbing material: 1 rad is the amount of radiation that deposits energy at the rate of 10^{-2} J/kg of material. (The earliest unit of dosage was the *roentgen* (R), which was defined in terms of the amount of ionization produced by the radiation: 1.6×10^{12} ion pairs per gram of dry air at standard conditions. Today, 1 R is defined as the amount of X- or γ radiation that deposits 0.878×10^{-2} J of energy per kilogram of air. This is quite close to the rad for X- and γ rays.)

The SI unit of absorbed dose is the *gray*:

$$1 \text{ Gy} = 1 \text{ J/kg} = 100 \text{ rad}. \tag{7.11}$$

The *effective dose* is often specified by the *rem* (rad equivalent man):

$$\text{effective dose (in rem)} = \text{dose (in rad)} \times \text{QF}, \tag{7.12}$$

where QF is the "quality factor" of a given type of radiation; 1 rem of any type of radiation does approximately the same amount of biological damage. The SI unit for effective dose is the *sievert* (Sv):

$$\text{effective dose (Sv)} = \text{dose (Gy)} \times \text{QF} = 10^2 \text{ effective dose (rem)}. \quad (7.13)$$

Example 7–10. What whole-body dose is received by a 70-kg laboratory worker exposed to a 40 mCi $^{60}_{27}$Co source, assuming the person's body has cross-sectional area 1.5 m^2 and is, on average, 4.0 m from the source for 4.0 h per day? $^{60}_{27}$Co emits γ rays of energy 1.33 MeV and 1.17 MeV in quick succession. Approximately 50 percent of the γ rays interact in the body and deposit all their energy. (The rest pass through.)

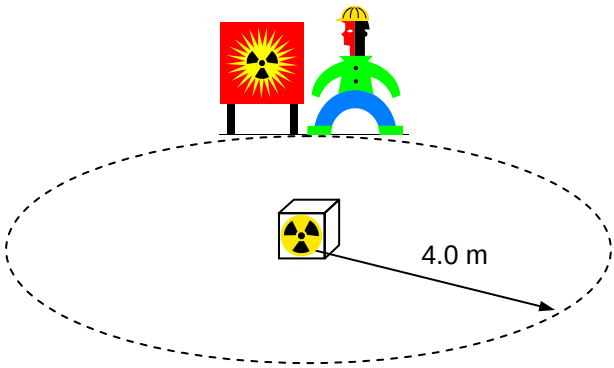


Figure 7.2. Example 7–10. Radiation spreads out in all directions. A person 4.0 m away intercepts only a fraction: his cross-sectional area divided by the area of a sphere of radius 4.0 m.

Solution. The total γ-ray energy per decay is $(1.33 + 1.17)$ MeV $= 2.50$ MeV, so the total energy emitted by the source is

$$(0.040 \text{ Ci})(3.7 \times 10^{10} \text{ decays/Ci} \cdot \text{s})(2.50 \text{ MeV}) = 3.7 \times 10^9 \text{ MeV/s}.$$

The proportion of this intercepted by the body is its 1.5 m^2 area divided by the area of a sphere of radius 4.0 m (Fig. 7.2)

$$\frac{1.5 \text{ m}^2}{4\pi r^2} = \frac{1.5 \text{ m}^2}{4\pi(4.0 \text{ m})^2} = 7.5 \times 10^{-3}.$$

So the rate energy is deposited in the body (remembering that only $\frac{1}{2}$ of the γ rays interact in the body) is

$$\begin{aligned} E &= \tfrac{1}{2}(7.5\times10^{-3})(3.7\times10^{9}\ \text{MeV/s})(1.6\times10^{-13}\ \text{J/MeV}) \\ &= 2.2\times10^{-6}\ \text{J/s}. \end{aligned}$$

Since 1 Gy = 1 J/kg, the whole-body dose rate for this 70-kg person is $(2.2\times10^{-6}\ \text{J/s})/(70\ \text{kg}) = 3.1\times10^{-8}$ Gy/s. In the space of 4.0 h, this amounts to a dose of $(4.0\ \text{h})(3\,600\ \text{s/h})(3.1\times10^{-8}\ \text{Gy/s}) = 4.5\times10^{-4}$ Gy. Since QF ≈ 1 for gammas, the effective dose (Eq. 7.13) is 450 μSv or (see Eq. 7.11):

$$(100\ \text{rad/Gy})(4.5\times10^{-4}\ \text{Gy})(1) = 45\ \text{mrem}.$$

This 45-mrem effective dose is nearly 10 percent of the normal allowed dose for a whole year (500 mrem/yr), or 1 percent of the yearly allowance for radiation workers. This worker should not receive such a dose every day and should seek ways to reduce it (shield the source, vary the work, work farther away, etc.)

Nuclear magnetic resonance (NMR) is a phenomenon which soon after its discovery in 1946 became a powerful research tool in a variety of fields from physics to chemistry and biochemistry. It is also an important medical imaging technique.

Spin angular momentum (and the associated magnetic moment) of the hydrogen ($^1_1$H) nucleus (which consists of a single proton) can take on only two values when placed in a magnetic field: spin up (parallel to the field) and spin down (antiparallel to the field) as suggested in Fig. 7.3. When a magnetic field is present, an energy state splits into two levels as shown in Fig. 7.4 with the spin-up state (parallel to field) having the lower energy. The spin-down state acquires an additional energy μB_{T} and the spin-up state $-\mu B_{\text{T}}$, where B_{T} is the total magnetic field at the nucleus. The difference in energy between the two states (Fig. 7.4) is thus

$$\Delta E = 2\mu_{\text{p}} B_{\text{T}},$$

where μ_{p} is the magnetic moment of the proton.

In a standard NMR setup, the sample to be examined is placed in a static magnetic field. A radiofrequency (RF) pulse of electromagnetic

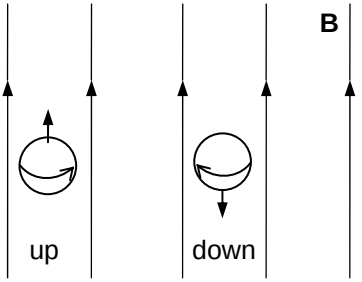


Figure 7.3. A proton in a magnetic field **B** with two possible states of spin, up and down.

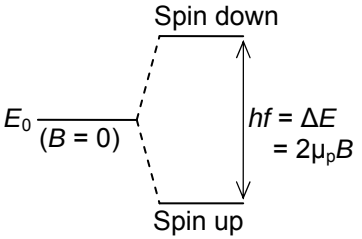


Figure 7.4. Energy E_0 in the absence of a magnetic field splits into two levels in the presence of a magnetic field.

radiation (that is, photons) is applied to the sample. If the frequency, f, of this pulse corresponds precisely to the energy difference between the two energy levels (Fig. 7.4), so that

$$2\pi\hbar f = \Delta E = 2\mu_{\mathrm{p}} B_{\mathrm{T}} \tag{7.14}$$

then the photons of the RF beam will be absorbed, exciting many of the nuclei from the lower state to the upper state. This is a resonance phenomenon since there is significant absorption only if f is very near $f = \mu_{\mathrm{p}} B_{\mathrm{T}}/\pi\hbar$.

Example 7–11. Calculate the resonant frequency for free protons in a 1.000-T magnetic field.

Solution. The magnetic moment of the proton is

$$\mu_{\mathrm{p}} = 2.7928\mu_{\mathrm{N}} = 2.7928\left(\frac{e\hbar}{2m_{\mathrm{p}}}\right).$$

Then

$$\begin{aligned} f &= \frac{\Delta E}{2\pi\hbar} = \frac{\mu_{\mathrm{p}} B}{\pi\hbar} = (2.7928)\left(\frac{eB}{2\pi m_{\mathrm{p}}}\right) \\ &= (2.7928)\,\frac{(1.6022\times10^{-19}\ \mathrm{C})(1.000\ \mathrm{T})}{2\pi(1.6726\times10^{-27}\ \mathrm{kg})} = 42.58\ \mathrm{MHz}. \end{aligned}$$

7.2 Nuclear reactions. Transmutation of elements

7–1. Natural aluminum is all $^{27}_{13}\mathrm{Al}$. If it absorbs a neutron, what does it become? Does it decay by β^+ or β^-? What will be the product nucleus?

SOLUTION. We find the product nucleus by balancing the mass and charge numbers:

$$\begin{aligned} Z(\mathrm{X}) &= Z(^{27}\mathrm{Al}) + Z(\mathrm{n}) = 13 + 0 = 13; \\ A(\mathrm{X}) &= A(^{27}\mathrm{Al}) + A(\mathrm{n}) = 27 + 1 = 28, \end{aligned}$$

so the product nucleus is $^{28}_{13}\mathrm{Al}$.

If $^{28}_{13}\mathrm{Al}$ were a β^+ emitter, the resulting nucleus would be $^{28}_{12}\mathrm{Mg}$, which has too many neutrons relative to the number of protons to be stable. Thus we have a β^- emitter.

The decay is

$$^{28}_{13}\mathrm{Al} \rightarrow {}^{28}_{14}\mathrm{Si} + {}^{0}_{-1}\mathrm{e} + \tilde{\nu},$$

so the product is $^{28}_{14}\mathrm{Si}$.

7–2. Determine whether the reaction $^{2}_{1}\mathrm{H}\,(\mathrm{d},\,\mathrm{n})\,^{3}_{2}\mathrm{He}$ requires a threshold energy. (d stands for deuterium, $^{2}_{1}\mathrm{H}$.)

SOLUTION. For the reaction $^{2}_{1}\mathrm{H}\,(\mathrm{d},\mathrm{n})\,^{3}_{2}\mathrm{He}$, we find the difference of the initial and the final masses:

$$\begin{aligned} \Delta M &= M(^2\mathrm{H}) + M(^2\mathrm{H}) - m(\mathrm{n}) - M(^3\mathrm{He}) \\ &= 2(2.014102\ \mathrm{u}) - (1.008665\ \mathrm{u}) - (3.016029\ \mathrm{u}) = +0.003510\ \mathrm{u}. \end{aligned}$$

Thus no threshold energy is required.

7–3. Is the reaction $^{238}_{92}\mathrm{U}\,(\mathrm{n},\,\gamma)\,^{239}_{92}\mathrm{U}$ possible with slow neutrons? Explain.

SOLUTION. For the reaction $^{238}_{92}\mathrm{U}\,(\mathrm{n},\,\gamma)\,^{239}_{92}\mathrm{U}$ with slow neutrons, whose kinetic energy is negligible, we find the difference of the initial and the final masses:

$$\begin{aligned} \Delta M &= M(^{238}\mathrm{U}) + m(\mathrm{n}) - M(^{239}\mathrm{U}) \\ &= (238.050782\ \mathrm{u}) + (1.008665\ \mathrm{u}) - (239.054287\ \mathrm{u}) \\ &= +0.005160\ \mathrm{u}. \end{aligned}$$

Thus no threshold energy is required, so the reaction is possible.

7–4. Does the reaction ${}^{7}_{3}\mathrm{Li}\,(\mathrm{p},\,\alpha)\,{}^{4}_{2}\mathrm{He}$ require energy or does it release energy? How much energy?

SOLUTION. For the reaction ${}^{7}_{3}\mathrm{Li}\,(\mathrm{p},\,\alpha)\,{}^{4}_{2}\mathrm{He}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{7}\mathrm{Li}) + M({}^{1}\mathrm{H}) - M({}^{4}\mathrm{He}) - M({}^{4}\mathrm{He})\right) c^2 \\ &= ((7.016004\ \mathrm{u}) + (1.007825\ \mathrm{u}) - 2(4.002603\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +17.35\ \mathrm{MeV}. \end{aligned}$$

Thus 17.35 MeV is released.

7–5. Calculate the energy released (or energy input required) for the reaction ${}^{9}_{4}\mathrm{Be}\,(\alpha,\,\mathrm{n})\,{}^{12}_{6}\mathrm{C}$.

SOLUTION. For the reaction ${}^{9}_{4}\mathrm{Be}\,(\alpha,\,\mathrm{n})\,{}^{12}_{6}\mathrm{C}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{9}\mathrm{Be}) + M({}^{4}\mathrm{He}) - m(\mathrm{n}) - M({}^{12}\mathrm{C})\right) c^2 \\ &= ((9.012182\ \mathrm{u}) + (4.002603\ \mathrm{u}) - (1.008665\ \mathrm{u}) - (12.000000\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +5.701\ \mathrm{MeV}. \end{aligned}$$

Thus 5.701 MeV is released.

7–6. (a) Can the reaction ${}^{24}_{12}\mathrm{Mg}\,(\mathrm{n},\,\mathrm{d})\,{}^{23}_{11}\mathrm{Na}$ occur if the bombarding particles have 10.00 MeV of kinetic energy? (b) If so, how much energy is released?

SOLUTION. (a) For the reaction ${}^{24}_{12}\mathrm{Mg}\,(\mathrm{n},\,\mathrm{d})\,{}^{23}_{11}\mathrm{Na}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{24}\mathrm{Mg}) + m(\mathrm{n}) - M({}^{2}\mathrm{H}) - M({}^{23}\mathrm{Na})\right) c^2 \\ &= ((23.985042\ \mathrm{u}) + (1.008665\ \mathrm{u}) - (2.014102\ \mathrm{u}) - (22.989770\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= -9.469\ \mathrm{MeV}. \end{aligned}$$

Because $K + Q > 0$, the reaction can occur.

(b) The energy released is

$$K + Q = 10.00\ \mathrm{MeV} - 9.469\ \mathrm{MeV} = 0.53\ \mathrm{MeV}.$$

7–7. (a) Can the reaction ${}^{7}_{3}\mathrm{Li}\,(\mathrm{p},\,\alpha)\,{}^{4}_{2}\mathrm{He}$ occur if the incident proton has kinetic energy = 2 500 keV? (b) If so, what is the total kinetic energy of the products?

SOLUTION. (a) For the reaction ${}^{7}_{3}\mathrm{Li}\,(\mathrm{p},\,\alpha)\,{}^{4}_{2}\mathrm{He}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{7}\mathrm{Li}) + M({}^{1}\mathrm{H}) - M({}^{4}\mathrm{He}) - M({}^{4}\mathrm{He})\right) c^2 \\ &= ((7.016004\ \mathrm{u}) + (1.007825\ \mathrm{u}) - 2(4.002603\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +17.35\ \mathrm{MeV}. \end{aligned}$$

Because $Q > 0$, the reaction can occur.

(b) The kinetic energy of the products is

$$K = K_{\mathrm{i}} + Q = 2.500\ \mathrm{MeV} + 17.35\ \mathrm{MeV} = 19.85\ \mathrm{MeV}.$$

7–8. In the reaction ${}^{14}_{7}\mathrm{N}\,(\alpha,\,\mathrm{p})\,{}^{17}_{8}\mathrm{O}$, the incident α particles have 7.68 MeV of kinetic energy. (a) Can this reaction occur? (b) If so, what is the total kinetic energy of the products? The mass of ${}^{17}_{8}\mathrm{O}$ is 16.999131 u.

SOLUTION. (a) For the reaction ${}^{14}_{7}\mathrm{N}\,(\alpha,\,\mathrm{p})\,{}^{17}_{8}\mathrm{O}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{14}\mathrm{N}) + M({}^{4}\mathrm{He}) - M({}^{1}\mathrm{H}) - M({}^{17}\mathrm{O})\right) c^2 \\ &= ((14.003074\ \mathrm{u}) + (4.002603\ \mathrm{u}) - (1.007825\ \mathrm{u}) - (16.999131\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= -1.191\ \mathrm{MeV}. \end{aligned}$$

Because $K + Q > 0$, the reaction can occur.

(b) The kinetic energy of the products is

$$K = K_{\mathrm{i}} + Q = 7.68\ \mathrm{MeV} - 1.191\ \mathrm{MeV} = 6.49\ \mathrm{MeV}.$$

7–9. Calculate the Q-value for the "capture" reaction ${}^{16}_{8}\mathrm{O}\,(\alpha,\,\gamma)\,{}^{20}_{10}\mathrm{Ne}$.

SOLUTION. For the reaction ${}^{16}_{8}\mathrm{O}\,(\alpha,\,\gamma)\,{}^{20}_{10}\mathrm{Ne}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{16}\mathrm{O}) + M({}^{4}\mathrm{He}) - M({}^{20}\mathrm{Ne})\right) c^2 \\ &= ((15.994915\ \mathrm{u}) + (4.002603\ \mathrm{u}) - (19.992440\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +4.730\ \mathrm{MeV}. \end{aligned}$$

7–10. Calculate the total kinetic energy of the products of the reaction ${}^{13}_{6}\mathrm{C}\,(\mathrm{d},\,\mathrm{n})\,{}^{14}_{7}\mathrm{N}$ if the incoming deuteron (d) has $K = 36.3$ MeV.

SOLUTION. For the reaction ${}^{13}_{6}\mathrm{C}\,(\mathrm{d},\,\mathrm{n})\,{}^{14}_{7}\mathrm{N}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{13}\mathrm{C}) + M({}^{2}\mathrm{H}) - m(\mathrm{n}) - M({}^{14}\mathrm{N})\right) c^2 \\ &= ((13.003355\ \mathrm{u}) + (2.014102\ \mathrm{u}) - (1.008665\ \mathrm{u}) - (14.003074\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +5.326\ \mathrm{MeV}. \end{aligned}$$

The kinetic energy of the products is

$$K = K_{\mathrm{i}} + Q = 36.3\ \mathrm{MeV} + 5.326\ \mathrm{MeV} = 41.6\ \mathrm{MeV}.$$

7–11. An example of a "stripping" nuclear reaction is ${}^{6}_{3}\mathrm{Li}\,(\mathrm{d},\,\mathrm{p})\,\mathrm{X}$. (a) What is X, the resulting nucleus? (b) Why is it called a "stripping" reaction? (c) What is the Q-value of this reaction? Is the reaction endothermic or exothermic?

SOLUTION. (a) We find the product nucleus by balancing the mass and charge numbers:

$$\begin{aligned} Z(\mathrm{X}) &= Z({}^{6}\mathrm{Li}) + Z({}^{2}\mathrm{H}) - Z({}^{1}\mathrm{H}) = 3 + 1 - 1 = 3; \\ A(\mathrm{X}) &= A({}^{6}\mathrm{Li}) + A({}^{2}\mathrm{H}) - A({}^{1}\mathrm{H}) = 6 + 2 - 1 = 7, \end{aligned}$$

so the product nucleus is ${}^{7}_{3}\mathrm{Li}$.

(b) It is a "stripping" reaction because a neutron is stripped from the deuteron.

(c) For the reaction ${}^{6}_{3}\mathrm{Li}\,(\mathrm{d},\mathrm{p})\,{}^{7}_{3}\mathrm{Li}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{6}\mathrm{Li}) + M({}^{2}\mathrm{H}) - M({}^{1}\mathrm{H}) - M({}^{7}\mathrm{Li})\right) c^2 \\ &= (6.015122\ \mathrm{u}) + (2.014102\ \mathrm{u}) - (1.007825\ \mathrm{u}) - (7.016004\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +5.025\ \mathrm{MeV}. \end{aligned}$$

Because $Q > 0$, the reaction is exothermic.

7–12. An example of a "pick-up" nuclear reaction is ${}^{12}_{6}\mathrm{C}\,({}^{3}_{2}\mathrm{He},\,\alpha)\,\mathrm{X}$. (a) Why is it called a "pickup" reaction? (b) What is the resulting nucleus?

(c) What is the Q-value of this reaction? Is the reaction endothermic or exothermic?

SOLUTION. (a) It is a "pickup" reaction because the ^{3}He picks up a neutron.

(b) We find the product nucleus by balancing the mass and charge numbers:

$$\begin{aligned} Z(X) &= Z(^{12}\mathrm{C}) + Z(^3\mathrm{He}) - Z(^4\mathrm{He}) = 6 + 2 - 2 = 6; \\ A(X) &= A(^{12}\mathrm{C}) + A(^3\mathrm{He}) - A(^4\mathrm{He}) = 12 + 3 - 4 = 11, \end{aligned}$$

so the product nucleus is $^{11}_{6}\mathrm{C}$.

(c) For the reaction $^{12}_{6}\mathrm{C}\,(^{3}_{2}\mathrm{He},\,\alpha)\,^{11}_{6}\mathrm{C}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{12}\mathrm{C}) + M(^3\mathrm{He}) - M(^4\mathrm{He}) - M(^{11}\mathrm{C})\right)c^2 \\ &= ((12.000000\ \mathrm{u}) + (3.016029\ \mathrm{u}) - (4.002603\ \mathrm{u}) - (11.011434\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) = +1.856\ \mathrm{MeV}. \end{aligned}$$

Because $Q > 0$, the reaction is exothermic.

7–13. (a) Complete the following nuclear reaction: $?\,(\mathrm{p},\,\gamma)\,^{32}_{16}\mathrm{S}$.
(b) What is the Q-value?

SOLUTION. (a) We find the initial nucleus by balancing the mass and charge numbers:

$$\begin{aligned} Z(\mathrm{X}) &= Z(^{32}\mathrm{S}) - Z(^1\mathrm{H}) = 16 - 1 = 15; \\ A(\mathrm{X}) &= A(^{32}\mathrm{S}) - A(^1\mathrm{H}) = 32 - 1 = 31, \end{aligned}$$

so the initial nucleus is $^{31}_{15}\mathrm{P}$. The reaction is $^{31}_{15}\mathrm{P}\,(\mathrm{p},\,\gamma)\,^{32}_{16}\mathrm{S}$.

(b) For the reaction, we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{31}\mathrm{P}) + M(^1\mathrm{H}) - M(^{32}\mathrm{S})\right)c^2 \\ &= ((30.973762\ \mathrm{u}) + (1.007825\ \mathrm{u}) - (31.972071\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +8.864\ \mathrm{MeV}. \end{aligned}$$

7–14. Use conservation of energy and momentum to show that a bombarding proton must have an energy of 3.23 MeV in order to make the reaction $^{13}_{6}\mathrm{C}\,(\mathrm{p},\,\mathrm{n})\,^{13}_{7}\mathrm{N}$ occur. (See Example 7–3.)

SOLUTION. For the reaction ${}^{13}_{6}\mathrm{C}\,(\mathrm{p},\,\mathrm{n})\,{}^{13}_{7}\mathrm{N}$, we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{13}\mathrm{C}) + M({}^{1}\mathrm{H}) - m(\mathrm{n}) - M({}^{13}\mathrm{N})\right) c^2 \\ &= ((13.003355\ \mathrm{u}) + (1.007825\ \mathrm{u}) - (1.008665\ \mathrm{u}) - (13.005739\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= -3.003\ \mathrm{MeV}. \end{aligned}$$

The kinetic energy of the products is

$$K_\mathrm{n} + K_\mathrm{N} = K_\mathrm{p} + Q.$$

Because the kinetic energies $\ll mc^2$, we can use a nonrelativistic treatment:

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}.$$

The least kinetic energy is required when the product particles move together with the same speed.

With the target at rest, for momentum conservation we have

$$p_\mathrm{p} = p_\mathrm{n} + p_\mathrm{N} = (m_\mathrm{n} + m_\mathrm{N})v,$$

or

$$K_\mathrm{p} = \frac{p_\mathrm{p}^2}{2m_\mathrm{p}} = \frac{(m_\mathrm{n} + m_\mathrm{N})^2}{2m_\mathrm{p}}\,v^2 = \frac{m_\mathrm{n} + m_\mathrm{N}}{m_\mathrm{p}}(K_\mathrm{n} + K_\mathrm{N}),$$

or

$$K_\mathrm{n} + K_\mathrm{N} = \frac{m_\mathrm{p}}{m_\mathrm{n} + m_\mathrm{N}}K_\mathrm{p}.$$

When we use this in the kinetic energy equation, we get

$$\begin{aligned} \frac{m_\mathrm{p}}{m_\mathrm{n} + m_\mathrm{N}}K_\mathrm{p} &= K_\mathrm{p} + Q; \\ \left(\frac{1\ \mathrm{u}}{1\ \mathrm{u} + 13\ \mathrm{u}} - 1\right)K_\mathrm{p} &= -3.003\ \mathrm{MeV}, \end{aligned}$$

which gives $K_\mathrm{p} = 3.23\ \mathrm{MeV}$.

7.3 Cross section

7–15. What is the effective cross section for the collision of two hard spheres of radius R_1 and R_2.

SOLUTION. From Fig. 7.5 we see that a collision will occur if

$$d \leqslant R_1 + R_2.$$

Thus the area of the effective circle presented by R_2 to the center of R_1 is

$$\sigma = \pi(R_1 + R_2)^2.$$

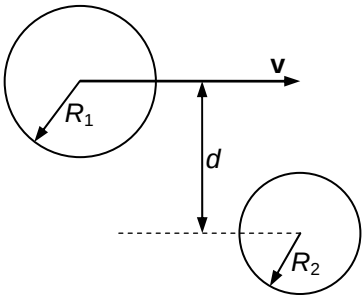


Figure 7.5. Problem 7–15.

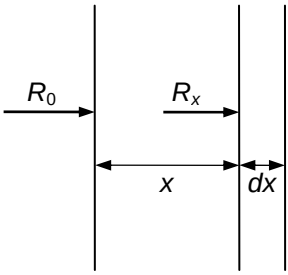


Figure 7.6. Problem 7–17.

7–16. The cross section for the reaction

$$\mathrm{n} + {}^{10}_{5}\mathrm{B} \rightarrow {}^{7}_{3}\mathrm{Li} + {}^{4}_{2}\mathrm{He}$$

is about 40 bn for an incident neutron of low energy (kinetic energy ≈ 0). The boron is contained in a gas with $n = 1.7 \times 10^{21}$ nuclei/m^3 and the target has thickness $t = 12.0$ cm. What fraction of incident neutrons will be scattered?

SOLUTION. We assume the density of the gas atoms is low enough that we can use the result for a thin target. Thus the fraction of neutrons that interact is

$$\frac{R}{R_0} = n\sigma t = (1.7 \times 10^{21}\ \mathrm{m}^{-3})(40 \times 10^{-28}\ \mathrm{m}^2)(0.120\ \mathrm{m}) = 8.2 \times 10^{-7}.$$

7–17. When the target is thick, the rate at which projectile particles collide with nuclei in the rear of the target is less than in the front of the target, since some scattering (i. e., collisions) takes place in the front layers. Let R_0 be the rate at which incident particles strike the front of the target, and R_x be the rate at a distance x into the target ($R_x = R_0$ at $x = 0$). Then show that the rate at which particles are scattered (and therefore lost from the incident beam) in a thickness dx is $-dR_x = R_x n\sigma\, dx$, where the minus sign means that R_x is decreasing and n is the number of nuclei per unit volume. Then show that $R_x = R_0 e^{-n\sigma x}$, where σ is the total cross section. If the thickness of the target is t, what does $R_x = R_0 e^{-n\sigma t}$ represent?

SOLUTION. The initial rate of incident particles is R_0, and the reduced rate a distance x inside the target is R_x (Fig. 7.6). The rate at which particles collide in the next small distance dx is

$$dR = R_x n\sigma\, dx.$$

Because each collision removes a particle from the beam, the change in the incident rate of particles in the beam a distance x inside the target is $-dR$, or

$$dR_x = -R_x n\sigma\, dx.$$

When we integrate this from the initial incidence, we get

$$\begin{aligned} \int_{R_0}^{R_x} \frac{dR_x}{R_x} &= -n\sigma \int_0^x dx; \\ \ln\left(\frac{R_x}{R_0}\right) &= -n\sigma x, \end{aligned}$$

or

$$R_x = R_0 e^{-n\sigma x}.$$

If $x = t$, we get

$$R_t = R_0 e^{-n\sigma t},$$

which represents the rate at which incident particles pass through the target without scattering.

7–18. A 1.0-cm-thick lead target reduces a beam of gamma rays to 30 percent of its original intensity. What thickness of lead will allow only one γ in 10^6 to penetrate (see Problem 7–17)?

SOLUTION. We use the result from Problem 7–17 in the form

$$\ln\left(\frac{R_t}{R_0}\right) = -n\sigma t.$$

We can find the value of $n\sigma$ from

$$\ln\left(\frac{R_1}{R_0}\right) = \ln(0.30) = -n\sigma t_1 = -n\sigma(0.010 \text{ m}),$$

which gives $n\sigma = 1.2 \times 10^2 \text{ m}^{-1}$.

We find the thickness to reduce the rate by $1/10^6$ from

$$\ln\left(\frac{R_2}{R_0}\right) = \ln(10^{-6}) = -n\sigma t_2 = -(1.2 \times 10^2 \text{ m}^{-1})t_2,$$

which gives $t_2 = 0.12 \text{ m} = 12 \text{ cm}$.

7–19. Use Fig. 7.1 to estimate what thickness of ^{113}Cd ($\rho = 8\,650$ kg/m^3) will cause a 1 percent reaction rate ($R/R_0 = 0.01$) for (a) 0.1-eV neutrons; (b) 10-eV neutrons.

SOLUTION. We assume a 1% reaction rate allows us to treat the target as thin. The density of Cd atoms is

$$\begin{aligned} n &= \frac{(8\,650 \text{ kg/m}^3)(10^3 \text{ g/kg})}{112.9 \text{ g/mol}}(6.02 \times 10^{23} \text{ atoms/mol}) \\ &= 4.61 \times 10^{28} \text{ m}^{-3}. \end{aligned}$$

(a) From Fig. 7.1 we find that the cross section for 0.1-eV neutrons is 3 000 bn. Thus we have

$$\begin{aligned} \frac{R}{R_0} &= n\sigma x; \\ 0.01 &= (4.61 \times 10^{28} \text{ m}^{-3})(3\,000 \times 10^{-28} \text{ m}^2)x, \end{aligned}$$

which gives $x = 7 \times 10^{-7} \text{ m} = 0.7 \ \mu\text{m}$.

(b) From Fig. 7.1 we find that the cross section for 10-eV neutrons is 0.3 bn. Thus we have

$$\begin{aligned} \frac{R}{R_0} &= n\sigma x; \\ 0.01 &= (4.61 \times 10^{28} \text{ m}^{-3})(0.3 \times 10^{-28} \text{ m}^2)x, \end{aligned}$$

which gives $x = 0.007 \text{ m} = 7 \text{ mm}$.

7.4 Nuclear fission. Nuclear reactors

7–20. Calculate the energy released in the fission reaction

$$\mathrm{n} + {}^{235}_{92}\mathrm{U} \rightarrow {}^{88}_{38}\mathrm{Sr} + {}^{136}_{54}\mathrm{Xe} + 12\mathrm{n}.$$

Use Appendix C, and assume the initial kinetic energy of the neutron is very small.

SOLUTION. If we neglect the initial kinetic energy of the neutron, the released energy is the Q-value:

$$\begin{aligned} Q &= \left(M(^{235}\mathrm{U}) + m(\mathrm{n}) - 12m(\mathrm{n}) - M(^{88}\mathrm{Sr}) - M(^{136}\mathrm{Xe})\right)c^2 \\ &= ((235.043923\ \mathrm{u}) - 11(1.008665\ \mathrm{u}) - (87.905614\ \mathrm{u}) \\ &\quad - (135.907220\ \mathrm{u}))\,c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= 126.5\ \mathrm{MeV}. \end{aligned}$$

7–21. What is the energy released in the fission reaction of Eq. 7.4? (The masses of ${}^{141}_{56}\mathrm{Ba}$ and ${}^{92}_{36}\mathrm{Kr}$ are 140.91440 u and 91.92630 u, respectively.)

SOLUTION. For the reaction

$$\mathrm{n} + {}^{235}_{92}\mathrm{U} \rightarrow {}^{141}_{56}\mathrm{Ba} + {}^{92}_{36}\mathrm{Kr} + 3\mathrm{n},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{235}\mathrm{U}) + m(\mathrm{n}) - 3m(\mathrm{n}) - M(^{141}\mathrm{Ba}) - M(^{92}\mathrm{Kr})\right)c^2 \\ &= ((235.043923\ \mathrm{u}) - 2(1.008665\ \mathrm{u}) - (140.91440\ \mathrm{u}) \\ &\quad - (91.92630\ \mathrm{u}))\,c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= 173.2\ \mathrm{MeV}. \end{aligned}$$

If the kinetic energy of the incident neutron is very small, the released energy is 173.2 MeV.

7–22. How many fissions take place per second in a 200-MW reactor? Assume 200 MeV is released per fission.

SOLUTION. If we assume 100% efficiency, we have

$$\begin{aligned} P &= \frac{E}{t}; \\ 200\ \mathrm{MW} &= (200\ \mathrm{MeV})(1.60 \times 10^{-19}\ \mathrm{J/eV})\left(\frac{n}{t}\right), \end{aligned}$$

which gives $n/t = 6.3 \times 10^{18}$ fissions/s.

7–23. Suppose that the average electric power consumption, day and night, in a typical house is 300 W. What initial mass of $^{235}_{92}$U would have to undergo fission to supply the electrical needs of such a house for a year? (Assume 200 MeV is released per fission, as well as 100% efficiency.)

SOLUTION. With an efficiency of 100% we find the number of fissions from

$$\begin{aligned} P &= \frac{E}{t}; \\ 300 \text{ W} &= \frac{(200 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV})n}{3.16 \times 10^7 \text{ s}}, \end{aligned}$$

which gives $n = 2.96 \times 10^{20}$ fissions.

Each fission uses one uranium atom, so the required mass is

$$m = \frac{2.96 \times 10^{20} \text{ atoms}}{6.02 \times 10^{23} \text{ atoms/mol}}(235 \text{ g/mol}) = 0.116 \text{ g}.$$

7–24. What initial mass of $^{235}_{92}$U is required to operate a 500-MW reactor for 1 yr? Assume 40 percent efficiency.

SOLUTION. We find the number of fissions from

$$\begin{aligned} P &= \frac{E}{t}; \\ \frac{500 \text{ MW}}{0.40} &= \frac{(200 \text{ MeV})(1.60 \times 10^{-19} \text{ J/eV})n}{3.16 \times 10^7 \text{ s}}, \end{aligned}$$

which gives $n = 1.23 \times 10^{27}$ fissions.

Each fission uses one uranium atom, so the required mass is

$$m = \frac{1.23 \times 10^{27} \text{ atoms}}{6.02 \times 10^{23} \text{ atoms/mol}}(235 \text{ g/mol}) = 4.82 \times 10^5 \text{ g} = 4.8 \times 10^2 \text{ kg}.$$

7–25. If a 1.0-MeV neutron emitted in a fission reaction loses one-half of its kinetic energy in each collision with moderator nuclei, how many collisions must it make to reach thermal energy ($\frac{3}{2} k_B T = 0.040$ eV)?

SOLUTION. We find the number of collisions from

$$\begin{aligned} E_n &= E_0 \left(\frac{1}{2}\right)^n; \\ 0.040 \text{ eV} &= (1.0 \times 10^6 \text{ eV}) \left(\frac{1}{2}\right)^n, \end{aligned}$$

which gives $n = 25$.

7–26. Suppose that the neutron multiplication factor is 1.0004. If the average time between successive fissions in a chain of reactions is 1.0 ms, by what factor will the reaction rate increase in 1.0 s?

SOLUTION. The number of fissions in one second is

$$n = \frac{t}{\Delta t} = \frac{1.0 \text{ s}}{1.0 \times 10^{-3} \text{ s}} = 1.0 \times 10^3.$$

For each fission the number of neutrons is 1.0004 times the number from the previous fission. Thus the reaction rate will increase by

$$(1.0004)n = (1.0004)(1000) = 1.49.$$

7–27. Estimate the ratio of the height of the Coulomb barrier for α decay to that for fission of $^{236}_{92}$U. (Both are described by a potential energy diagram of the shape shown in Fig. 3.11.)

SOLUTION. When ^{236}U decays by α emission, the resulting nucleus is ^{232}Th. Thus the radii of the two particles are

$$\begin{aligned} r_\alpha &= (1.2 \text{ fm}) \sqrt[3]{4} = 1.90 \text{ fm}; \\ r_{\text{Th}} &= (1.2 \text{ fm}) \sqrt[3]{232} = 7.37 \text{ fm}. \end{aligned}$$

At the instant of separation, the two particles are in contact, so the Coulomb energy is

$$U_\alpha = k_{\text{e}} \frac{Z_\alpha Z_{\text{Th}} e^2}{r_\alpha + r_{\text{Th}}}.$$

If we assume fission into equal parts, the resulting nucleus has $A = 118$. Thus each radius is

$$r_{\text{f}} = (1.2 \text{ fm}) \sqrt[3]{118} = 5.89 \text{ fm}.$$

At the instant of separation, the two particles are in contact, so the Coulomb energy is

$$U_\mathrm{f} = k_\mathrm{e}\frac{Z_\mathrm{f}^2 e^2}{2r_\mathrm{f}}.$$

The ratio is

$$\frac{U_\alpha}{U_\mathrm{f}} = \frac{Z_\alpha Z_\mathrm{Th}(2r_\mathrm{f})}{Z_\mathrm{f}^2(r_\alpha + r_\mathrm{Th})} = \frac{(2)(90)(2)(5.89\ \mathrm{fm})}{(46)^2(1.90\ \mathrm{fm} + 7.37\ \mathrm{fm})} = 0.11.$$

7–28. Assuming a fission of ${}^{236}_{92}\mathrm{U}$ into two roughly equal fragments, estimate the electric potential energy just as the fragments separate from each other. Assume that the fragments are spherical (see Eq. 6.1) and compare your calculation to the nuclear fission energy released, about 200 MeV.

SOLUTION. If we assume fission into equal parts, the resulting nucleus has $A = 118$. Thus each radius is

$$r_\mathrm{f} = (1.2\ \mathrm{fm})\sqrt[3]{118} = 5.89\ \mathrm{fm}.$$

At the instant of separation, the two particles are in contact, so the Coulomb energy is

$$U_\mathrm{f} = k_\mathrm{e}\frac{Z_\mathrm{f}^2 e^2}{2r_\mathrm{f}} = \frac{(46)^2(1.44\ \mathrm{MeV}\cdot\mathrm{fm})}{(2)(5.89\ \mathrm{fm})} = 259\ \mathrm{MeV}.$$

This is greater than the fission energy release of 200 MeV, which is probably an indication of the complexity of the fission process.

7.5 Fusion

7–29. What is the average kinetic energy of protons at the center of a star where the temperature is 10^7 K?

SOLUTION. We find the average kinetic energy from

$$K = \frac{3}{2}k_\mathrm{B}T = \frac{\frac{3}{2}(1.38\times10^{-23}\ \mathrm{J/K})(10^7\ \mathrm{K})}{1.60\times10^{-19}\ \mathrm{J/eV}} = 1.29\times10^3\ \mathrm{eV} = 1.3\ \mathrm{keV}.$$

7–30. Show that the energy released in the fusion reaction

$${}^2_1\mathrm{H} + {}^3_1\mathrm{H} \rightarrow {}^4_2\mathrm{He} + \mathrm{n}$$

is 17. 59 MeV.

SOLUTION. For the reaction

$$^2_1\mathrm{H} + ^3_1\mathrm{H} \rightarrow ^4_2\mathrm{He} + \mathrm{n},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^2\mathrm{H}) + M(^3\mathrm{H}) - m(\mathrm{n}) - M(^4\mathrm{He})\right) c^2 \\ &= ((2.014102\ \mathrm{u}) + (3.016049\ \mathrm{u}) - (1.008665\ \mathrm{u}) - (4.002603\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= +17.59\ \mathrm{MeV}. \end{aligned}$$

Thus 17. 59 MeV is released.

7–31. Show that the energy released when two deuterium nuclei fuse to form $^3_2$He with the release of a neutron is 3. 27 MeV.

SOLUTION. For the reaction

$$^2_1\mathrm{H} + ^2_1\mathrm{H} \rightarrow ^3_2\mathrm{He} + \mathrm{n},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^2\mathrm{H}) + M(^2\mathrm{H}) - m(\mathrm{n}) - M(^3\mathrm{He})\right) c^2 \\ &= (2(2.014102\ \mathrm{u}) - (1.008665\ \mathrm{u}) - (3.016029\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= +3.27\ \mathrm{MeV}. \end{aligned}$$

Thus 3. 27 MeV is released.

7–32. Verify the Q-value stated for each of the reactions of Eqs. 7.6. [*Hint*: Be careful with electrons.]

SOLUTION. For the reaction

$$^1_1\mathrm{H} + ^1_1\mathrm{H} \rightarrow ^2_1\mathrm{H} + ^0_{+1}\mathrm{e} + \nu,$$

we must add two electron masses to the left hand side to use atomic masses. We need one electron mass on the right hand side to use the atomic mass for hydrogen. Thus we have two extra electron masses on the right hand

side. We determine the Q-value:

$$\begin{aligned} Q &= \left(M(^1\mathrm{H}) + M(^1\mathrm{H}) - M(^2\mathrm{H}) - 2m(\mathrm{e})\right)c^2 \\ &= (2(1.007825\ \mathrm{u}) - (2.014102\ \mathrm{u}) - 2(0.0005486\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= 0.42\ \mathrm{MeV}. \end{aligned}$$

For the reaction

$$^1_1\mathrm{H} + {}^2_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + \gamma,$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^1\mathrm{H}) + M(^2\mathrm{H}) - M(^3\mathrm{He})\right)c^2 \\ &= ((1.007825\ \mathrm{u}) + (2.014102\ \mathrm{u}) - (3.016029\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= 5.49\ \mathrm{MeV}. \end{aligned}$$

For the reaction

$$^3_2\mathrm{He} + {}^3_2\mathrm{He} \rightarrow {}^4_2\mathrm{He} + {}^1_1\mathrm{H} + {}^1_1\mathrm{H},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^3\mathrm{He}) + M(^3\mathrm{He}) - M(^4\mathrm{He}) - M(^1\mathrm{H}) - M(^1\mathrm{H})\right)c^2 \\ &= (2(3.016029\ \mathrm{u}) - (4.002603\ \mathrm{u}) - 2(1.007825\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= 12.86\ \mathrm{MeV}. \end{aligned}$$

7–33. Calculate the energy release per gram of fuel for the reactions of Eqs. 7.9a, b, and c. Compare to the energy release per gram of uranium in fission.

SOLUTION. For the reaction

$$^2_1\mathrm{H} + {}^2_1\mathrm{H} \rightarrow {}^3_1\mathrm{H} + {}^1_1\mathrm{H},$$

two atoms of ^{2}H, or 4 u, are used as fuel. The energy release is

$$\begin{aligned} \frac{4.03\ \mathrm{MeV/reaction}}{(4\ \mathrm{u/reaction})(1.66 \times 10^{-27}\ \mathrm{kg/u})} &= 6.1 \times 10^{26}\ \mathrm{MeV/kg} \\ &= 6.1 \times 10^{23}\ \mathrm{MeV/g}. \end{aligned}$$

For the reaction

$$^2_1\mathrm{H} + ^2_1\mathrm{H} \rightarrow ^3_2\mathrm{He} + \mathrm{n},$$

two atoms of ^{2}H, or 4 u, are used as fuel. The energy release is

$$\begin{aligned}\frac{3.27 \text{ MeV/reaction}}{(4 \text{ u/reaction})(1.66 \times 10^{-27} \text{ kg/u})} &= 4.9 \times 10^{26} \text{ MeV/kg} \\ &= 4.9 \times 10^{23} \text{ MeV/g}.\end{aligned}$$

For the reaction

$$^2_1\mathrm{H} + ^3_1\mathrm{H} \rightarrow ^4_2\mathrm{He} + \mathrm{n},$$

one atom of ^{2}H and one atom of ^{3}H, or 5 u, are used as fuel. The energy release is

$$\begin{aligned}\frac{17.59 \text{ MeV/reaction}}{(5 \text{ u/reaction})(1.66 \times 10^{-27} \text{ kg/u})} &= 2.1 \times 10^{27} \text{ MeV/kg} \\ &= 2.1 \times 10^{24} \text{ MeV/g}.\end{aligned}$$

In the fission reaction, one atom of ^{235}U, or 235 u, is used as fuel. The energy release is

$$\begin{aligned}\frac{200 \text{ MeV/reaction}}{(235 \text{ u/reaction})(1.66 \times 10^{-27} \text{ kg/u})} &= 5.1 \times 10^{26} \text{ MeV/kg} \\ &= 5.1 \times 10^{23} \text{ MeV/g}.\end{aligned}$$

Thus most fusion reactions yield more energy per unit mass.

Note that ^{235}U is only 0.72% of natural uranium. Thus approximately every gram of ^{235}U would come from $1/(0.0072) = 1.4 \times 10^2$ g ^{238}U. On this basis the fission yield is

$$\frac{5.1 \times 10^{23} \text{ MeV}/(1 \text{ g } ^{235}\text{U})}{1.4 \times 10^2 \text{ g } ^{238}\text{U}/(1 \text{ g } ^{235}\text{U})} = 3.6 \times 10^{21} \text{ MeV}/(1 \text{ g } ^{238}\text{U}).$$

7–34. If a typical house requires 300 W of electric power on average, what minimum amount of deuterium fuel would have to be used in a year to supply these electrical needs? Assume the reaction of Eq. 7.9b.

SOLUTION. We find the minimum number of fusions by assuming 100% efficiency:

$$\begin{aligned} P &= \frac{E}{t}; \\ 300\ \text{W} &= \frac{(3.27\ \text{MeV})(1.60\times10^{-13}\ \text{J/MeV})n}{3.16\times10^{7}\ \text{s/yr}}, \end{aligned}$$

which gives $n = 1.81\times10^{22}$ fusions/yr.

Each fusion uses two deuterium atoms, so the required mass is

$$m = \frac{1.81\times10^{22}\ \text{atoms/yr}}{6.02\times10^{23}\ \text{atoms/mol}}(2)(2\ \text{g/mol}) = 0.120\ \text{g/yr}.$$

7–35. Show that the energies carried off by the $^4_2$He nucleus and the neutron for the reaction of Eq. 7.9c are about 3.5 MeV and 14 MeV, respectively. Are these fixed values, independent of the plasma temperature?

SOLUTION. Because the kinetic energies $\ll mc^2$, we can use a nonrelativistic treatment:

$$K = \frac{mv^2}{2} = \frac{p^2}{2m}.$$

We assume the kinetic energy of the deuterium and the tritium can be neglected, so for momentum conservation we have

$$p_{\text{He}} = p_{\text{n}}.$$

The kinetic energy of the particles is

$$\begin{aligned} Q &= K_{\text{He}} + K_{\text{n}} = \frac{p_{\text{He}}^2}{2m_{\text{He}}} + \frac{p_{\text{n}}^2}{2m_{\text{n}}} = \frac{p_{\text{He}}^2(m_{\text{n}} + m_{\text{He}})}{2m_{\text{He}}m_{\text{n}}}; \\ 17.59\ \text{MeV} &= \frac{p_{\text{He}}^2(1\ \text{u} + 4\ \text{u})}{2(4\ \text{u})(1\ \text{u})}, \end{aligned}$$

which gives $p_{\text{He}}^2 = 28.1\ \text{MeV}\cdot\text{u}$.

The kinetic energy of ^{4}He is

$$K_{\text{He}} = \frac{p_{\text{He}}^2}{2m_{\text{He}}} = \frac{28.1\ \text{MeV}\cdot\text{u}}{2(4\ \text{u})} = 3.5\ \text{MeV}.$$

The kinetic energy of n is

$$K_{\mathrm{n}} = \frac{p_{\mathrm{n}}^2}{2m_{\mathrm{n}}} = \frac{28.1\ \mathrm{MeV}\cdot\mathrm{u}}{2(1\ \mathrm{u})} = 14\ \mathrm{MeV}.$$

This result is not independent of the plasma temperature, which is a measure of the initial kinetic energies.

7–36. Suppose a fusion reactor ran on "d-d" reactions, Eqs. 7.9a and b. Estimate how much water, for fuel, would be needed per hour to run a 1 000-MW reactor, assuming 30 percent efficiency.

SOLUTION. The rate at which input energy is needed by the reactor is

$$P = \frac{1\,000\times10^6\ \mathrm{W}}{0.30}\,\frac{3\,600\ \mathrm{s/h}}{1.60\times10^{-13}\ \mathrm{J/MeV}} = 7.50\times10^{25}\ \mathrm{MeV/h}.$$

If we assume equal contributions from the two equations, four deuterium nuclei release

$$4.03\ \mathrm{MeV} + 3.27\ \mathrm{MeV} = 7.30\ \mathrm{MeV}.$$

Each water molecule has two H atoms. If 0.015% of them are deuterium nuclei, the number of water molecules needed for fuel is

$$\begin{aligned} N &= \frac{7.50\times10^{25}\ \mathrm{MeV/h}}{((7.30\ \mathrm{MeV})/(4\ \mathrm{nuclei}))\,(2\ \mathrm{nuclei/molecule})(0.015\times10^{-2})} \\ &= 1.37\times10^{29}\ \mathrm{molecules/h}. \end{aligned}$$

The required mass of water is

$$\begin{aligned} m &= \frac{1.37\times10^{29}\ \mathrm{molecules/h}}{6.02\times10^{23}\ \mathrm{molecules/mol}}(18\ \mathrm{g/mol}) = 4.1\times10^6\ \mathrm{g/h} \\ &= 4.1\times10^3\ \mathrm{kg/h}. \end{aligned}$$

7–37. How much energy (J) is contained in 1.00 kg of water if its natural deuterium is used in the fusion reaction of Eq. 7.9a? Compare to the energy obtained from the burning of 1.0 kg of gasoline, about 5×10^7 J.

SOLUTION. The number of deuterium nuclei in 1.00 kg of water is

$$\begin{aligned} N &= \frac{1.00\times10^{3}\text{ g}}{18\text{ g/mol}}(6.02\times10^{23}\text{ molecules/mol})(2\text{ nuclei/molecule}) \\ &\quad\times(0.015\times10^{-2}) \\ &= 1.00\times10^{22}\text{ nuclei.} \end{aligned}$$

Two deuterium nuclei release 4.03 MeV, so the total energy is

$$\begin{aligned} E &= \frac{(1.00\times10^{22}\text{ nuclei})(4.03\text{ MeV})(1.60\times10^{-13}\text{ J/MeV})}{2\text{ nuclei}} \\ &= 3.23\times10^{9}\text{ J.} \end{aligned}$$

This is 65× the energy from burning 1.0 kg of gasoline.

7–38. The energy output of massive stars is believed to be due to the *carbon cycle* (Eqs. 7.8). (a) Show that no carbon is consumed in this cycle and that the net effect is the same as for the proton-proton cycle. (b) What is the total energy release? (c) Determine the energy output for each reaction and decay. (d) Why does the carbon cycle require a higher temperature ($\approx 2\times10^{7}$ K) than the proton-proton cycle ($\approx 1.5\times10^{7}$ K)?

SOLUTION. (a) If we look at the reactions for the carbon cycle, we see that one carbon atom is used in the first reaction and one carbon atom is produced in the last reaction. If we add all of the reactions, we get

$$4\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + 2e^{+} + 2\nu.$$

(b) If we ignore the gamma rays and the pair annihilation, the Q-value is

$$\begin{aligned} Q &= \left(4M({}^{1}\mathrm{H}) - M({}^{4}\mathrm{He}) - 4m(\mathrm{e})\right)c^{2} \\ &= (4(1.007825\text{ u}) - (4.002603\text{ u}) - 4(0.0005486\text{ u})) \\ &\quad\times c^{2}(931.5\text{ MeV/uc}^{2}) \\ &= +24.69\text{ MeV.} \end{aligned}$$

When we include the energy from the electron-positron annihilation, the total energy release is

$$24.69\text{ MeV} + 2(1.022\text{ MeV}) = 26.73\text{ MeV.}$$

(c) For the reaction

$$^{12}_{6}\mathrm{C} + ^{1}_{1}\mathrm{H} \rightarrow ^{13}_{7}\mathrm{N},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{12}\mathrm{C}) + M(^{1}\mathrm{H}) - M(^{13}\mathrm{N})\right) c^2 \\ &= ((12.000000\ \mathrm{u}) + (1.007825\ \mathrm{u}) - (13.005739\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +1.94\ \mathrm{MeV}. \end{aligned}$$

For the decay

$$^{13}_{7}\mathrm{N} \rightarrow ^{13}_{6}\mathrm{C} + \mathrm{e}^{+},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{13}\mathrm{N}) - M(^{13}\mathrm{C}) - 2m(\mathrm{e})\right) c^2 \\ &= ((13.005739\ \mathrm{u}) - (13.003355\ \mathrm{u}) - 2(0.0005486\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +1.20\ \mathrm{MeV}. \end{aligned}$$

For the reaction

$$^{13}_{6}\mathrm{C} + ^{1}_{1}\mathrm{H} \rightarrow ^{14}_{7}\mathrm{N},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{13}\mathrm{C}) + M(^{1}\mathrm{H}) - M(^{14}\mathrm{N})\right) c^2 \\ &= ((13.003355\ \mathrm{u}) + (1.007825\ \mathrm{u}) - (14.003074\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +7.55\ \mathrm{MeV}. \end{aligned}$$

For the reaction

$$^{14}_{7}\mathrm{N} + ^{1}_{1}\mathrm{H} \rightarrow ^{15}_{8}\mathrm{O},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{14}\mathrm{N}) + M(^{1}\mathrm{H}) - M(^{15}\mathrm{O})\right) c^2 \\ &= ((14.003074\ \mathrm{u}) + (1.007825\ \mathrm{u}) - (15.003065\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +7.30\ \mathrm{MeV}. \end{aligned}$$

For the decay

$$^{15}_{8}\mathrm{O} \rightarrow {}^{15}_{7}\mathrm{N} + \mathrm{e}^{+},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{15}\mathrm{O}) - M(^{15}\mathrm{N}) - 2m(\mathrm{e})\right)c^2 \\ &= ((15.003065\ \mathrm{u}) - (15.000108\ \mathrm{u}) - 2(0.0005486\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +1.73\ \mathrm{MeV}. \end{aligned}$$

For the reaction

$$^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{15}\mathrm{N}) + M(^{1}\mathrm{H}) - M(^{12}\mathrm{C}) - M(^{4}\mathrm{He})\right)c^2 \\ &= ((15.000108\ \mathrm{u}) + (1.007825\ \mathrm{u}) - (12.000000\ \mathrm{u}) - (4.002603\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= +4.96\ \mathrm{MeV}. \end{aligned}$$

(d) For the nuclei with higher atomic number, there is a greater repulsion between the positive charges, so higher kinetic energies are required. Thus higher temperatures are required.

7–39. (a) Compare the energy needed for the first reaction of the carbon cycle to that for a deuteron-tritium reaction (Example 7–8). (b) If a deuteron-tritium reaction requires $T \approx 3\times10^8$ K, estimate the temperature needed for the first carbon-cycle reaction.

Solution. (a) To initiate the reaction, the kinetic energy of the initial nuclei must be sufficient to overcome the Coulomb barrier. The radii of the two initial nuclei for the first reaction of the carbon cycle are

$$\begin{aligned} r_{\mathrm{C}} &= (1.2\ \mathrm{fm})\sqrt[3]{12} = 2.75\ \mathrm{fm}; \\ r_{\mathrm{p}} &= (1.2\ \mathrm{fm})\sqrt[3]{1} = 1.2\ \mathrm{fm}. \end{aligned}$$

We assume the total kinetic energy equals the Coulomb energy when the nuclei are in contact:

$$K_{\mathrm{C-p}} = U_{\mathrm{C-p}} = k_{\mathrm{e}}\frac{Z_{\mathrm{C}}Z_{\mathrm{p}}e^2}{r_{\mathrm{C}} + r_{\mathrm{p}}}.$$

The radii of the two initial nuclei for the deuteron-tritium reaction are

$$\begin{aligned} r_{\mathrm{d}} &= (1.2\ \mathrm{fm})\sqrt[3]{2} = 1.51\ \mathrm{fm}; \\ r_{\mathrm{t}} &= (1.2\ \mathrm{fm})\sqrt[3]{3} = 1.73\ \mathrm{fm}. \end{aligned}$$

We assume the total kinetic energy equals the Coulomb energy when the nuclei are in contact:

$$K_{\mathrm{d-t}} = U_{\mathrm{d-t}} = k_{\mathrm{e}}\frac{Z_{\mathrm{d}}Z_{\mathrm{t}}e^2}{r_{\mathrm{d}} + r_{\mathrm{t}}}.$$

Thus the ratio is

$$\frac{K_{\mathrm{C-p}}}{K_{\mathrm{d-t}}} = \frac{Z_{\mathrm{C}}Z_{\mathrm{p}}(r_{\mathrm{d}} + r_{\mathrm{t}})}{Z_{\mathrm{d}}Z_{\mathrm{t}}(r_{\mathrm{C}} + r_{\mathrm{p}})} = \frac{(6)(1)(1.51\ \mathrm{fm} + 1.73\ \mathrm{fm})}{(1)(1)(2.75\ \mathrm{fm} + 1.2\ \mathrm{fm})} = 4.9\times .$$

(b) Because the kinetic energy is proportional to the temperature, we have

$$\begin{aligned} \frac{K_{\mathrm{C-p}}}{K_{\mathrm{d-t}}} &= \frac{T_{\mathrm{C-p}}}{T_{\mathrm{d-t}}}; \\ 4.9 &= \frac{T_{\mathrm{C-p}}}{3\times 10^8\ \mathrm{K}}, \end{aligned}$$

which gives $T_{\mathrm{C-p}} = 1.5\times 10^9$ K.

7–40. The deuterium-tritium pellet in a laser fusion device contains equal numbers of $^2_1$H and $^3_1$H atoms raised to a density of 200×10^3 kg/m^3 by the laser pulses. Estimate (a) the density of particles in this compressed state, and (b) the length of time τ the pellet must be confined to meet the Lawson criterion (Eq. 7.10) for ignition.

SOLUTION. (a) If n is the density of particles, for equal numbers of ^{2}H and ^{3}H atoms we have

$$\begin{aligned} \rho &= 200\times 10^3\ \mathrm{kg/m^3} \\ &= \tfrac{1}{2}n(2\ \mathrm{u})(1.66\times 10^{-27}\ \mathrm{kg/u}) + \tfrac{1}{2}n(3\ \mathrm{u})(1.66\times 10^{-27}\ \mathrm{kg/u}), \end{aligned}$$

which gives $n = 4.8\times 10^{31}$ particles/m^3.

(b) To meet the Lawson criterion (Eq. 7.10), we have

$$\begin{aligned} n\tau &\geqslant 3\times 10^{20}\ \mathrm{s/m^3}; \\ (4.8\times 10^{31}\ \mathrm{particles/m^3})\tau &\geqslant 3\times 10^{20}\ \mathrm{s/m^3}, \end{aligned}$$

which gives $\tau \geqslant 6\times 10^{-12}$ s.

7.6 Dosimetry

7–41. A dose of 4.0 Sv of γ rays in a short period would be lethal to about half the people subjected to it. How many grays is this?

SOLUTION. Because the quality factor for gamma rays is ≈ 1, we have effective dose

$$(\text{Gy}) = \frac{\text{effective dose (Sv)}}{\text{QF}} = \frac{4.0 \text{ Sv}}{1} = 4.0 \text{ Gy}.$$

7–42. Fifty rads of α-particle radiation is equivalent to how many rads of X-rays in terms of biological damage?

SOLUTION. Because the quality factor for α-particle radiation is ≈ 20, we have

$$\begin{aligned} \text{effective dose (rem)} &= \text{effective dose (rad)} \times \text{QF}; \\ \text{rad}_\alpha \times 20 &= \text{rad}_\text{X} \times 1; \\ \text{rad}_\text{X} &= (50 \text{ rad})(20) = 1\,000 \text{ rad}. \end{aligned}$$

7–43. How many rads of slow neutrons will do as much biological damage as 50 rads of fast neutrons?

SOLUTION. Because the quality factor for slow neutrons is ≈ 3 and for fast neutrons is ≈ 10, we have

$$\begin{aligned} \text{effective dose (rem)} &= \text{effective dose (rad)} \times \text{QF}; \\ \text{rad}_\text{slow} \times 3 &= \text{rad}_\text{fast} \times 10; \\ \text{rad}_\text{slow} &= \frac{(50 \text{ rad})(10)}{3} = 167 \text{ rad}. \end{aligned}$$

7–44. How much energy is deposited in the body of a 70-kg adult exposed to a 50-rad dose?

SOLUTION. The energy deposited is

$$E = (50 \text{ rad})(1.00 \times 10^{-2} \text{ J/kg} \cdot \text{rad})(70 \text{ kg}) = 35 \text{ J}.$$

7–45. A 0.025-μCi sample of $^{32}_{15}\text{P}$ is injected into an animal for tracer studies. If a Geiger counter intercepts 20 percent of the emitted β particles, what will be the counting rate, assumed 90 percent efficient.

SOLUTION. If the counter counts 90% of the intercepted β particles, we have

$$\begin{aligned} n &= (0.90)(0.20)(0.025 \times 10^{-6}\ \text{Ci})(3.7 \times 10^{10}\ \text{decays/s} \cdot \text{Ci}) \\ &= 1.7 \times 10^{2}\ \text{counts/s}. \end{aligned}$$

7–46. A 1.0-mCi source of $^{32}_{15}\text{P}$ (in $NaHPO_4$), a β^- emitter, is implanted in an organ where it is to administer 36 Gy. The half-life of $^{32}_{15}\text{P}$ is 14.3 days and 1 mCi delivers about 10 mGy/min. Approximately how long should the source remain implanted?

SOLUTION. If the decay rate were constant, the time required would be

$$t_1 = \frac{36\ \text{Gy}}{(1.0 \times 10^{-2}\ \text{Gy/min})(60\ \text{min/h})(24\ \text{h/day})} = 2.50\ \text{days}.$$

This time is

$$\frac{2.50\ \text{days}}{14.3\ \text{days}} = 0.175\ \text{half-lives}.$$

The activity of the source at this time would be

$$(1.0 \times 10^{-2}\ \text{Gy/min}) \left(\frac{1}{2}\right)^{0.175} = 0.886 \times 10^{-2}\ \text{Gy/min}.$$

If we approximate the exponential decay as linear, and use the average activity, we get

$$\begin{aligned} t_2 &= \frac{36\ \text{Gy}}{\frac{1}{2}\,(1.0 \times 10^{-2}\ \text{Gy/min} + 0.886 \times 10^{-2}\ \text{Gy/min})\,(1\,440\ \text{min/day})} \\ &= 2.65\ \text{days} \approx 2.7\ \text{days}. \end{aligned}$$

7–47. About 35 eV is required to produce one ion pair in air. Show that this is consistent with the two definitions of the roentgen given in the text.

SOLUTION. If we start with the current definition of the roentgen, we get

$$\begin{aligned} 1\ \text{R} &= \frac{0.878 \times 10^{-2}\ \text{J/kg}}{(1.60 \times 10^{-19}\ \text{J/eV})(1\,000\ \text{g/kg})(35\ \text{eV/pair})} \\ &= 1.57 \times 10^{12}\ \text{pairs/g} \approx 1.6 \times 10^{12}\ \text{pairs/g}. \end{aligned}$$

7–48. $^{57}_{27}$Co emits 122-keV γ rays. If a 70-kg person swallowed $1.85\ \mu$Ci of $^{57}_{27}$Co, what would be the dose rate (Gy/day) averaged over the whole body? Assume that 50 percent of the γ-ray energy is deposited in the body.

SOLUTION. Because each decay gives one gamma ray, the rate at which energy is emitted is

$$\begin{aligned} P &= (1.85\times10^{-6}\ \text{Ci})(3.7\times10^{1}0\ \text{decays/s}\cdot\text{Ci})(122\times10^{3}\ \text{eV/decay}) \\ &\quad \times(1.60\times10^{-19}\ \text{J/eV}) \\ &= 1.34\times10^{-9}\ \text{J/s}. \end{aligned}$$

If 50% of the energy is absorbed by the body, the dose rate is

$$\text{dose rate} = \frac{(0.50)(1.34\times10^{-9}\ \text{J/s})(86\,400\ \text{s/day})}{(1.00\ \text{J/kg}\cdot\text{Gy})(70\ \text{kg})} = 8.2\times10^{-7}\ \text{Gy/day}.$$

7–49. What is the mass of a 1.00-μCi $^{14}_{6}$C source?

SOLUTION. The decay constant is

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{5\,730\ \text{yr}} = 1.209\times10^{-4}\ \text{yr}^{-1} = 3.83\times10^{-12}\ \text{s}^{-1}.$$

We find the number of ^{14}C atoms from

$$\begin{aligned} \text{Activity} &= \lambda N; \\ (1.00\times10^{-6}\ \text{Ci})(3.7\times10^{10}\ \text{decays/s}\cdot\text{Ci}) &= (3.83\times10^{-12}\ \text{s}^{-1})N, \end{aligned}$$

which gives $N = 9.67\times10^{15}$ atoms.

The mass is

$$m = \frac{9.67\times10^{15}\ \text{atoms}}{6.02\times10^{23}\ \text{atoms/mol}}(14\ \text{g/mol}) = 2.25\times10^{-7}\ \text{g} = 0.225\ \mu\text{g}.$$

7–50. Huge amounts of radioactive $^{131}_{53}$I were released in the accident at Chernobyl in 1986. Chemically, iodine goes to the human thyroid. (Doctors can use it for diagnosis and treatment of thyroid problems.) In a normal thyroid, $^{131}_{53}$I absorption can cause damage to the thyroid. (a) Write down the decay scheme for $^{131}_{53}$I. (b) Its half-life is 8.0 d; how long would

it take for ingested ${}^{131}_{53}\mathrm{I}$ to become 10 percent of the initial value? (c) Absorbing 1 mCi of ${}^{131}_{53}\mathrm{I}$ can be harmful; what mass of iodine is this?

SOLUTION. (a) ${}^{131}_{53}\mathrm{I} \rightarrow {}^{131}_{54}\mathrm{Xe} + {}^{0}_{-1}\mathrm{e} + \nu + \gamma$.

(b) We find the number of half-lives from

$$\begin{aligned} \frac{N}{N_0} &= \left(\frac{1}{2}\right)^n; \\ (0.10) &= \left(\frac{1}{2}\right)^n, \end{aligned}$$

or

$$n \lg 2 = \lg 10,$$

which gives $n = 3.32$.

Thus the elapsed time is

$$\Delta t = nT_{1/2} = (3.32)(8.0 \text{ days}) = 27 \text{ days}.$$

(c) We find the number of atoms from

$$\begin{aligned} \text{Activity} &= \lambda N; \\ (1.00 \times 10^{-3} \text{ Ci}) & \\ \times (3.7 \times 10^{10} \text{ decays/s} \cdot \text{Ci}) &= \frac{(0.693)}{(8.0 \text{ days})(86\,400 \text{ s/day})} N, \end{aligned}$$

which gives $N = 3.69 \times 10^{13}$ atoms.

The mass is

$$m = \frac{3.69 \times 10^{13} \text{ atoms}}{6.02 \times 10^{23} \text{ atoms/mol}} (131 \text{ g/mol}) = 8.0 \times 10^{-9} \text{ g}.$$

7–51. Assume a liter of milk typically has an activity of 2 000 pCi due to ${}^{40}_{19}\mathrm{K}$. If a person drinks two glasses (0.5 L) per day, estimate the total dose (in Sv and in rem) received in a year. As a crude model, assume the milk stays in the stomach 12 hr and is then released. Assume also that very roughly 10 percent of the 1.5 MeV released per decay is absorbed by the body. Compare your result to the normal allowed dose per year. Make your estimate for (a) a 50-kg adult, and (b) a 5-kg baby.

SOLUTION. Because each decay gives one gamma ray, the rate at which energy is emitted is

$$\begin{aligned} P &= (2\,000 \times 10^{-12}\ \text{Ci/L})(3.7 \times 10^{10}\ \text{decays/s} \cdot \text{Ci})(1.5\ \text{MeV/decay}) \\ &\quad \times (1.60 \times 10^{-13}\ \text{J/MeV}) \\ &= 1.78 \times 10^{-11}\ \text{J/s} \cdot \text{L}. \end{aligned}$$

If 10% of the energy is absorbed by the body for half a year (12 h/day), the total absorbed energy rate is

$$\begin{aligned} \text{rate} &= (0.10)(1.78 \times 10^{-11}\ \text{J/s} \cdot \text{L})(0.5\ \text{L})\tfrac{1}{2}(3.16 \times 10^{7}\ \text{s/yr}) \\ &= 1.4 \times 10^{-5}\ \text{J/yr}. \end{aligned}$$

For beta particles and gamma rays, QF = 1.

(a) For an adult, the total dose in one year is

$$\begin{aligned} \text{dose} &= \frac{(1.4 \times 10^{-5}\ \text{J/yr})(1)}{(1.00\ \text{J/kg} \cdot \text{Gy})(50\ \text{kg})} = 2.8 \times 10^{-7}\ \text{Sv} = 2.8 \times 10^{-5}\ \text{rem} \\ &= 0.03\ \text{mrem} \approx 0.006\,\%\ \text{of allowed dose.} \end{aligned}$$

(b) For a baby, the total dose in one year is

$$\begin{aligned} \text{dose} &= \frac{(1.4 \times 10^{-5}\ \text{J/yr})(1)}{(1.00\ \text{J/kg} \cdot \text{Gy})(5\ \text{kg})} = 2.8 \times 10^{-6}\ \text{Sv} = 2.8 \times 10^{-4}\ \text{rem} \\ &= 0.3\ \text{mrem} \approx 0.06\,\%\ \text{of allowed dose.} \end{aligned}$$

7–52. Radon gas, $^{222}_{86}\text{Rn}$, is considered a serious hazard. It decays by α-emissioa. (a) What is the daughter nucleus? (b) Is the daughter nucleus stable or radioactive? If the latter, how does it decay, and what is its lifetime? (c) Is the daughter nucleus also a noble gas, or is it chemically reacting? (d) Suppose 1.0 ng of $^{222}_{86}\text{Rn}$ seeps into a basement. What will be its activity? If the basement is then sealed, what will be the activity 1 month later?

SOLUTION. (a) We find the daughter nucleus by balancing the mass and charge numbers:

$$\begin{aligned} Z(\text{X}) &= Z(^{222}\text{Rn}) - Z(^{4}\text{He}) = 86 - 2 = 84; \\ A(\text{X}) &= A(^{222}\text{Rn}) - A(^{4}\text{He}) = 222 - 4 = 218, \end{aligned}$$

so the product nucleus is ${}^{218}_{84}\mathrm{Po}$.

(b) From Fig. 6.2, we see that ${}^{218}_{84}\mathrm{Po}$ is radioactive, and decays by both α and β^- emission:

$$ {}^{218}_{84}\mathrm{Po} \rightarrow {}^{214}_{82}\mathrm{Pb} + {}^{4}_{2}\mathrm{He}; \qquad {}^{218}_{84}\mathrm{Po} \rightarrow {}^{218}_{85}\mathrm{At} + {}^{0}_{-1}\mathrm{e} + \nu. $$

The half life for both decays is 3.1 min.

(c) Both daughter nuclei are chemically reactive.

(d) The decay constant is

$$ \lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{(3.8 \text{ days})(86\,400 \text{ s/day})} = 2.11 \times 10^{-6} \text{ s}^{-1}. $$

The number of radon atoms in 1.0 ng is

$$ N_0 = \frac{1.0 \times 10^{-9} \text{ g}}{222 \text{ g/mol}}(6.02 \times 10^{23} \text{ atoms/mol}) = 2.71 \times 10^{12} \text{ atoms}. $$

The initial activity is

$$ \lambda N_0 = \frac{(2.11 \times 10^{-6} \text{ s}^{-1})(2.71 \times 10^{12})}{3.7 \times 10^{10} \text{ decays/s} \cdot \text{Ci}} = 1.5 \times 10^{-4} \text{ Ci} = 150 \ \mu\text{Ci}. $$

After 1 month, the number of half-lives is

$$ n = \frac{30 \text{ days}}{3.8 \text{ days}} = 7.89. $$

The activity is

$$ \lambda N = \lambda N_0 \left(\frac{1}{2}\right)^n = (150 \ \mu\text{Ci}) \left(\frac{1}{2}\right)^{7.89} = 0.63 \ \mu\text{Ci}. $$

7.7 Nuclear magnetic resonance

7–53. The radio frequency at which the nucleus displays a resonance absorption between spin states is called the *Larmor precessional frequency* and is given by Eq. 7.14. Calculate the Larmor frequency for free neutrons in a magnetic field of 1.00 T, and (b) free protons in the Earth's magnetic field at a location where the magnitude of the field is 50.0 μT.

SOLUTION. (a)

$$f_{\mathrm{n}} = \frac{\mu_{\mathrm{n}} B}{\pi\hbar} = \frac{(1.9135\,\mu_{\mathrm{N}})B}{\pi\hbar}$$
$$= \frac{(1.9135)(5.051\times10^{-27}\ \mathrm{J/T})(1.00\ \mathrm{T})}{\pi(1.055\times10^{-34}\ \mathrm{J\cdot s})} = 29.2\ \mathrm{MHz}.$$

The existence of a magnetic moment for the neutron is surprising in view of the fact that the neutron is uncharged. This suggests that the neutron is not a fundamental particle but rather has an underlying structure consisting of charged constituents. This is explored in Chapter 8.

(b)

$$f_{\mathrm{Earth}} = \frac{\mu_{\mathrm{p}} B}{\pi\hbar} = \frac{(2.7928\,\mu_{\mathrm{N}})B}{\pi\hbar}$$
$$= \frac{(2.7928)(5.051\times10^{-27}\ \mathrm{J/T})(50.0\times10^{-6}\ \mathrm{T})}{\pi(1.055\times10^{-34}\ \mathrm{J\cdot s})} = 2.13\ \mathrm{kHz}.$$

7–54. Calculate the wavelength of photons needed to produce NMR transitions in free protons in a 1.000-T field. In what region of the spectrum does it lie?

SOLUTION. From the Example 7–11 we know that in a magnetic field of 1.000 T, resonance occurs when the photon has a frequency of 42.58 MHz. The wavelength is

$$\lambda = \frac{c}{f} = \frac{3.00\times10^{8}\ \mathrm{m/s}}{42.58\times10^{6}\ \mathrm{Hz}} = 7.05\ \mathrm{m},$$

FM radio wave.

7.8 General problems

7–55. Chadwick discovered the neutron by bombarding $^{9}_{4}$Be with the popular projectile of the day, α particles. (a) If one of the reaction products was the then unknown neutron, what was the other product? (b) What is the Q of this reaction?

SOLUTION. (a) We find the other nucleus by balancing the mass and charge numbers:

$$Z(\mathrm{X}) = Z(^{9}\mathrm{Be}) + Z(^{4}\mathrm{He}) - Z(\mathrm{n}) = 4 + 2 - 0 = 6;$$
$$A(\mathrm{X}) = A(^{9}\mathrm{Be}) + A(^{4}\mathrm{He}) - A(\mathrm{n}) = 9 + 4 - 1 = 12,$$

so the other nucleus is ${}^{12}_{6}\mathrm{C}$.

(b) For the reaction

$$ {}^{9}_{4}\mathrm{Be} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{12}_{6}\mathrm{C} + {}^{1}_{0}\mathrm{n}, $$

we determine the Q-value:

$$ \begin{aligned} Q &= \left(M({}^{9}\mathrm{Be}) + M({}^{4}\mathrm{He}) - M({}^{12}\mathrm{C}) - m(\mathrm{n})\right) c^2 \\ &= ((9.012182\ \mathrm{u}) + (4.002603\ \mathrm{u}) - (12.000000\ \mathrm{u}) - (1.008665\ \mathrm{u})) \\ &\quad \times c^2 (931.5\ \mathrm{MeV/u}c^2) \\ &= +5.70\ \mathrm{MeV}. \end{aligned} $$

7–56. Fusion temperatures are often given in keV. Determine the conversion factor from kelvins to keV using, as is common in this field, $\overline{K} = k_{\mathrm{B}}T$ without the factor $\frac{3}{2}$.

SOLUTION. We find the conversion from

$$ K = k_{\mathrm{B}}T $$

by converting the units for k_{B}:

$$ 1.38 \times 10^{-23}\ \mathrm{J/K} = \frac{1.38 \times 10^{-23}\ \mathrm{J/K}}{1.60 \times 10^{-16}\ \mathrm{J/keV}} = 8.63 \times 10^{-8}\ \mathrm{keV/K}. $$

7–57. One means of enriching uranium is by diffusion of the gas UF_6. Calculate the ratio of the speeds of molecules of this gas containing ${}^{235}_{92}\mathrm{U}$ and ${}^{238}_{92}\mathrm{U}$, on which this process depends.

SOLUTION. The average kinetic energy is a function of the temperature:

$$ K = \tfrac{1}{2}mv^2 = \tfrac{3}{2}k_{\mathrm{B}}T. $$

When we form the ratio for the molecules of the two isotopes at the same temperature, we get

$$ \frac{v_{235}}{v_{238}} = \sqrt{\frac{m_{238}}{m_{235}}} = \sqrt{\frac{238\ \mathrm{u} + 6(19\ \mathrm{u})}{235\ \mathrm{u} + 6(19\ \mathrm{u})}} = 1.004. $$

7–58. (a) What mass of ${}^{235}_{92}\mathrm{U}$ was actually fissioned in the first atomic bomb, whose energy was the equivalent of about 20 kilotons of TNT (1

kiloton of TNT releases 5×10^{12} J)? (b) What was the actual mass transformed to energy?

SOLUTION. (a) We find the number of fissions from

$$n = \frac{(20 \text{ kt})(5 \times 10^{12} \text{ J/kt})}{(200 \text{ MeV/fission})(1.60 \times 10^{-13} \text{ J/MeV})} = 3.13 \times 10^{24} \text{ fissions.}$$

Each fission uses one uranium atom, so the required mass is

$$M = \frac{3.13 \times 10^{24} \text{ atoms}}{6.02 \times 10^{23} \text{ atoms/mol}}(235 \text{ g/mol}) = 1.2 \times 10^{3} \text{ g} = 1.2 \text{ kg.}$$

(b) We find the mass transformation from

$$m = \frac{E}{c^2} = \frac{(20 \text{ kt})(5 \times 10^{12} \text{ J/kt})}{(3.00 \times 10^{8} \text{ m/s})^2} = 1.1 \times 10^{-3} \text{ kg} = 1.1 \text{ g.}$$

7–59. In a certain town the average yearly background radiation consists of 21 mrad of X-rays and γ rays plus 3.0 mrad of particles having a QF of 10. How many rem will a person receive per year on the average?

SOLUTION. The effective dose in rem = effective dose (rad) $\times$ QF. For the two radiations, we get

$$\begin{aligned} \text{dose (rem)} &= \sum \text{dose (rad)} \times \text{QF} \\ &= (21 \text{ mrad/yr})(1) + (3.0 \text{ mrad/yr})(10) \\ &= 51 \text{ mrem/yr.} \end{aligned}$$

7–60. Deuterium makes up 0.015 percent of natural hydrogen. Make a rough estimate of the total deuterium in the Earth's oceans and estimate the total energy released if all of it were used in fusion reactors.

SOLUTION. If we assume the oceans cover 70% of the Earth's surface and the average depth is 3 km, the mass of the ocean water is

$$\begin{aligned} M &= \rho V = (1\,000 \text{ kg/m}^3)(0.70)4\pi(6.38 \times 10^{6} \text{ m})^2(3 \times 10^{3} \text{ m}) \\ &= 1 \times 10^{21} \text{ kg.} \end{aligned}$$

Each water molecule has two H atoms, so the number of deuterium atoms is

$$\begin{aligned} N &= \frac{(1\times10^{21}\text{ kg})(10^3\text{ g/kg})}{18\text{ g/mol}}(6.02\times10^{23}\text{ atoms/mol})(2) \\ &\quad\times(0.015\times10^{-2}) \\ &= 1\times10^{43}\text{ atoms.} \end{aligned}$$

The mass of deuterium is

$$m=\frac{(1\times10^{43}\text{ atoms})(2\text{ g/mol})}{6.02\times10^{23}\text{ atoms/mol}}=3\times10^{19}\text{ g}=3\times10^{16}\text{ kg.}$$

If we assume equal contributions from the reactions in Eq. 7.9a and Eq. 7.9b, the average energy released by a deuterium nucleus is

$$\frac{4.03\text{ MeV}+3.27\text{ MeV}}{2+2}=1.8\text{ MeV/nucleus.}$$

Thus the total energy released is

$$\begin{aligned} E &= (1.8\text{ MeV/nucleus})(1\times10^{43}\text{ nuclei})=1.8\times10^{43}\text{ MeV} \\ &= 3\times10^{30}\text{ J.} \end{aligned}$$

7–61. A shielded γ-ray source yields a dose rate of 0.052 rad/h at a distance of 1.0 m for an average-sized person. If workers are allowed a maximum dose rate of 5.0 rem/yr, how close to the source may they operate, assuming a 40-h work week? Assume that the intensity of radiation falls off as the square of the distance. (It actually falls off more rapidly than r^{-2} because of absorption in the air, so the answer above will give a better-than-permissible value.)

SOLUTION. Because the QF for gamma rays is 1, the dose in rem is the dose in rad. The allowed dose rate is

$$\frac{5.0\text{ rem/yr}}{(50\text{ wk/yr})(40\text{ h/wk})}=2.5\times10^{-3}\text{ rem/h}=2.5\times10^{-3}\text{ rad/h.}$$

If the dose rate falls off as the square of the distance, we form the ratio:

$$\begin{aligned}\frac{(\text{dose rate})_2}{(\text{dose rate})_1} &= \left(\frac{r_1}{r_2}\right)^2;\\ \frac{2.5\times10^{-3}\text{ rad/h}}{0.052\text{ rad/h}} &= \left(\frac{1.0\text{ m}}{r_2}\right)^2,\end{aligned}$$

which gives $r_2 = 4.6$ m.

7–62. Radon gas, $^{222}_{86}$Rn, is formed by α decay. (a) Write the decay equation, (b) Ignoring the kinetic energy of the daughter nucleus (it's so massive), estimate the kinetic energy of the α particle produced. (c) Estimate the momentum of the α and of the daughter nucleus. (d) Estimate the kinetic energy of the daughter, and show that your approximation in (b) was valid.

SOLUTION. (a) $^{226}_{88}\text{Ra} \rightarrow {}^{222}_{86}\text{Rn} + {}^{4}_{2}\text{He}$.

(b) If we ignore the K of the daughter, the K of the α particle is the Q-value:

$$\begin{aligned}K_\alpha &= Q = \left(M(^{226}\text{Ra}) - M(^{222}\text{Rn}) - M(^4\text{He})\right)c^2\\ &= ((226.025402\text{ u}) - (222.017570\text{ u}) - (4.002603\text{ u}))\\ &\quad \times c^2(931.5\text{ MeV/u}c^2)\\ &= 4.871\text{ MeV}.\end{aligned}$$

(c) From momentum conservation, the momenta of the α particle and the daughter will have equal magnitudes:

$$\begin{aligned}p_{\text{Rn}} &= p_\alpha = \sqrt{2m_\alpha K_\alpha}\\ &= \sqrt{2(4\text{ u})(1.66\times10^{-27}\text{ kg/u})(4.87\text{ MeV})(1.6\times10^{-13}\text{ J/MeV})}\\ &= 1.02\times10^{-19}\text{ kg}\cdot\text{m/s}.\end{aligned}$$

(d) The kinetic energy of the daughter is

$$\begin{aligned}K_{\text{Rn}} &= \frac{p_{\text{Rn}}^2}{2m_{\text{Rn}}} = \frac{(1.02\times10^{-19}\text{ kg}\cdot\text{m/s})^2}{2(222\text{ u})(1.66\times10^{-27}\text{ kg/u})} = 1.4\times10^{-14}\text{ J}\\ &= 0.088\text{ MeV}.\end{aligned}$$

Because this is less than 2% of the Q-value, our approximation is valid.

7–63. The reaction $^{18}_{8}\mathrm{O}\,(\mathrm{p},\,\mathrm{n})\,^{18}_{9}\mathrm{F}$ requires an input of energy equal to 2.453 MeV. What is the mass of $^{18}_{9}\mathrm{F}$?

SOLUTION. Because the reaction $^{18}_{8}\mathrm{O}\,(\mathrm{p},\mathrm{n})\,^{18}_{9}\mathrm{F}$ requires an input of energy, the Q-value is negative:

$$\begin{aligned} Q &= \left(M(^{18}\mathrm{O}) + M(^{1}\mathrm{H}) - m(\mathrm{n}) - M(^{18}\mathrm{F})\right)c^2; \\ -2.453\ \mathrm{MeV} &= ((17.999160\ \mathrm{u}) + (1.007825\ \mathrm{u}) - (1.008665\ \mathrm{u}) \\ &\quad - M(^{18}\mathrm{F}))\,c^2(931.5\ \mathrm{MeV/u}c^2), \end{aligned}$$

which gives $M(^{18}\mathrm{F}) = 18.000953$ u.

7–64. Consider a system of nuclear power plants that produce 3 400 MW. (a) What total mass of $^{235}_{92}\mathrm{U}$ fuel would be required to operate these plants for 1 yr, assuming that 200 MeV is released per fission? (b) Typically 6 percent of the $^{235}_{92}\mathrm{U}$ nuclei that fission produce $^{90}_{38}\mathrm{Sr}$, a β^- emitter with a half-life of 29 yr. What is the total radioactivity of the $^{90}_{38}\mathrm{Sr}$, in curies, produced in 1 yr? (Neglect the fact that some of it decays during the 1-yr period.)

SOLUTION. (a) We find the number of fissions from

$$\begin{aligned} P &= \frac{E}{t}; \\ 3\,400\ \mathrm{MW} &= \frac{(200\ \mathrm{MeV})(1.60\times10^{-19}\ \mathrm{J/eV})n}{3.16\times10^{7}\ \mathrm{s}}, \end{aligned}$$

which gives $n = 3.36\times10^{27}$ fissions.

Each fission uses one uranium atom, so the required mass is

$$m = \frac{3.36\times10^{27}\ \mathrm{atoms}}{6.02\times10^{23}\ \mathrm{atoms/mol}}(235\ \mathrm{g/mol}) = 1.31\times10^{6}\ \mathrm{g} = 1.31\times10^{3}\ \mathrm{kg}.$$

(b) We find the activity from the number of Sr atoms produced:

$$\begin{aligned} \text{Activity} &= \lambda N_{\mathrm{Sr}} = \frac{0.693}{(29\ \mathrm{yr})(3.16\times10^{7}\ \mathrm{s/yr})}\,\frac{(0.06)(3.36\times10^{27})}{3.7\times10^{10}\ \mathrm{decays/s\cdot Ci}} \\ &= 4.1\times10^{6}\ \mathrm{Ci}. \end{aligned}$$

7–65. In the net reaction, Eq. 7.7, for the proton-proton cycle in the Sun, the neutrinos escape from the Sun with energy of about 0.5 MeV.

The remaining energy, 26. 2 MeV, is available within the Sun. Use this value to calculate the "heat of combustion" per kilogram of hydrogen fuel and compare it to the heat of combustion of coal, about 3×10^7 J/kg.

SOLUTION. In the net reaction four protons produce 26. 2 MeV. Thus the heat of combustion is

$$\frac{26.2 \text{ MeV}}{4 \text{ u}} \frac{1.6 \times 10^{-13} \text{ J/MeV}}{1.66 \times 10^{-27} \text{ kg/u}} = 6.31 \times 10^{14} \text{ J/kg}.$$

This is $\approx 10^7\times$ the heat of combustion of coal.

7–66. Energy reaches the Earth from the Sun at a rate of about 1 400 W/m^2. Calculate (a) the total energy output of the Sun, and (b) the number of protons consumed per second in the reaction of Eq. 7.7, assuming that this is the source of all the Sun's energy. (c) Assuming that the Sun's mass of 2.0×10^{30} kg was originally all protons and that all could be involved in nuclear reactions in the Sun's core, how long would you expect the Sun to "glow" at its present rate?

SOLUTION. (a) The energy is radiated uniformly over a sphere, so the total energy rate is

$$P = (1\,400 \text{ W/m}^2)4\pi(1.50\times10^{11} \text{ m})^2 = 4.0\times10^{26} \text{ J/s} = 2.5\times10^{39} \text{ MeV/s}.$$

(b) In the net reaction four protons produce 26. 2 MeV. (The neutrinos escape with 0. 5 MeV.) Thus the consumption of protons is

$$n = \frac{2.5 \times 10^{39} \text{ MeV/s}}{(26.2 \text{ MeV})/(4 \text{ protons})} = 3.8 \times 10^{38} \text{ protons/s}.$$

(c) We find the time from

$$t = \frac{N}{n} = \frac{(2.0 \times 10^{30} \text{ kg})/(1.66 \times 10^{-27} \text{ kg/proton})}{3.8 \times 10^{38} \text{ protons/s}} = 3.2 \times 10^{18} \text{ s} \approx 10^{11} \text{ yr}.$$

7–67. Estimate how many solar neutrinos pass through a 100 m^2 ceiling of a room, at latitude 40°, in a year. [*Hint*: See Problems 7–65 and 7–66.]

SOLUTION. In the net proton-proton cycle, four protons produce two neutrinos. If we use the result from Problem 7–66 for the rate at which

protons are consumed, for the rate at which neutrinos are produced we have

$$\begin{aligned} n_\nu &= (3.8 \times 10^{38} \text{ protons/s}) \left(\frac{2 \text{ neutrinos}}{4 \text{ protons}} \right) (3.16 \times 10^7 \text{ s/yr}) \\ &= 6.0 \times 10^{45} \text{ neutrinos/yr}. \end{aligned}$$

The neutrinos are spread uniformly over a sphere centered at the Sun, so the number that would pass through an area of 100 m^2 at the Earth that is always perpendicular to the neutrino flux is

$$N_{\nu 0} = \frac{6.0 \times 10^{45} \text{ neutrinos/yr}}{4\pi (1.50 \times 10^{11} \text{ m})^2} (100 \text{ m}^2) = 2.1 \times 10^{24} \text{ neutrinos/yr}.$$

At a latitude of 40° the ceiling is not always perpendicular to the neutrino flux, so the number is reduced by a factor of $\cos 40°$, which we assume as an average value for the annual variation in the elevation of the Sun. During the daily rotation of the Earth, the neutrino flux will vary sinusoidally from zero to the maximum. (We take both directions through the ceiling as positive.) To average this variation, we use a factor of $\cos 45°$, so we have

$$\begin{aligned} N_\nu &= N_{\nu 0} \cos 40° \cos 45° = (2.1 \times 10^{24} \text{ neutrinos/yr}) \cos 40° \cos 45° \\ &\approx 1 \times 10^{24} \text{ neutrinos/yr}. \end{aligned}$$

7–68. Show, using the laws of conservation of energy and momentum, that for a nuclear reaction requiring energy, the minimum kinetic energy of the bombarding particle (the *threshold energy*) is equal to

$$-\frac{Q m_{\text{pr}}}{m_{\text{pr}} - m_{\text{b}}},$$

where $-Q$ is the energy required (difference in total mass between products and reactants), m_{b} is the rest mass of the bombarding particle, and m_{pr} the total rest mass of the products. Assume the target nucleus is at rest before an interaction takes place, and that all particles are nonrelativistic.

Solution. The kinetic energy of the products is

$$K_{\text{pr}} = K_{\text{b}} + Q.$$

Because the kinetic energies $\ll mc^2$, we can use a non relativistic treatment:

$$K = \frac{mv^2}{2} = \frac{p^2}{2m}.$$

The least kinetic energy is required when the product particles move together with the same speed. With the target at rest, for momentum conservation we have

$$p_{\mathrm{b}} = p_{\mathrm{pr}} = m_{\mathrm{pr}} v,$$

or

$$K_{\mathrm{b}} = \frac{p_{\mathrm{b}}^2}{2m_{\mathrm{b}}} = \frac{m_{\mathrm{pr}}^2}{2m_{\mathrm{b}}} v^2 = \frac{m_{\mathrm{pr}}}{m_{\mathrm{b}}} K_{\mathrm{pr}},$$

or

$$K_{\mathrm{pr}} = \frac{m_{\mathrm{b}}}{m_{\mathrm{pr}}} K_{\mathrm{b}}.$$

When we use this in the kinetic energy equation, we get

$$\begin{aligned} \frac{m_{\mathrm{b}}}{m_{\mathrm{pr}}} K_{\mathrm{b}} &= K_{\mathrm{b}} + Q; \\ \left(\frac{m_{\mathrm{b}}}{m_{\mathrm{pr}}} - 1 \right) K_{\mathrm{b}} &= Q, \end{aligned}$$

which gives

$$K_{\mathrm{b}} = -\frac{Q m_{\mathrm{pr}}}{m_{\mathrm{pr}} - m_{\mathrm{b}}}.$$

7–69. The early scattering experiments performed around 1910 in Ernest Rutherford's laboratory in England produced the first evidence that an atom consists of a heavy nucleus surrounded by electrons. In one of these experiments, α particles struck a gold-foil target 4.0×10^{-5} cm thick in which there were 5.9×10^{28} gold atoms per cubic meter. Although most α particles either passed straight through the foil or were scattered at small angles, approximately 1.6×10^{-3} percent were scattered at angles greater than 90°—that is, in the backward direction. (a) Calculate the cross section, in barns, for backward scattering. (b) Rutherford concluded that such backward scattering could occur only if an atom consisted of a very tiny, massive, and positively charged nucleus with electrons orbiting

some distance away. Assuming that backward scattering occurs for nearly direct collisions (i. e., $\sigma \approx$ area of nucleus), estimate the diameter of a gold nucleus.

SOLUTION. (a) If we assume a thin target, we find the cross section for backward scattering from

$$\begin{aligned} \frac{R}{R_0} &= n\sigma x; \\ 1.6 \times 10^{-5} &= (5.9 \times 10^{28}\ \mathrm{m}^{-3})\sigma(4.0 \times 10^{-7}\ \mathrm{m}), \end{aligned}$$

which gives

$$\sigma = 6.8 \times 10^{-28}\ \mathrm{m}^2 = 6.8\ \mathrm{bn}.$$

(b) If we assume the cross section is the area presented by the nucleus, we have

$$\sigma = 6.8 \times 10^{-28}\ \mathrm{m}^2 = \pi r^2,$$

which gives $r = 1.5 \times 10^{-14}$ m, so the diameter is 3×10^{-14} m.

7–70. Some stars, in a later stage of evolution, may begin to fuse two ${}^{12}_{6}\mathrm{C}$ nuclei into one ${}^{24}_{12}\mathrm{Mg}$ nucleus. (a) How much energy would be released in such a reaction? (b) What kinetic energy must two carbon nuclei each have when far apart, if they can then approach each other to within 6.0 fm, center-to-center? (c) Approximately what temperature would this require?

SOLUTION. (a) For the reaction

$${}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \rightarrow {}^{24}_{12}\mathrm{Mg},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{12}\mathrm{C}) + M({}^{12}\mathrm{C}) - M({}^{24}\mathrm{Mg})\right)c^2 \\ &= (2(12.000000\ \mathrm{u}) - (23.985042\ \mathrm{u}))\,c^2(931.5\ \mathrm{MeV/uc^2}) \\ &= +13.93\ \mathrm{MeV}. \end{aligned}$$

Thus 13.93 MeV is released.

(b) If the two nuclei are just touching, the Coulomb potential energy must be the initial kinetic energies of the two nuclei:

$$2K = U = k_{\mathrm{e}}\frac{Z_{\mathrm{C}}Z_{\mathrm{C}}e^2}{R} = \frac{(6)(6)(1.44\ \mathrm{MeV \cdot fm})}{6.0\ \mathrm{fm}} = 8.6\ \mathrm{MeV}.$$

Thus each carbon nucleus has a kinetic energy of 4.3 MeV.

(c) We find the temperature from

$$\begin{aligned} K &= \tfrac{3}{2}k_{\mathrm{B}}T; \\ (4.3\text{ MeV})(1.60\times10^{-13}\text{ J/MeV}) &= \tfrac{3}{2}(1.38\times10^{-23}\text{ J/K})T, \end{aligned}$$

which gives $T = 3\times10^{10}$ K.

7–71. An average adult body contains about 0.10 Ci of $^{40}_{19}$K, which comes from food. (a) How many decays occur per second? (b) The potassium decays produce β particles with energies of around 1.4 MeV. Calculate the dose per year in sieverts for a 50-kg adult. Is this a significant fraction of the 3.6 mSv/year background rate?

SOLUTION. (a) The rate of decay is

$$\text{Activity} = (0.10\times10^{-6}\text{ Ci})(3.7\times10^{10}\text{ decays/s}\cdot\text{Ci}) = 3.7\times10^{3}\text{ decays/s}.$$

(b) Because the QF for beta particles is 1, the dose in Sv is the dose in Gy:

$$\begin{aligned} \text{dose rate} &= \frac{(3.7\times10^{3}\text{ decays/s})(1.4\text{ MeV/decay})}{(50\text{ kg})(1.00\text{ J/kg}\cdot\text{Gy})} \\ &\quad\times(3.16\times10^{7}\text{ s/yr})(1.6\times10^{-13}\text{ J/MeV}) \\ &= 5.2\times10^{-4}\text{ Sv/yr} \approx 0.15\text{ background}. \end{aligned}$$

7–72. When the nuclear reactor accident occurred at Chernobyl in 1986, 2.0×10^{7} Ci were released into the atmosphere. Assuming that this radiation was distributed uniformly over the surface of the Earth, what was the activity per square meter? (The actual activity was not uniform; even within Europe wet areas received more radioactivity from rainfall).

SOLUTION. The activity released into the atmosphere was

$$\text{activity} = (2.0\times10^{7}\text{ Ci})(3.70\times10^{10}\text{ Bq/Ci}) = 7.4\times10^{17}\text{ Bq}.$$

If we assume uniform spreading over the surface of the Earth, we have

$$\text{activity/m}^2 = \frac{7.4\times10^{17}\text{ Bq}}{4\pi(6.38\times10^{6}\text{ m})^2} = 1.45\times10^{3}\text{ Bq/m}^2.$$

8 Elementary Particles

8.1 Review

Particle accelerators are used to accelerate charged particles, such as electrons and protons, to very high energy. High-energy particles have short wavelength and so can be used to probe the structure of matter at very small distances in great detail. The wavelength of projectile particles is given by de Broglie's wavelength formula (Eq. 2.6),

$$\lambda = \frac{2\pi\hbar}{p}. \tag{8.1}$$

Example 8–1. What is the wavelength, and hence the expected resolution, for a beam of 1.3-GeV electrons?

SOLUTION. The 1.3 GeV = 1 300 MeV refers to the kinetic energy, and is about 2 500 times the rest energy (mass) of the electron ($mc^2 = 0.51$ MeV). We are clearly dealing with relativistic speeds here, and it is easily shown (see Eqs. 1.11 and 1.13) that the speed of the electron is nearly $c = 3.0 \times 10^8$ m/s. Therefore

$$K = (\gamma - 1)mc^2 \approx \gamma(mc^2) = 1\,300 \text{ MeV},$$

and

$$\begin{aligned} \lambda &= \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\gamma mv} \approx \frac{2\pi\hbar}{\gamma mc} = \frac{2\pi\hbar c}{\gamma mc^2} \approx \frac{2\pi\hbar c}{K} \\ &= \frac{(6.6 \times 10^{-34} \text{ J}\cdot\text{s})(3.0 \times 10^8 \text{ m/s})}{(1.3 \times 10^9 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})} = 0.96 \times 10^{-15} \text{ m} = 0.96 \text{ fm}. \end{aligned}$$

The maximum possible resolution of this beam of electrons is far greater than for a light beam in a light microscope ($\lambda \approx 500$ nm). Indeed, this resolution of about 1 fm is on the order of the size of nuclei (see Eq. 6.1).

High kinetic energy also allows the creation of new particles through collision (via $E = mc^2$).

Van De Graaff and linear accelerators use high voltage to accelerate particles along a line. Cyclotrons and synchrotrons use a magnetic field to keep the particles in a circular path and accelerate them at intervals by

high voltage. In a cyclotron (Fig. 8.1, charged particles starting at point A with some initial velocity travel in circular orbits in the magnetic field **B**. The particles are accelerated to higher speeds each time they pass in the gap between the metal "dees," where there is an electric field **E**. (There is no electric field within the cavity of the metal dees.) The electric filed changes direction each half-cycle, owing to an ac voltage $V = V_0 \sin(2\pi f t)$, so that the particles are increased in speed each passage through the gap. It can be shown that the frequency f of the voltage must be

$$f = \frac{qB}{2\pi m}, \tag{8.2}$$

where q and m are the charge and mass of the particles moving in the magnetic field B. This is called the *cyclotron frequency*. The kinetic energy of the particles increases by $2qV_0$ each revolution, assuming that the gap is small.

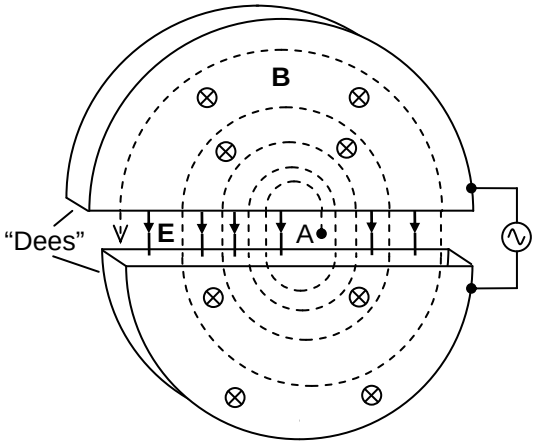


Figure 8.1. Diagram of a cyclotron. The magnetic field, applied by a large electromagnet, points into the page. A is the ion source. The field lines shown are for the electric field in the gap.

Example 8–2. A small cyclotron of maximum radius $R = 0.25$ m accelerates protons in a 1.7-T magnetic field. Calculate (a) what frequency

is needed for the applied alternating voltage, and (b) the kinetic energy of protons when they leave the cyclotron.

SOLUTION. (a) From Eq. 8.2,

$$f = \frac{qB}{2\pi m} = \frac{1.6 \times 10^{-19}\ \text{C})(1.7\ \text{T})}{(6.28)(1.67 \times 10^{-27}\ \text{kg})} = 2.6 \times 10^{7}\ \text{Hz} = 26\ \text{MHz},$$

which is in the radio wave region of the EM spectrum.

(b) The protons leave the cyclotron at $r = R = 0.25$ m. From $F = mg$, we have $qvB = mv^2/r$, so $v = qBr/m$, and the kinetic energy is

$$\begin{aligned} K &= \frac{1}{2}mv^2 = \frac{1}{2}m\frac{q^2B^2R^2}{m^2} = \frac{q^2B^2R^2}{2m} \\ &= \frac{(1.6 \times 10^{-19}\ \text{C})^2(1.7\ \text{T})^2(0.25\ \text{m})^2}{(2)(1.67 \times 10^{-27}\ \text{kg})} = 1.4 \times 10^{-2}\ \text{J} = 8.7\ \text{MeV}. \end{aligned}$$

Note that the magnitude of the voltage applied to the dees does affect the final energy. But the higher this voltage, the fewer revolutions are required to bring the protons to full energy.

Originally, high-energy physics experiments were carried out by allow of particles from an accelerator to strike a stationary target. A newer, very powerful technique of *colliding beams* allows for higher interaction energy.

Example 8–3. Determine the energy required to accelerate a proton in a high-energy accelerator (a) from rest to $v = 0.900\,c$, and (b) from $v = 0.900\,c$ to $v = 0.999\,c$. (c) What is the kinetic energy achieved by each proton?

SOLUTION. We use the work-energy principle, which is still valid relativistically as we saw in Chapter 1: $W = \Delta K$. The kinetic energy of a proton is given by Eq. 1.11

$$K = (\gamma - 1)mc^2,$$

where the relativistic factor γ is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \left(\beta = \frac{v}{c}\right).$$

The work-energy theorem becomes

$$W = \Delta K = (\Delta\gamma)mc^2$$

since m (the rest mass) and c are constant.

(a) For $v = 0$, $\gamma = 1$; and for $v = 0.900\,c$

$$\gamma = \frac{1}{\sqrt{1-(0.900)^2}} = 2.29.$$

Hence, with rest mass of the proton $mc^2 = 938$ MeV, the work needed is

$$W = \Delta K = (\Delta\gamma)mc^2 = (2.29 - 1.00)(938\text{ MeV}) = 1.21\text{ GeV}.$$

(b) For $v = 0.999\,c$,

$$\gamma = \frac{1}{\sqrt{1-(0.999)^2}} = 22.4.$$

So the work needed to accelerate a proton from $0.900\,c$ to $0.999\,c$ is

$$W = \Delta K = (\Delta\gamma)mc^2 = (22.4 - 2.29)(938\text{ MeV}) = 18.9\text{ GeV}.$$

(c) The kinetic energy reached by the proton in (a) is just equal to the work done on it, $K = 1.21$ GeV. The final kinetic energy of the proton in (b), moving at $v = 0.999\,c$, is

$$K = (\gamma - 1)mc^2 = (21.4)(938\text{ MeV}) = 20.1\text{ GeV},$$

which makes sense since, starting from rest, we did work $W = 1.21$ GeV + 18.9 GeV = 20.1 GeV on it.

Particle *detectors* that can image particle tracks include photographic emulsions, cloud chambers, bubble chambers, and today wire drift chambers.

For the electromagnetic force, it is photons that are exchanged between two charged particles that give rise to the force between them. A simple diagram describing this photon exchange is shown in Fig. 8.2. Such a diagram, called a *Feynman diagram*, is based on the theory of *quantum electrodynamics* (QED). Figure 8.2 represents the simplest case, in which a single photon is exchanged. One of the charged particles emits the photon and recoils somewhat as a result; and the second particle absorbs the photon. In any collision or interaction, energy and momentum are transferred from one charged particle to the other, carried by the photon. Because the

photon is absorbed by the second particle very shortly after it is emitted by the first, it is not observable, and is referred to as a *virtual* photon, in contrast to one that is free and can be detected by instruments. The photon is said to *mediate*, or *carry*, the electromagnetic force.

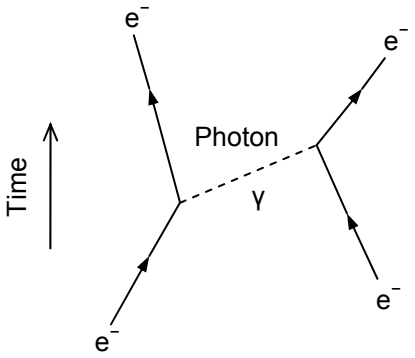


Figure 8.2. Feynman diagram showing a photon acting as the carrier of the electromagnetic force between two electrons. This is sort of an x vs. t graph, with t increasing upward. Starting at the bottom, two electrons approach each other (the distance between them decreases in time). As they get close, momentum and energy get transferred from one to the other, carried by a photon (or, perhaps, by more than one), and the two electrons bounce apart.

By analogy with photon exchange that mediates the electromagnetic force Hideki Yukawa argued in his early theory (1935) that there ought to be a particle that mediates the strong nuclear force—the force that holds nucleons together in the nucleus. Yukawa called this predicted particle a *meson*. We can make a rough estimate of the mass of the meson for a proton-neutron interaction as follows. Suppose the proton is at rest. For it to emit a meson would require energy (to make the meson's mass) that would have to come from nowhere, which would violate conservation of energy. But the uncertainty principle allows nonconservation of energy by an amount ΔE if it occurs only for a time Δt given by $(\Delta E)(\Delta t) \approx \hbar/2$. We set ΔE equal to the energy needed to create the mass m of the meson: $\Delta E = mc^2$. Conservation of energy is violated only as long as the meson exists, which is the time Δt required for the meson to pass from one nucleon

to the other. If we assume the meson travels at relativistic speed, close to the speed of light c, then Δt need be at most about $\Delta t = d/c$, where d is the maximum distance that can separate the interacting nucleons. Thus we can write

$$\begin{aligned} \Delta E\,\Delta t &\approx \frac{\hbar}{2} \\ mc^2\left(\frac{d}{c}\right) &\approx \frac{\hbar}{2}, \quad \text{or} \\ mc^2 &\approx \frac{\hbar c}{2d}\,. \end{aligned} \tag{8.3}$$

The range of the strong nuclear force (the maximum distance away it can be felt is small—not much more than the size of a nucleon or small nucleus (see Eq. 6.1)—so let us take $d \approx 1 \times 10^{-15}$ m. Then from Eq. 8.3,

$$mc^2 \approx \frac{\hbar c}{2d} = \frac{(0.658 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.0 \times 10^{8}\ \text{m/s})}{(2)(1 \times 10^{-15}\ \text{m})} \approx 100\ \text{MeV}.$$

The mass of the predicted meson is roughly 100 MeV/c^2 or about 200 times the electron mass of 0.51 MeV/c^2. [Note, incidentally, that since the electromagnetic force has infinite range ($d = \infty$), Eq. 8.3 tells us that the exchanged particle for the electromagnetic force, the photon, will have zero rest mass.]

The particle predicted by Yukawa was discovered in cosmic rays in 1947. It is called the π meson, or the *pion*. It comes in three charge states: +, −, or 0. The π^+ and π^- have mass of 139.6 MeV/c^2 and the π^0 a mass of 135.0 MeV/c^2. All three interact strongly with matter. Reactions observed in the laboratory, using a particle accelerator, include

$$\begin{aligned} \mathrm{p} + \mathrm{p} &\rightarrow \mathrm{p} + \mathrm{p} + \pi^0, \\ \mathrm{p} + \mathrm{p} &\rightarrow \mathrm{p} + \mathrm{n} + \pi^+. \end{aligned} \tag{8.4}$$

The incident proton from the accelerator must have sufficient energy to produce the additional mass of the free pion.

Yukawa's theory of pion exchange as carrier of the strong force is now out of date, and has been replaced by *quantum chromodynamics* in which the basic entities are *quarks*, and between them are exchanged *gluons* as the

basic carriers of the strong force. But the basic idea of the earlier theory, that forces can be understood as the exchange of particles, remains valid, as does Eq. 8.3 for estimating the mass of exchange particles.

An *antiparticle* has the same mass as a particle but opposite charge. Certain other properties may also be opposite: for example, the antiproton has *baryon number* (nucleon number) opposite to that for the proton.

In all nuclear and particle reactions, the following conservation laws hold: momentum, angular momentum, mass-energy, electric charge, baryon number, and the three *lepton numbers*.

Conservation laws are ordering principles: they help to explain why some reactions occur and others do not. For example, the following reaction has never been found to occur:

$$\mathrm{p} + \mathrm{n} \not\rightarrow \mathrm{p} + \mathrm{p} + \tilde{\mathrm{p}}$$

even though charge, energy, and so on, are conserved ($\tilde{\mathrm{p}}$ means an antiproton). To understand why such a reaction does not occur, physicists hypothesized the conservation of baryon number. An important addition to this law is the proposal that whereas all nucleons have baryon number $B = +1$, all antinucleons (antiprotons, antineutrons) have $B = -1$. The reaction above does not conserve baryon number since on the left side we have $B = (+1)+(+1) = +2$, and on the right $B = (+1)+(+1)+(-1) = +1$. On the other hand, the following reaction does conserve B and does occur if the incoming proton has sufficient energy:

$$\begin{aligned} \mathrm{p} + \mathrm{p} \quad &\rightarrow \quad \mathrm{p} + \mathrm{p} + \tilde{\mathrm{p}} + \mathrm{p}, \\ B = +1 + 1 \quad &= \quad +1 + 1 - 1 + 1. \end{aligned}$$

As indicated, $B = +2$ on both sides of this equation. From these and other reactions, the conservation of baryon number has been established as a basic law of physics.

Also useful are the conservation laws for the three lepton numbers, associated with weak interactions including decays. In ordinary β decay, an electron or positron is emitted along with a neutrino or antineutrino. In a similar type of decay, a particle known as a *muon* can be emitted instead of an electron. A muon seems to be much like an electron, except its mass is 207 times as large (106 MeV/c^2). The neutrino (ν_e) that accompanies

an emitted electron is found to be different from the neutrino (ν_μ) that accompanies an emitted muon. Each of these neutrinos has an antiparticle: $\tilde{\nu}_e$ and $\tilde{\nu}_\mu$. In ordinary β decay we have, for example,

$$n \rightarrow p + e^- + \tilde{\nu}_e$$

but never $n \rightarrow p + e^- + \tilde{\nu}_\mu$ nor $n \rightarrow p + e^- + \tilde{\nu}_e + \nu_e$. To explain why these do not occur, the concept of electron lepton number, L_e, was invented. If the electron (e^-) and the electron neutrino (ν_e) are assigned $L_e = +1$, and e^+ and $\tilde{\nu}_e$ are assigned $L_e = -1$, whereas all other particles have $L_e = 0$, then all observed decays conserve L_e. For example, in $n \rightarrow p + e^- + \tilde{\nu}_e$, $L_e = 0$ initially, and $L_e = 0 + (+1) + (-1) = 0$ after the decay. Decays that do not conserve L_e but would obey the other conservation laws are not observed to occur. Hence it is believed that L_e is conserved in all interactions.

In a decay involving muons, such as

$$\pi^+ \rightarrow \mu^+ + \nu_\mu,$$

a second quantum number, muon lepton number (L_μ), is conserved. The μ^- and ν_μ are assigned $L_\mu = +1$, and μ^+ and $\tilde{\nu}_\mu$ have $L_\mu = -1$, whereas other particles have $L_\mu = 0$. It is believed that L_μ is also conserved in all interactions or decays. Similar assignments can be made for a third lepton number, L_τ, associated with the more recently discovered τ lepton and its neutrino, ν_τ.

Keep in mind that antiparticles have not only opposite electric charge from their particles, but also opposite B, L_e, L_μ, and L_τ.

Example 8–4. Which of the following decay schemes is possible for muon decay: (a) $\mu^- \rightarrow e^- + \tilde{\nu}_e$; (b) $\mu^- \rightarrow e^- + \tilde{\nu}_e + \nu_\mu$; (c) $\mu^- \rightarrow e^- + \nu_e$? All of these particles have $L_\tau = 0$.

SOLUTION. A μ^- has $L_\mu = +1$ and $L_e = 0$. This is the initial state, and the final state (after decay) must also have $L_\mu = +1$, $L_e = 0$.

In (a), the final state has $L_\mu = 0 + 0 = 0$, and $L_e = +1 - 1 = 0$; L_μ would not be conserved and indeed this decay is not observed to occur.

The final state of (b) has $L_\mu = 0 + 0 + 1 = +1$ and $L_e = +1 - 1 + 0 = 0$, so both L_μ and L_e are conserved. This is in fact the most common decay mode of the μ^-.

Finally, (c) does not occur because L_e (= +2 in the final state) is not conserved, nor is L_μ.

Example 8–5. In addition to the "number" conservation laws which help explain the decay schemes of particles, we can also apply the laws of conservation of energy and momentum. The decay of a Σ^+ particle with a rest mass of 1 189 MeV/c^2 (see Table B.1) commonly yields a proton (rest mass of 938 MeV/c^2) and a neutral pion, π^0 (rest mass of 135 MeV/c^2). What are the kinetic energies of the products, assuming the Σ^+ parent particle was at rest?

Solution. The energy released through the decay can be calculated from change in the rest energies, as we did for nuclear processes (Eq. 6.3 or 7.1):

$$Q = (m_{\Sigma^+} - (m_p + m_{\pi^0}))\, c^2 = (1\,189 - (938 + 135))\ \text{MeV} = 116\ \text{MeV}.$$

This energy Q becomes the kinetic energy of the resulting decay particles p and π^0:

$$Q = K_p + K_{\pi^0}$$

with each particle's kinetic energy related to its momentum by (Eqs. 1.12 and 1.13):

$$K_p = E_p - m_p c^2 = \sqrt{(p_p c)^2 + (m_p c^2)^2} - m_p c^2,$$

and similarly for the pion. From momentum conservation, the proton and pion have the same magnitude of momentum since the original particle was at rest: $p_p = p_{\pi^0} = p$. Then

$$\begin{aligned} Q = 116\ \text{MeV} \quad &= \quad \left(\sqrt{(pc)^2 + (938\ \text{MeV})^2} - 938\ \text{MeV}\right) \\ &\quad + \left(\sqrt{(pc)^2 + (135\ \text{MeV})^2} - 135\ \text{MeV}\right). \end{aligned}$$

We solved this for pc, which gives $pc = 189$ MeV. Substituting into the expression for the kinetic energy, first for the proton, then for the pion we obtain $K_p = 19$ MeV and $K_{\pi^0} = 97$ MeV.

It is useful to arrange the particles in categories according to their properties. One way of doing this is according to their interactions, since not all particles interact by means of all four of the forces known in nature

(though all interact via gravity). Table B.1 lists some of the more common particles classified in this way along many of their properties. The particles listed are those that are stable, and many that are unstable. At the top of the table are the *gauge bosons*, which include the photon, and the W and Z particles, that mediate the electromagnetic and weak interactions, respectively.

Other particles can be classified as either *leptons* or *hadrons*. Leptons participate in the weak and electromagnetic interactions. Hadrons, which today are considered to be made up of *quarks*, participate in the strong interaction as well. The hadrons can be classified as *mesons*, with baryon number zero, and *baryons*, with nonzero baryon number.

All particles, except for the photon, electron, neutrinos, and proton, decay with measurable half-lives varying from 10^{-25} s to 10^{3} s. The half-life depends on which force is predominant in the decay. Weak decays usually have half-lives greater than about 10^{-13} s. Electromagnetic decays have half-lives on the order of 10^{-16} to 10^{-19} s.

The shortest lived particles, called *resonances*, decay via the strong interaction and live typically for only about 10^{-23} s. In fact their lifetimes are so short that they do not travel far enough to be detected before decaying. Their decay products can be detected, however, and it is from them that the existence of such short-lived particles is inferred. To see how this is done, consider the Δ^{++} particle observed in a scattering of π^{+} mesons by protons. A graph of the integral cross section versus the pion's kinetic energy is shown in Fig. 8.3. The peak within 150–200 MeV was much higher than expected and certainly much than the number of interactions at neighboring energies. This led to conclusion that the π^{+} and proton combined momentarily to form a short-lived particle before coming apart again, or at least that they resonated together for a short time. Indeed, the large peak in Fig. 8.3 resembles a resonance curve, and that is why this new particle—now called the Δ^{++}—is referred to as a resonance. Hundreds of other resonances have been found in a similar way. Many resonances are regarded as excited states of other particles such as of the nucleon (proton or neutron).

Example 8–6. Use Fig. 8.3 to estimate a mass of the Δ resonance.

Solution. To see the processes in their simplest form kinematically it is convenient to transform to a coordinate frame K', where the projectile

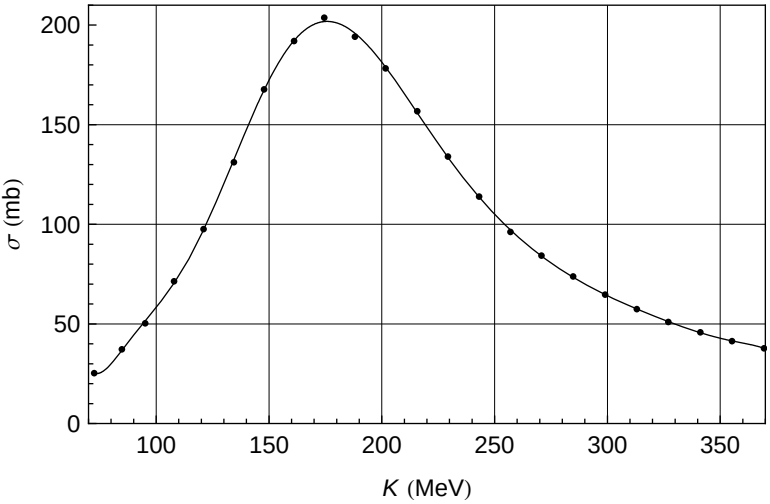


Figure 8.3. Integral cross section of π^+ particles scattered by a proton target as a function of the incident π^+ kinetic energy in the lab reference frame. The solid line is the least-square fit. The resonance shape represents the formation of a short-lived particle, the Δ^{++}, which has a charge in this case of $+2e$.

(π^+) and the target (p) have equal and oppositely directed momenta. This frame is called the *center of momentum* (CM) frame. The total energy E' of the system is conserved in the CM frame and is just the rest energy of the resonance particle. For the sake of brevity we temporarily set $c = 1$, so that the relativistic invariant $E^2 - p^2c^2 = m^2c^4$ becomes simply $E^2 - p^2 = m^2$. In the lab frame, before the collision, E_π and E_p are the respective total energies of the pion and proton, p_π and $p_\mathrm{p}(= 0)$ are the respective momenta of the pion and proton, and K_π is the kinetic energy of the incident pion. We have

$$\begin{aligned} E'^2 &= E^2 - p^2 = (E_\pi + E_\mathrm{p})^2 - (p_\pi + p_\mathrm{p})^2 \\ &= E_\pi^2 + E_\mathrm{p}^2 + 2E_\pi E_\mathrm{p} - p_\pi^2 - p_\mathrm{p}^2 - 2p_\pi p_\mathrm{p}\cos\theta \\ &= (E_\pi^2 - p_\pi^2) + (E_\mathrm{p}^2 - p_\mathrm{p}^2) + 2E_\pi E_\mathrm{p} - 2p_\pi p_\mathrm{p}\cos\theta \\ &= m_\pi^2 + m_\mathrm{p}^2 + 2(m_\pi + K_\pi)m_\mathrm{p} \\ &= (m_\pi + m_\mathrm{p})^2 + 2m_\mathrm{p}K_\pi. \end{aligned} \tag{8.5}$$

From Fig. 8.3 we see that the cross section peaks at about $K_\pi \approx 175$ MeV. From Table B.1 we find $m_\pi = 139.6$ MeV and $m_\mathrm{p} = 938.3$ MeV. Hence,

$$\begin{aligned} E'^2 &= \sqrt{(139.6\ \mathrm{MeV} + 938.3\ \mathrm{MeV})^2 + 2(938.3\ \mathrm{MeV})(175\ \mathrm{MeV})} \\ &\approx 1\,221\ \mathrm{MeV}. \end{aligned}$$

To let you to taste the flavor of the real particle physics, we would say that the rest energy of the Δ^{++} particle is not the same as the CM energy at which the cross section is a maximum. In fact, the cross section σ of the reaction involved is given by the Breit–Wigner formula

$$\sigma = \frac{8\pi}{k'^2_\pi} \frac{(\frac{1}{2}\Delta E')^2}{(E' - E'_0)^2 + (\frac{1}{2}\Delta E')^2}, \tag{8.6}$$

where $k'_\pi = p'_\pi/\hbar$ is the relativistic CM wavenumber of the incident pion. It can be shown that the CM momentum p'_π of the pion is related to its lab momentum p_π by the formula

$$p'^2_\pi = \frac{m^2_\mathrm{p}}{E'^2}\, p^2_\pi = \frac{K_\pi(K_\pi + 2m_\pi)m^2_\mathrm{p}}{(m_\pi + m_\mathrm{p})^2 + 2m_\mathrm{p}K_\pi}.$$

E'_0 is the energy at which $k'^2_\pi\sigma$ reaches a maximum, and $\Delta E'$ is the total width of the resonance. When data are plotted in such a way as to separate out the resonance behavior as suggested by Eq. 8.6, the peak does occur at 1 232 MeV. This is shown in Fig. 8.4, where $k'^2\sigma/8\pi$ is plotted versus total CM energy. Notice also that the maximum in this plot has a value of one.

The width of a resonance is an interesting application of the uncertainty principle. If a particle's mass (i. e., its rest energy) is uncertain by an amount ΔE, then its lifetime will be

$$\Delta t \approx \frac{\hbar}{2\,\Delta E}.$$

Certain particles have a property, called *strangeness*, which is conserved by the strong force but not by the weak force. Strange particles, K, Λ and Σ, behave rather strangely in two ways. First, they are always produced in pairs. For example, the reaction $\pi^- + \mathrm{p} \to \mathrm{K}^0 + \Lambda^0$ occurs with high

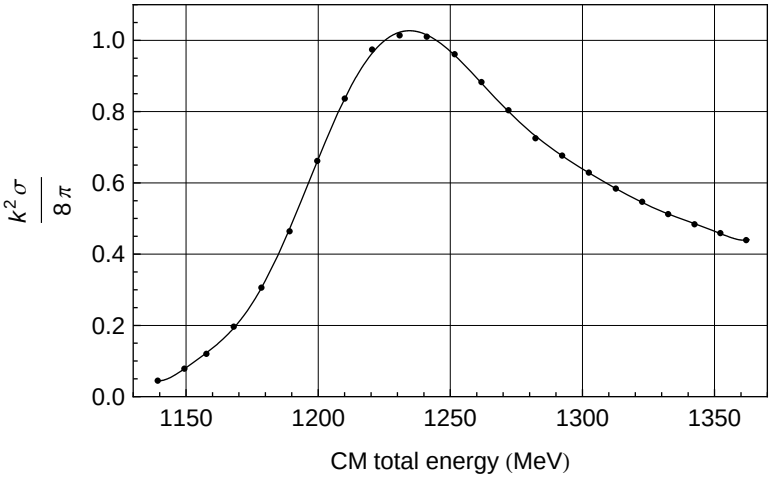


Figure 8.4. Integral cross section data of Fig. 8.3 plotted in such a way as to separate out the resonance behavior.

probability, but the reaction $\pi^- + \mathrm{p} \to \mathrm{K}^0 + \mathrm{n}$ is never observed to occur. The second feature of these particles is that, although they are clearly produced via the strong interaction (that is, at a high rate), they do not decay at a rate characteristic of the strong interaction even though they decay into strongly interacting particles (for example, $\mathrm{K} \to 2\,\pi$, $\Sigma^+ \to \mathrm{p} + \pi^0$). Instead of lifetimes of 10^{-23} s as expected for strongly interacting particles, strange particles have lifetimes of 10^{-10} to 10^{-8} s, which are characteristic of the weak interaction.

By assigning the strangeness numbers (S) indicated in Table B.1, the production of strange particles in pairs is readily explained. Antiparticles are assigned opposite strangeness from their particles: one of each pair is assigned $S = +1$ and the other $S = -1$ (see Table B.1). For example, in the reaction $\pi^- + \mathrm{p} \to \mathrm{K}^0 + \Lambda^0$, the initial state has strangeness $S = 0 + 0 = 0$, and the final state has $S = +1 - 1 = 0$, so strangeness is conserved. But for $\pi^- + \mathrm{p} \to \mathrm{K}^0 + \mathrm{n}$, the initial state has $S = 0$ and the final state has $S = +1 + 0 = +1$, so strangeness would not be conserved; and this reaction is not observed.

To explain the decay of strange particles, it is assumed that strangeness is conserved in the strong interaction but is *not* conserved in the weak interaction. Thus, although strange particles were forbidden by strangeness conservation to decay to nonstrange particles of lower mass via the strong interaction, they could undergo such decay by means of the weak interaction. This would occur much more slowly, of course, which accounts for their longer lifetimes of 10^{-10} to 10^{-8} s.

The conservation of strangeness was the first example of a "partially conserved" quantity. The more recently noted properties, *charm*, *bottomness*, and *topness*, also are believed to be conserved by the strong force but not by the weak.

Example 8–7. Using the conservation laws for particle interactions, determine the other particle as a result of the reaction

$$\pi^- + p \rightarrow \mathrm{K}^0 + \,?,$$

in addition to $\mathrm{K}^0 + \Lambda^0$.

SOLUTION. We write equations for the conserved numbers in this reaction, with B, L_e, S, and Q as unknowns whose determination will reveal what the possible particle might be:

$$\begin{array}{lrcl} \text{Baryon number:} & 0+1 & = & 0+B \\ \text{Lepton number:} & 0+0 & = & 0+L_\mathrm{e} \\ \text{Charge:} & -1+1 & = & 0+Q \\ \text{Strangeness:} & 0+0 & = & 1+S. \end{array}$$

The unknown product particle would have to have these characteristics:

$$B = +1 \qquad L_\mathrm{e} = 0 \qquad Q = 0 \qquad S = -1.$$

In addition to Λ^0, a neutral sigma particle, Σ^0, is also consistent with these numbers.

Today's standard model of elementary particles considers *quarks* as the basic building blocks of the hadrons. The six quarks are called *up*, *down*, *strange*, *charmed*, *bottom*, and *top* quarks. It is expected that there are the same number of quarks as leptons (six of each), and that quarks and leptons are the truly elementary particles along with the gauge bosons (γ, W, Z, gluons). See Table B.4 where the quarks and leptons are arranged in

three groups (generations). Quarks are assumed to have fractional charge ($\frac{1}{3}$ or $\frac{2}{3}$ the charge on the electron), than the previously thought smallest charge). Other properties of quarks and antiquarks are indicated in Table B.2. All hadrons could be constructed in theory from these types of quark. Mesons would consist of quark-antiquark pair. For example, a π^+ meson is considered a $u\tilde{d}$ pair (note that for the $u\tilde{d}$ pair, $Q = \frac{2}{3}e + \frac{1}{3}e = +1e$, $B = \frac{1}{3} - \frac{1}{3} = 0$, $S = 0 + 0 = 0$, as they must for a π^+). A π^0 meson is a combination of $d\tilde{d}$ and $u\tilde{u}$. A K^+ meson is $u\tilde{s}$, with $Q = +1$, $B = 0$, $S = +1$. Baryons, on the other hand, would consist of three quarks. For example, a neutron is n = ddu, whereas an antiproton is $\tilde{p} = \tilde{u}\tilde{u}\tilde{d}$.

Charm, bottomness and topness are conserved in strong, but not weak, interactions. Table B.3 lists some of mesons and baryons associated with charm and bottomness. The convincing evidence for the top quark came only recently. Its extremely high mass of about 175 GeV/c^2 contributed to its elusiveness because of the huge energy required to produce it.

Example 8–8. Find the baryon number, charge, and strangeness for the following quark combinations and identify the hadron particle that is made up of these quark combinations: (a) udd, (b) $u\tilde{u}$, (c) uss, (d) sdd, and (e) $b\tilde{u}$.

Solution. We use Table B.2 to get the properties of the quarks, then Table B.1 or B.3 to find the particle that has these properties.

(a) udd has

$$\begin{aligned} Q &= +\tfrac{2}{3}\,e - \tfrac{1}{3}\,e - \tfrac{1}{3}\,e = 0, \\ B &= 1 + 1 + 1 = 1, \\ S &= 0 + 0 + 0 = 0, \\ C &= 0, \\ \text{bottomness} &= 0, \\ \text{topness} &= 0. \end{aligned}$$

The only baryon ($B = +1$) that has $Q = 0$, $S = 0$, etc., is the neutron (Table B.1).

(b) $u\tilde{u}$ has $Q = \frac{2}{3}\,e - \frac{2}{3}\,e = 0$, $B = 0$, and all other quantum numbers = 0. Sounds like a π^0.

(c) uss has $Q = 0$, $B = +1$, $S = -2$, others = 0. This is a Ξ^0.

(d) sdd has $Q = -1$, $B = +1$, $S = -1$, so must be a Σ^-.

(e) $b\tilde{u}$ has $Q = -1$, $B = 0$, $S = 0$, $C = 0$, bottomness $= -1$, topness $= 0$. This must be a B^- meson (Table B.3).

The distinction between the six quarks (u, d, s, c, b, t) was referred to as flavor. According to theory, each of the flavors of quark can have three colors, usually designated red, green, and blue. The antiquarks are colored antired, antigreen, and antiblue. Baryons are made up of three quarks, one of each color. Mesons consist of a quark-antiquark pair of a particular color and its anticolor. Thus both baryons and mesons are colorless.

Originally, the idea of quark color was proposed to preserve the Pauli exclusion principle. Not all particles obey the exclusion principle. Those that do, such as electrons, protons, and neutrons, are called fermions. Those that don't are called bosons. These two categories are distinguished also in their spin: bosons have integer spin (0, 1, 2, etc.) whereas fermions have half-integer spin, usually $\frac{1}{2}$, as for electrons and nucleons, but can also be $\frac{3}{2}$, $\frac{5}{2}$, etc. Matter is made up mainly of fermions, but the carriers of the forces (γ, W, Z, and gluons) are all bosons. Quarks are fermions (they have spin $\frac{1}{2}$) and therefore should obey the exclusion principle. Yet for three particular baryons (uuu, ddd, and sss), all three quarks would have the same quantum numbers, and at least two of them have their spin in the same direction (since there are only two choices, spin up $[m_s = +\frac{1}{2}]$ or spin down $[m_s = -\frac{1}{2}]$). This would seem to violate the exclusion principle; but if quarks have an additional quantum number (color), which could be different for each quark, it would serve to distinguish them and the exclusion principle would hold. Each quark is assumed to carry a color charge, analogous to electric charge, and the strong force between quarks is often referred to as the *color force*. The strong color force is transmitted by *gluons*. They are included in Table B.4. There are eight gluons, according to the theory, all massless, and six of them have color charge. Because gluons have color charge, they could attract each other and form composite particles. Such *glueballs* are being searched for. Thus gluons have replaced mesons as the particles carrying the strong (color) force. The theory of the strong force is called quantum chromodynamics, or QCD, to indicate that the force acts between color charges.

The color force has the interesting property that its strength increases with increasing distance (as does the force exerted by a coiled spring); as

two quarks approach each other very closely (equivalently, have high energy), the force between them becomes very small. This aspect is referred to as *asymptotic freedom*.

The weak force, as we have seen, is thought to be mediated by the W^+, W^-, and Z^0 particles. It acts between the weak charges that each particle has. Each elementary particle can thus have electric charge, weak charge, color charge, and gravitational mass, although one or more of these could be zero. For example, all leptons have color charge of zero, so they do not interact via the strong force.

Electroweak theory views the weak and electromagnetic forces as two aspects of a single underlying interaction. QCD plus the electroweak theory are referred to as the *Standard Model*. Theoreticians have wondered why the W and Z have large masses rather than being massless like the photon. Electroweak theory suggests an explanation by means of a new *Higgs field* and its particle, the *Higgs boson*, which interact with the W and Z to "slow them down." In being forced to go slower than the speed of light, they must acquire mass.

With the success of the unified electroweak theory, attempts have recently been made to incorporate it and QCD for the strong (color) force into a so-called *grand unified theory* (GUT). One type of such a grand unified theory of the electromagnetic, weak, and strong forces has been worked out in which there is only one class of particle—leptons and quarks belong to the same family and are able to change freely from one type to the other—and the three forces are different aspects of a single underlying force. The unity is predicted to occur, however, only on a scale of less than about 10^{-30} m. If two elementary particles (leptons or quarks) approach each other to within this *unification scale*, the apparently fundamental distinction between them would not exist at this level, and a quark could readily change to a lepton, or vice versa. Baryon and lepton number would not be conserved. The weak, electromagnetic, and strong (color) force would blend to a force of a single strength.

The theory predial existence of particles, called X bosons, that can be exchanged between a quark and lepton allowing one to change into the other. The mass of the proposed X boson, consistent with the uncertainty principle as applied earlier in this chapter (see Eq. 8.3 with $d \approx 10^{-30}$ m) would be about 10^{14} GeV/c^2, or 10^{14} times the proton mass. With

such an incredibly large mass, there is little hope of seeing them in the laboratory. It is also this huge mass that would keep baryon and lepton numbers conserved in observed reactions, since the likelihood of producing such a massive particle, even as a virtual exchange particle, is extremely small at even the highest laboratory energies.

What happens between the unification distance of 10^{-30} m and more larger distances is referred to as *symmetry breaking*. As an analogy, consider an atom in a crystal. Deep within the atom, there is much symmetry —in the innermost regions the electron cloud is spherically symmetric (Chapter 4). Farther out, this symmetry breaks down—the electron clouds are distributed preferentially along the lines (bonds) joining the atoms in the crystal. In a similar way, at 10^{-30} m the force between elementary particles is theorized to be a single force—it is symmetrical and does not single out one type of "charge" over another. But at larger distances, that symmetry is broken and we see three distinct forces. (In the standard model of electroweak interactions the symmetry breaking between the electromagnetic and the weak interactions occurs at about 10^{-18} m.)

Since unification occurs at such tiny distances and huge energies, the theory is difficult to test experimentally. But it is not completely impossible. One possibly testable prediction is the basis for the idea that the proton may decay (via, for example, $\mathrm{p} \rightarrow \pi^0 + \mathrm{e}^+$) and violate conservation of baryon number. This could happen if two quarks approached to within 10^{-31} m of each other. But it is very unlikely at normal temperature and energy, so the decay of proton can only be an unlikely process. In the simplest form of GUT, the theoretical estimate of the proton lifetime for the decay mode $\mathrm{p} \rightarrow \pi^0 + \mathrm{e}^+$ is $\approx 10^{31}$ yr, and this has just come within the realm of testability. Proton decays have still not been seen and experiments put the lower limit on the proton lifetime for the above mode to be about 2.5×10^{32} yr, an order of magnitude greater than this prediction. This may seem disappointment, but on the other hand, it presents a challenge. Indeed, more complex GUTs are not affected by this result.

Example 8–9. An experiment uses 3 300 tons of water waiting to see a proton decay of the type

$$p \rightarrow \pi^0 + \mathrm{e}^+.$$

If the experiment is run for four years without detecting a decay, estimate

the lower limit on the proton half-life.

SOLUTION. As with radioactive decay, the number of decays is proportional to the number of parent species (N), the time interval (Δt), and the decay constant (λ) which is related to the half-life $T_{1/2}$ by (see Eqs. 6.5 and 6.9):

$$\Delta N = -\lambda N\,\Delta t = -\left(\frac{\ln 2}{T_{1/2}}\right) N\,\Delta t.$$

Thus, dealing only with magnitudes,

$$T_{1/2} = \frac{N}{\Delta N}\,\Delta t \ln 2$$

which tells us that, for $\Delta N < 1$ over the four-year trial:

$$T_{1/2} > N(4\text{ yr})(0.69)$$

where N is the number of protons in 3 300 tons of water. To determine N, we note that each molecule of H_2O contains $(2 + 8 =)$ 10 protons. So one mole of water (18 g) contains $10 \times 6 \times 10^{23}$ protons in 18 g of water, or about 3×10^{26} protons per kilogram. One ton is 10^3 kg, so the chamber contains $(3.3 \times 10^6\text{ kg})(3 \times 10^{26}\text{ protons/kg}) \approx 1 \times 10^{33}$ protons. Then our very rough estimate for a lower limit on the proton half-life is

$$T_{1/2} > (10^{33})(4\text{ yr})(0.7) \approx 3 \times 10^{33}\text{ yr}.$$

An interesting prediction of unified theories relates to cosmology (see Chapter 9). It is thought that during the first 10^{-35} s after the theorized big bang that created the universe, the temperature was so extremely high that particles had energies corresponding to the unification scale. Baryon number would not have been conserved then, thus allowing an imbalance that might account for the observed predominance of matter ($B > 0$) over antimatter ($B < 0$) in the universe.

This last example is interesting, for it illustrates a deep connection between investigations at either end of the size scale: theories about the tiniest objects (elementary particles) have a strong bearing on the understanding of the universe as a whole.

Even more ambitious than grand unified theories are attempts to also incorporate gravity, and thus unify all four forces in nature into a single

theory. The only consistent theory so far that attempts to unify all four forces is called *string theory*, in which the elementary particles (Table B.4) are imagined not as points but as one-dimensional strings about 10^{-35} m long.

A related idea is *supersymmetry*, which applied to strings is known as *superstring theory*. Supersymmetry predicts that interactions exist that would change fermions into bosons and vice versa, and that all known fermions have supersymmetric boson partners. Thus, for every quark there would be a *squark*, and for every lepton there would be a *slepton*. Likewise, for every known boson (photons and gluons, for example), there would be a supersymmetric fermion (*photinos* and *gluinos*). Supersymmetric particles might be heavier than their conventional counterparts, perhaps too heavy to have been produced in today's accelerators. Until a supersymmetric particle is found, however, supersymmetry is just an elegant guess.

8.2 High-energy particles. Accelerators and detectors

8–1. What is the total energy of a proton whose kinetic energy is 6.35 GeV?

SOLUTION. The total energy of the proton is

$$E = K + m_{\mathrm{p}}c^2 = 6.35\ \mathrm{GeV} + 0.938\ \mathrm{GeV} = 7.29\ \mathrm{GeV}.$$

8–2. Calculate the wavelength of 35-GeV electrons.

SOLUTION. The total energy of the electron is

$$E = K + m_{\mathrm{e}}c^2 = 35\ \mathrm{GeV} + 0.511\ \mathrm{MeV} = 35\ \mathrm{GeV}.$$

Because the mass is negligible, the momentum is $p = E/c$. We find the wavelength from

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar c}{E} = \frac{1.24 \times 10^{-12}\ \mathrm{MeV \cdot m}}{(35\ \mathrm{GeV})(10^3\ \mathrm{MeV/GeV})} = 3.5 \times 10^{-17}\ \mathrm{m}.$$

8–3. What strength of magnetic field is used in a cyclotron in which protons make 2.8×10^7 revolutions per second?

SOLUTION. We find the magnetic field from the cyclotron frequency:

$$\begin{aligned} f &= \frac{qB}{2\pi m}; \\ 2.8 \times 10^7\ \mathrm{Hz} &= \frac{(1.60 \times 10^{-19}\ \mathrm{C})B}{2\pi(1.67 \times 10^{-27}\ \mathrm{kg})}, \end{aligned}$$

which gives $B = 1.8$ T.

8–4. What is the time for one complete revolution for a very high-energy proton in the 1.0-km-radius Fermilab accelerator?

SOLUTION. Very-high-energy protons will have a speed $v \approx c$. Thus the time for one revolution is

$$t = \frac{2\pi r}{v} = \frac{2\pi(1.0 \times 10^3 \text{ m})}{3.00 \times 10^8 \text{ m/s}} = 2.1 \times 10^{-5} \text{ s} = 21 \ \mu\text{s}.$$

8–5. If α particles are accelerated by the cyclotron of Example 8–2, what must be the frequency of the voltage applied to the dees?

SOLUTION. The cyclotron frequency is

$$f = \frac{qB}{2\pi m}.$$

If we form the ratio for the two particles, we get

$$\begin{aligned} \frac{f_2}{f_1} &= \frac{q_2}{q_1}\frac{m_1}{m_2}; \\ \frac{f_2}{26 \text{ MHz}} &= (2)\left(\frac{1}{4}\right), \end{aligned}$$

which gives $f_2 = 13$ MHz.

8–6. (a) If the cyclotron of Example 8–2 accelerated α particles, what maximum energy could they attain? What would their speed be? (b) Repeat for deuterons ($^2_1$H). (c) In each case, what frequency of voltage is required?

SOLUTION. (a) The maximum kinetic energy is

$$K = \frac{mv^2}{2} = \frac{q^2B^2R^2}{2m}.$$

If we form the ratio for the two particles, we get

$$\begin{aligned} \frac{K_\alpha}{K_\text{p}} &= \left(\frac{q_\alpha}{q_\text{p}}\right)^2\left(\frac{m_\text{p}}{m_\alpha}\right); \\ \frac{K_\alpha}{8.7 \text{ MeV}} &= (2)^2\left(\frac{1}{4}\right) = 1, \end{aligned}$$

which gives $K_\alpha = 8.7$ MeV.

We find the speed from

$$\begin{aligned} K_\alpha &= \tfrac{1}{2}mv_\alpha^2; \\ (8.7 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV}) &= \tfrac{1}{2}(4)(1.67 \times 10^{-27} \text{ kg})v_\alpha^2, \end{aligned}$$

which gives $v_\alpha = 2.0 \times 10^7$ m/s.

(b) For deuterons we get

$$\begin{aligned} \frac{K_\mathrm{d}}{K_\mathrm{p}} &= \left(\frac{q_\mathrm{d}}{q_\mathrm{p}}\right)^2 \left(\frac{m_\mathrm{p}}{m_\mathrm{d}}\right); \\ \frac{K_\mathrm{d}}{8.7 \text{ MeV}} &= (1)^2 \left(\frac{1}{2}\right) = \frac{1}{2}, \end{aligned}$$

which gives $K_\mathrm{d} = 4.3$ MeV.

We find the speed from

$$\begin{aligned} K_\mathrm{d} &= \frac{mv_\mathrm{d}^2}{2}; \\ (4.3 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV}) &= \frac{(2)(1.67 \times 10^{-27} \text{ kg})v_\mathrm{d}^2}{2}, \end{aligned}$$

which gives $v_\mathrm{d} = 2.0 \times 10^7$ m/s.

Note that the α particle and the deuteron have the same q/m.

(c) The cyclotron frequency is

$$f = \frac{qB}{2\pi m}.$$

If we form the ratio for the two particles, we get

$$\frac{f_\mathrm{d}}{f_\alpha} = \frac{q_\mathrm{d}}{q_\alpha}\,\frac{m_\alpha}{m_\mathrm{d}} = \left(\frac{1}{2}\right)\left(\frac{4}{2}\right) = 1,$$

so they require the same frequency. We find the frequency from

$$\begin{aligned} \frac{f_\mathrm{d}}{f_\mathrm{p}} &= \frac{q_\mathrm{d}}{q_\mathrm{p}}\,\frac{m_\mathrm{p}}{m_\mathrm{d}}; \\ \frac{f_\mathrm{d}}{26 \text{ MHz}} &= (1)\left(\frac{1}{2}\right), \end{aligned}$$

which gives $f_{\mathrm{d}} = f_\alpha = 13$ MHz.

8–7. Which is better for picking out details of the nucleus: 30-MeV α particles or 30-MeV protons? Compare each of their wavelengths with the size of a nucleon in a nucleus.

SOLUTION. The size of a nucleon is

$$d \approx 2(1.2 \times 10^{-15}\ \mathrm{m}) = 2.4 \times 10^{-15}\ \mathrm{m}.$$

Because

$$30\ \mathrm{MeV} \ll mc^2,$$

we find the momentum from

$$K = \frac{p^2}{2m},$$

so the wavelength is

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mK}}.$$

For the α particle we have

$$\begin{aligned} \lambda_\alpha &= \frac{6.63 \times 10^{-34}\ \mathrm{J \cdot s}}{\sqrt{2(4)(1.67 \times 10^{-27}\ \mathrm{kg})(30\ \mathrm{MeV})(1.60 \times 10^{-13}\ \mathrm{J/MeV})}} \\ &= 2.6 \times 10^{-15}\ \mathrm{m} \approx \text{ size of nucleon.} \end{aligned}$$

For the proton we have

$$\begin{aligned} \lambda_{\mathrm{p}} &= \frac{6.63 \times 10^{-34}\ \mathrm{J \cdot s}}{\sqrt{2(1.67 \times 10^{-27}\ \mathrm{kg})(30\ \mathrm{MeV})(1.60 \times 10^{-13}\ \mathrm{J/MeV})}} \\ &= 5.2 \times 10^{-15}\ \mathrm{m} \approx\ 2(\text{size of nucleon}). \end{aligned}$$

Thus the α particle is better.

8–8. The voltage across the dees of a cyclotron is 55 kV. How many revolutions do protons make to reach a kinetic energy of 25 MeV?

SOLUTION. The alternating frequency means the proton is accelerated by the voltage twice in each revolution. The number of revolutions is

$$n = \frac{E}{2eV} = \frac{(25\ \mathrm{MeV})(10^3\ \mathrm{keV/MeV})}{2(1\ \mathrm{e})(55\ \mathrm{kV})} = 2.3 \times 10^2\ \mathrm{rev}.$$

8–9. What is the wavelength, and maximum resolving power attainable, using 900-GeV protons at Fermilab?

SOLUTION. The total energy of the proton is

$$E = K + mc^2 = 900 \text{ GeV} + 0.938 \text{ GeV} = 900 \text{ GeV}.$$

Thus $v \approx c$ and the momentum is $p = E/c$.
We find the wavelength from

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar c}{E} = \frac{1.24 \times 10^{-12} \text{ MeV} \cdot \text{m}}{(900 \text{ GeV})(10^3 \text{ MeV/GeV})} = 1.4 \times 10^{-18} \text{ m}.$$

This is the minimum resolving distance, or the maximum resolving power.

8–10. A cyclotron with a radius of 1.0 m is to accelerate deuterons ($^2_1$H) to an energy of 10 MeV. (a) What is the required magnetic field? (b) What frequency is needed for the voltage between the dees? (c) If the potential difference between the dees averages 22 kV, how many revolutions will the particles make before exiting? (d) How much time does it take for one deuteron to go from start to exit, and (e) how far does it travel during this time?

SOLUTION. (a) We find the magnetic field from the maximum kinetic energy:

$$\begin{aligned} K &= \frac{mv^2}{2} = \frac{q^2B^2R^2}{2m}; \\ (10 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV}) &= \frac{(1.60 \times 10^{-19} \text{ C})^2 B^2 (1.0 \text{ m})^2}{2(2)(1.67 \times 10^{-27} \text{ kg})}, \end{aligned}$$

which gives $B = 0.65$ T.

(b) The cyclotron frequency is

$$f = \frac{qB}{2\pi m} = \frac{(1.60 \times 10^{-19} \text{ C})(0.65 \text{ T})}{2\pi(2)(1.67 \times 10^{-27}\text{kg})} = 4.9 \times 10^6 \text{ Hz} = 4.9 \text{ MHz}.$$

(c) The deuteron is accelerated by the voltage twice in each revolution. The number of revolutions is

$$n = \frac{K}{2eV} = \frac{(10 \text{ MeV})(10^3 \text{ keV/MeV})}{2(1 \text{ e})(22 \text{ kV})} = 227 \text{ rev} = 2.3 \times 10^2 \text{ rev}.$$

(d) Because the time for each revolution is the same, we have

$$t = nT = \frac{n}{f} = \frac{227}{4.9 \times 10^6 \text{ Hz}} = 4.6 \times 10^{-5} \text{ s} = 46 \ \mu\text{s}.$$

(e) If the initial speed is zero, after N revolutions the speed is given by

$$\frac{mv_N^2}{2} = 2NeV,$$

with

$$\frac{mv_{\max}^2}{2} = 2neV = K,$$

or

$$v_N = \sqrt{\frac{4NeV}{m}} = v_{\max}\sqrt{\frac{N}{n}}\,.$$

Because each revolution takes the same time, the average speed is

$$v_{\text{av}} = \frac{\displaystyle\int_0^n v_N \, dN}{n} = \frac{v_{\max}}{n^{3/2}} \int_0^n \sqrt{N}\, dN = \frac{2}{3} v_{\max}.$$

Thus for the total distance we have

$$\begin{aligned} d &= v_{\text{av}} t = \frac{2}{3} v_{\max} t = \frac{2}{3}\sqrt{\frac{2K}{m}}\, t \\ &= \frac{2}{3}\sqrt{\frac{2(10 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV})}{(2)(1.67 \times 10^{-27} \text{ kg})}}\,(4.6 \times 10^{-5} \text{ s}) \\ &= 9.5 \times 10^2 \text{ m} = 0.95 \text{ km}. \end{aligned}$$

8–11. Protons are injected into the 1.0-km-radius Fermilab Tevatron with an energy of 8.0 GeV. If they are accelerated by 2.5 MV each revolution, how far do they travel and approximately how long does it take for them to reach 900 GeV?

Solution. The number of revolutions is

$$n = \frac{\Delta E}{2eV} = \frac{(900 \text{ GeV} - 8.0 \text{ GeV})(10^3 \text{ MeV/GeV})}{(1 \text{ e})(2.5 \text{ MV})} = 3.57 \times 10^5 \text{ rev}.$$

The total distance traveled is

$$d = n2\pi R = (3.57 \times 10^5)2\pi(1.0 \text{ km}) = 2.2 \times 10^6 \text{ km}.$$

Very-high-energy protons will have a speed $v \approx c$. Thus the time is

$$t = \frac{d}{v} = \frac{2.2 \times 10^9 \text{ m}}{3.00 \times 10^8 \text{ m/s}} = 7.5 \text{ s}.$$

8–12. The Fermilab Tevatron takes about 20 seconds to bring the energies of the stored protons from 150 GeV to 900 GeV. The acceleration is done once per turn. Estimate the energy given to the protons on each turn. (You can assume that the speed of the protons is essentially c the whole time.)

SOLUTION. Very-high-energy protons will have a speed $v \approx c$. Thus the time for one revolution is

$$T = \frac{2\pi R}{v} = \frac{2\pi(1.0 \times 10^3 \text{ m}}{3.00 \times 10^8 \text{ m/s}} = 2.09 \times 10^{-5} \text{ s}.$$

The number of turns is

$$n = \frac{t}{T} = \frac{20 \text{ s}}{2.09 \times 10^{-5} \text{ s}} = 9.57 \times 10^5 \text{ turns}.$$

We find the energy provided on each turn from

$$\frac{\Delta E}{n} = \frac{900 \text{ GeV} - 150 \text{ GeV}}{9.57 \times 10^5 \text{ turns}} = 7.8 \times 10^{-4} \text{ GeV/turn} = 0.78 \text{ MeV/turn}.$$

8–13. Show that the energy of a particle (charge e) in a synchrotron, in the relativistic limit ($v \approx c$), is given by

$$E(\text{in eV}) = Brc,$$

where B is magnetic field strength and r the radius of the orbit (SI units).

SOLUTION. In the relativistic limit $v \approx c$, and $E = pc$. For Newton's law we have

$$F = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{v}) = (\gamma m)\frac{d\mathbf{v}}{dt}$$

because v is constant. The magnetic field provides the radial acceleration:

$$qvB = \gamma m \frac{v^2}{r},$$

or

$$B = \gamma m \frac{v}{qr} = \frac{p}{qr} = \frac{E}{qrc}.$$

Thus the energy is

$$E = qBrc.$$

If the energy is in eV, the charge is $1\,e$, so we have

$$E(\text{eV}) = Brc.$$

8–14. What magnetic field intensity is needed at the 1.0-km-radius Fermilab synchrotron for 900-GeV protons?

SOLUTION. Because the kinetic energy is much greater than the rest energy, $E = pc$ and $v \approx c$. We use the result from Problem 8–13:

$$\begin{aligned} E(\text{eV}) &= Brc; \\ 900\ \text{GeV} = 900 \times 10^9\ \text{eV} &= B(1.0 \times 10^3\ \text{m})(3.00 \times 10^8\ \text{m/s}), \end{aligned}$$

which gives $B = 3.0$ T.

8.3 Particle exchange. Antiparticles. Interactions and conservation laws. Particle classification

8–15. How much energy is released in the decay

$$\pi^+ \to \mu^+ + \nu_\mu?$$

SOLUTION. For the reaction

$$\pi^+ \to \mu^+ + \nu_\mu,$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(m(\pi^+) - m(\mu^+)\right)c^2 = \left((139.6\ \text{MeV}/c^2) - (105.7\ \text{MeV}/c^2)\right)c^2 \\ &= 33.9\ \text{MeV}. \end{aligned}$$

Thus 33.9 MeV is released.

8–16. About how much energy is released when a Λ^0 decays to $n+\pi^0$? (See Table B.1.)

SOLUTION. For the reaction

$$\Lambda^0 \to \mathrm{n} + \pi^0,$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(m(\Lambda^0) - m(\mathrm{n}) - m(\pi^0)\right) c^2 \\ &= \left((1\,115.7 \text{ MeV}/c^2) - (939.6 \text{ MeV}/c^2) - (135.0 \text{ MeV}/c^2)\right) c^2 \\ &= 41.1 \text{ MeV}. \end{aligned}$$

Thus 41.1 MeV is released.

8–17. How much energy is required to produce a neutron-antineutron pair?

SOLUTION. The minimum energy must provide the rest energy of the pair:

$$E = 2m_{\mathrm{n}}c^2 = 2(939.6 \text{ MeV}) = 1\,879 \text{ MeV} = 1.879 \text{ GeV}.$$

8–18. Estimate the range of the strong force if the mediating particle were the kaon instead of the pion.

SOLUTION. We estimate the range from

$$\begin{aligned} mc^2 &= \frac{2\pi\hbar c}{2d}; \\ 497.7 \text{ MeV} &= \frac{(0.658 \times 10^{-15} \text{ eV} \cdot \text{s})(3.00 \times 10^8 \text{ m/s})}{2d}, \end{aligned}$$

which gives $d = 2.0 \times 10^{-16}$ m $= 0.2$ fm.

8–19. Two protons are heading toward each other with equal speeds. What minimum kinetic energy must each have if a π^0 meson is to be created in the process? (See Table B.1.)

SOLUTION. Because two protons are present before and after the process, and the total momentum is zero, the minimum kinetic energy will produce all three particles at rest. Thus the total initial kinetic energy must provide the rest energy of the π^0 meson:

$$2K = m_{\pi^0}c^2 = 135 \text{ MeV},$$

which gives $K = 67.5$ MeV.

8–20. Which of the following decays are possible? For those that are forbidden, explain which laws are violated.

$$\begin{array}{lrcl} \text{(a)} & \Xi^0 & \to & \Sigma^+ + \pi^- \\ \text{(b)} & \Omega^- & \to & \Sigma^0 + \pi^- + \nu \\ \text{(c)} & \Sigma^0 & \to & \Lambda^0 + \gamma + \gamma. \end{array}$$

Solution. (a) For the decay

$$\Xi^0 \to \Sigma^+ + \pi^-,$$

the conservation laws are

$$\begin{array}{rl} \text{Charge}: & 0 = +1 - 1; \\ \text{Energy (mass)}: & 1\,314.9 \text{ MeV} < 1\,189.4 \text{ MeV} + 139.6 \text{ MeV}. \end{array}$$

Thus this decay is forbidden, energy is not conserved.

(b) For the decay

$$\Omega^- \to \Sigma^0 + \pi^- + \nu,$$

the conservation laws are

$$\begin{array}{rl} \text{Charge}: & -1 = 0 - 1; \\ \text{Energy (mass)}: & 1\,672.5 \text{ MeV} > 1\,192.6 \text{ MeV} + 139.6 \text{ MeV}; \\ \text{Baryon number}: & +1 = +1 + 0 + 0; \\ \text{Lepton number}: & 0 \neq 0 + 0 + 1. \end{array}$$

Thus this decay is forbidden, lepton number is not conserved.

(c) For the decay

$$\Sigma^0 \to \Lambda^0 + \gamma + \gamma,$$

the conservation laws are

$$\begin{array}{rl} \text{Charge}: & 0 = 0 + 0 + 0; \\ \text{Energy (mass)}: & 1\,192.6 \text{ MeV} > 1\,115.7 \text{ MeV} + 0 + 0; \\ \text{Baryon number}: & +1 = +1 + 0 + 0; \\ \text{Lepton number}: & -1 = -1 + 0 + 0. \end{array}$$

Thus this decay is possible.

8–21. Estimate the range of the weak force using Eq. 8.3, given the masses of the W and Z particles as about 80 to 90 GeV/c^2.

SOLUTION. If we use the average mass, we estimate the range from

$$\begin{aligned} mc^2 &= \frac{\hbar c}{2d}; \\ (85\text{ GeV})(10^3\text{ MeV/GeV}) &= \frac{1.24\times10^{-12}\text{ MeV}\cdot\text{m}}{2d}, \end{aligned}$$

which gives $d = 7.3\times10^{-18}$ m.

8–22. What are the wavelengths of the two photons produced when a proton and antiproton at rest annihilate?

SOLUTION. The energy of the two photons must be the rest energy of the proton and antiproton:

$$\begin{aligned} 2m_\text{p}c^2 &= 4\pi\hbar f = \frac{4\pi\hbar c}{\lambda}; \\ 2(938.3\text{ MeV}/c^2)c^2 &= \frac{2(1.24\times10^{-12}\text{ MeV}\cdot\text{m})}{\lambda}, \end{aligned}$$

which gives $\lambda = 1.32\times10^{-15}$ m.

8–23. (a) Show, by conserving momentum and energy, that it is impossible for an isolated electron to radiate only a single photon. (b) With this result in mind, how can you defend the photon exchange diagram in Fig. 8.2?

SOLUTION. (a) For the reaction

$$e^- \to e^- + \gamma,$$

the isolated electron is at rest. For the photon, we have $E_\gamma = p_\gamma c$. For energy conservation we have

$$m_\text{e}c^2 = \sqrt{(p_\text{e}c)^2 + (m_\text{e}c^2)^2} + E_\gamma.$$

For momentum conservation we have

$$0 = p_e - p_\gamma.$$

When we eliminate p_e in the energy equation and rearrange, we have

$$m_\text{e}c^2 - E_\gamma = \sqrt{E_\gamma^2 + (m_\text{e}c^2)^2}.$$

When we square both sides, we have

$$(m_e c^2)^2 + E_\gamma^2 - 2E_\gamma m_e c^2 = E_\gamma^2 + (m_e c^2)^2,$$

which gives $E_\gamma = 0$. Thus no photon is emitted.

(b) For the photon exchange in Fig. 8.2, the photon exists for such a short time that the uncertainty principle allows energy to not be conserved during the exchange.

8–24. What would be the wavelengths of the two photons produced when an electron and a positron, each with 420 keV of kinetic energy, annihilate?

SOLUTION. We assume the momentum of the electron-positron pair is zero. The two photons must have opposite and equal momenta, therefore, equal energies. The energy of the two photons must be the total energy of the pair:

$$\begin{aligned} 2(K + m_e c^2) &= 4\pi\hbar f = \frac{4\pi\hbar c}{\lambda}; \\ 2\left((0.420\text{ MeV}) + (0.511\text{ MeV}/c^2)c^2\right) &= \frac{2(1.24 \times 10^{-12}\text{ MeV}\cdot\text{m})}{\lambda}, \end{aligned}$$

which gives

$$\lambda = 1.33 \times 10^{-12}\text{ m} = 1.33 \times 10^{-3}\text{ nm}.$$

8–25. In the rare decay

$$\pi^+ \to e^+ + \nu_e,$$

what is the kinetic energy of the positron? Assume the π^+ decays from rest.

SOLUTION. The total kinetic energy after the decay of the stationary π^+ is the Q-value:

$$\begin{aligned} Q &= \left(m(\pi^+) - m(e^+) - m(\nu_e)\right)c^2 \\ &= \left((139.6\text{ MeV}/c^2) - (0.511\text{ MeV}/c^2) - 0\right)c^2 = 139.1\text{ MeV}. \end{aligned}$$

For momentum conservation we have

$$0 = p_e - p_\nu,$$

or

$$(p_\nu c)^2 = (p_\mathrm{e} c)^2 = E_\mathrm{e}^2 - (m_\mathrm{e} c^2)^2 = K_\mathrm{e}^2 + 2K_\mathrm{e} m_\mathrm{e} c^2.$$

For energy conservation we have

$$Q = K_\mathrm{e} + p_\nu c = K_\mathrm{e} + p_\mathrm{e} c,$$

or

$$(p_\mathrm{e} c)^2 = (Q - K_\mathrm{e})^2 = Q^2 - 2K_\mathrm{e} Q + K_\mathrm{e}^2.$$

When we combine this with the result from momentum conservation, we get

$$K_\mathrm{e} = \frac{Q^2}{2(Q + m_\mathrm{e} c^2)} = \frac{(139.1\ \mathrm{MeV})^2}{2(139.1\ \mathrm{MeV} + 0.511\ \mathrm{MeV})} = 69.3\ \mathrm{MeV}.$$

8–26. What minimum kinetic energy must a neutron and proton each have if they are traveling at the same speed toward each other, collide, and produce a $\mathrm{K^+K^-}$ pair in addition to themselves? (See Table B.1.)

Solution. The minimum initial kinetic energy of the neutron and proton must provide the rest energy of the $\mathrm{K^+K^-}$ pair:

$$K_\mathrm{p} + K_\mathrm{n} = 2m_\mathrm{K} c^2 = 2(493.7\ \mathrm{MeV}) = 987.4\ \mathrm{MeV}.$$

Because the neutron and proton have the same speed but different masses, they have slightly different kinetic energies:

$$K = (\gamma - 1)\, mc^2 \propto m.$$

Thus we have

$$\frac{K_\mathrm{n}}{K_\mathrm{p} + K_\mathrm{n}} = \frac{m_\mathrm{n}}{m_\mathrm{p} + m_\mathrm{n}};$$
$$\frac{K_\mathrm{n}}{987.4\ \mathrm{MeV}} = \frac{939.6\ \mathrm{MeV}}{938.3\ \mathrm{MeV} + 939.6\ \mathrm{MeV}},$$

which gives $K_\mathrm{n} = 494.0$ MeV;

$$\frac{K_\mathrm{p}}{K_\mathrm{p} + K_\mathrm{n}} = \frac{m_\mathrm{p}}{m_\mathrm{p} + m_\mathrm{n}};$$
$$\frac{K_\mathrm{p}}{987.4\ \mathrm{MeV}} = \frac{938.3\ \mathrm{MeV}}{938.3\ \mathrm{MeV} + 939.6\ \mathrm{MeV}},$$

which gives $K_{\mathrm{p}} = 493.4$ MeV.

Note that we have ignored the small total momentum of the system.

8–27. Calculate the kinetic energy of each of the two products in the decay

$$\Xi^- \to \Lambda^0 + \pi^-.$$

Assume the Ξ^- decays from rest.

SOLUTION. The total kinetic energy after the decay of the stationary Ξ^- is the Q-value:

$$\begin{aligned} K_\Lambda + K_{\pi^-} &= Q = \big(m(\Xi^-) - m(\Lambda^0) - m(\pi^-)\big)\, c^2 \\ &= \big((1\,321.3 \text{ MeV}/c^2) - (1\,115.7 \text{ MeV}/c^2) \\ &\quad - (139.6 \text{ MeV}/c^2)\big)\, c^2 \\ &= 66.0 \text{ MeV}. \end{aligned}$$

For energy conservation we have

$$m_\Xi c^2 = E_\Lambda + E_{\pi^-},$$

or

$$E_{\pi^-} = m_\Xi c^2 - E_\Lambda.$$

For momentum conservation we have

$$0 = p_{\pi^-} - p_\Lambda,$$

or

$$(p_\Lambda c)^2 = (p_{\pi^-} c)^2 = E_\Lambda^2 - (m_\Lambda c^2)^2 = E_{\pi^-}^2 - (m_{\pi^-} c^2)^2.$$

When we combine this with the result from energy conservation, we get

$$\begin{aligned} E_\Lambda &= \frac{(m_\Xi c^2)^2 + (m_\Lambda c^2)^2 - (m_{\pi^-} c^2)^2}{2 m_\Xi c^2} \\ &= \frac{(1\,321.3 \text{ MeV})^2 + (1\,115.7 \text{ MeV})^2 - (139.6 \text{ MeV})^2}{2(1\,321.3 \text{ MeV})} \\ &= 1\,124.3 \text{ MeV}. \end{aligned}$$

For the kinetic energies we have

$$\begin{aligned} K_\Lambda &= E_\Lambda - m_\Lambda c^2 = 1\,124.3 \text{ MeV} - 1\,115.7 \text{ MeV} = 8.6 \text{ MeV}; \\ K_{\mathrm{p}} &= Q - K_\Lambda = 66.0 \text{ MeV} - 8.6 \text{ MeV} = 57.4 \text{ MeV}. \end{aligned}$$

8–28. Could a π^+ meson be produced if a 100-MeV proton struck a proton at rest? What minimum kinetic energy must the incoming proton have?

SOLUTION. No, because the kinetic energy of the incoming proton is less than the rest energy of the π^+ meson, which is 139. 6 MeV.
The reaction is

$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{n} + \pi^+.$$

The minimum initial kinetic energy produces the three particles moving together with the same speed, which we can consider to be a single particle with mass

$$M = m_\mathrm{p} + m_\mathrm{n} + m_\pi.$$

For energy conservation we have

$$E_\mathrm{p} + m_\mathrm{p}c^2 = E_M.$$

For momentum conservation we have

$$p_\mathrm{p} + 0 = p_M,$$

or

$$(p_\mathrm{p}c)^2 = (p_M c)^2 = E_\mathrm{p}^2 - (m_\mathrm{p}c^2)^2 = E_M^2 - (Mc^2)^2.$$

When we combine this with the result from energy conservation, we get

$$\begin{aligned} E_\mathrm{p} &= \frac{(Mc^2)^2 - 2(m_\mathrm{p}c^2)^2}{2m_\mathrm{p}c^2} = \frac{\left((m_\mathrm{p} + m_\mathrm{n} + m_\pi)c^2\right)^2 - 2(m_\mathrm{p}c^2)^2}{2m_\mathrm{p}c^2} \\ &= \frac{(938.3\ \mathrm{MeV} + 939.6\ \mathrm{MeV} + 139.6\ \mathrm{MeV})^2 - 2(938.3\ \mathrm{MeV})^2}{2(938.3\ \mathrm{MeV})} \\ &= 1\,230.7\ \mathrm{MeV}. \end{aligned}$$

For the kinetic energy we have

$$K_\mathrm{p} = E_\mathrm{p} - m_\mathrm{p}c^2 = 1\,230.7\ \mathrm{MeV} - 938.3\ \mathrm{MeV} = 292.4\ \mathrm{MeV}.$$

8–29. Calculate the maximum kinetic energy of the electron in the decay

$$\mu^- \to \mathrm{e}^- + \tilde{\nu}_\mathrm{e} + \nu_\mu.$$

[*Hint*: In what direction do the two neutrinos move relative to the electron in order to give the latter the maximum kinetic energy? Both energy and momentum are conserved; use relativistic formulas.]

SOLUTION. For the decay

$$\mu^- \to \mathrm{e}^- + \tilde{\nu}_\mathrm{e} + \nu_\mu,$$

the electron will have maximum kinetic energy when the two neutrinos have the same momentum and move opposite to the direction of the electron.
For a neutrino $E_\nu = p_\nu c$. For energy conservation we have

$$m_\mu c^2 = E_\mathrm{e} + 2E_\nu,$$

or

$$2E_\nu = m_\mu c^2 - E_\mathrm{e}.$$

For momentum conservation we have

$$0 = p_\mathrm{e} - 2p_\nu,$$

or

$$(2p_\nu c)^2 = 4E_\nu^2 = (p_\mathrm{e}c)^2 = E_\mathrm{e}^2 - (m_\mathrm{e}c^2)^2.$$

When we combine this with the result from energy conservation, we get

$$\begin{aligned} E_\mathrm{e} &= \frac{(m_\mu c^2)^2 + (m_\mathrm{e}c^2)^2}{2m_\mu c^2} = \frac{(105.7\ \mathrm{MeV})^2 + (0.511\ \mathrm{MeV})^2}{2(105.7\ \mathrm{MeV})} \\ &= 52.85\ \mathrm{MeV}. \end{aligned}$$

For the maximum kinetic energy we have

$$K_\mathrm{e} = E_\mathrm{e} - m_\mathrm{e}c^2 = 52.85\ \mathrm{MeV} - 0.511\ \mathrm{MeV} = 52.3\ \mathrm{MeV}.$$

8.4 Stability and resonances. Strangeness. Quarks.

8–30. Use Fig. 8.4 to estimate the energy width and then the lifetime of the Δ resonance using the uncertainty principle.

SOLUTION. Using Fig. 8.4 we estimate the energy width as 130 MeV. We estimate the lifetime from

$$\Delta t = \frac{\hbar}{2\,\Delta E} = \frac{0.658 \times 10^{-15}\ \text{eV} \cdot \text{s}}{2(1.3 \times 10^{8}\ \text{eV})} = 2.5 \times 10^{-24}\ \text{s}.$$

8–31. The measured width of the J/ψ meson is 88 keV. Estimate its lifetime.

SOLUTION. We estimate the lifetime from

$$\Delta t = \frac{\hbar}{2\,\Delta E} = \frac{0.658 \times 10^{-15}\ \text{eV} \cdot \text{s}}{2(8.8 \times 10^{4}\ \text{eV})} = 3.7 \times 10^{-21}\ \text{s}.$$

8–32. The measured width of the $\psi(3685)$ meson is 277 keV. Estimate its lifetime.

SOLUTION. We estimate the lifetime from

$$\Delta t = \frac{\hbar}{2\,\Delta E} = \frac{0.658 \times 10^{-15}\ \text{eV} \cdot \text{s}}{2(2.77 \times 10^{5}\ \text{eV})} = 1.19 \times 10^{-21}\ \text{s}.$$

8–33. What is the energy width (or uncertainty) of (a) η^0, and (b) Σ^0?

SOLUTION. We find the energy width from the lifetime given in Table B.1.
(a) For η^0 we have

$$\Delta E = \frac{\hbar}{2\,\Delta t} = \frac{0.658 \times 10^{-15}\ \text{eV} \cdot \text{s}}{2(5 \times 10^{-19}\ \text{s})} = 659\ \text{eV}.$$

(b) For Σ^0 we have

$$\Delta E = \frac{\hbar}{2\,\Delta t} = \frac{0.658 \times 10^{-15}\ \text{eV} \cdot \text{s}}{2(7.4 \times 10^{-20}\ \text{s})} = 4.45\ \text{keV}.$$

8–34. The B^- meson is presumed to be a $b\tilde{u}$ quark combination. (a) Show that this is consistent for all quantum numbers. (b) What are the quark combinations for B^+, B^0, $\tilde{B}^0$?

SOLUTION. (a) For $B^- = b\tilde{u}$ we have

$$\begin{aligned}
\text{Charge}: \quad & -1 = -\tfrac{1}{3} - \tfrac{3}{2}; \\
\text{Spin}: \quad & 0 = +\tfrac{1}{2} - \tfrac{1}{2}; \\
\text{Baryon number}: \quad & 0 = +\tfrac{1}{3} - \tfrac{1}{3}; \\
\text{Strangeness}: \quad & 0 = 0 + 0; \\
\text{Charm}: \quad & 0 = 0 + 0; \\
\text{Bottomness}: \quad & -1 = -1 + 0; \\
\text{Topness}: \quad & 0 = 0 + 0.
\end{aligned}$$

(b) Because B^+ is the antiparticle of B^-, we have $B^+ = \tilde{b}u$.

For B^0, to make the charge zero, we need the antiquark with the same properties as $\tilde{u}$, but with a charge of $+\frac{1}{3}$. Thus we have $B^0 = b\tilde{d}$.

Because $\tilde{B}^0$ is the antiparticle of B^0, we have $\tilde{B}^0 = \tilde{b}d$.

8–35. What are the quark combinations that can form (a) a neutron, (b) an antineutron, (c) a Λ^0, (d) a $\tilde{\Sigma}^0$?

SOLUTION. (a) For the neutron we must have charge, strangeness, charm, bottomness, and topness = 0. For the baryon number to be 1, we need three quarks: n = ddu.

(b) For the antineutron we have $\tilde{n} = \tilde{d}\tilde{d}\tilde{u}$.

(c) For the Λ^0 we must have charge, charm, bottomness, and topness = 0. To get strangeness = −1, we need an s quark. To get a baryon number of 1, we need three quarks. To get a charge = 0, we have $\Lambda^0 = uds$.

(d) For the $\tilde{\Sigma}^0$ we must have charge, charm, bottomness, and topness = 0. To get strangeness = +1, we need an $\tilde{s}$ quark. To get a baryon number of −1, we need three antiquarks. To get a charge = 0, we have $\tilde{\Sigma}^0 = \tilde{u}\tilde{d}\tilde{s}$.

8–36. What particles do the following quark combinations produce? (a) uud, (b) $\tilde{u}\tilde{u}\tilde{s}$, (c) $\tilde{u}s$, (d) $d\tilde{u}$, (e) $\tilde{c}s$.

SOLUTION. (a) For uud we have charge +1, baryon number = +1, while strangeness, charm, bottomness, and topness = 0. Thus we have p.

(b) For $\tilde{u}\tilde{u}\tilde{s}$ we have charge = −1, baryon number = −1, strangeness = +1, while charm, bottomness, and topness = 0. Thus we have $\tilde{\Sigma}^+$.

(c) For $\tilde{u}s$ we have charge = −1, baryon number = 0, strangeness = −1, while charm, bottomness, and topness = 0. Thus we have K^-.

(d) For $\mathrm{d\tilde{u}}$ we have charge $= -1$, baryon number $= 0$, strangeness $= 0$, while charm, bottomness, and topness $= 0$. Thus we have π^-.

(e) For $\mathrm{\tilde{c}s}$ we have charge $= -1$, baryon number $= 0$, strangeness $= -1$, charm $= -1$, while bottomness, and topness $= 0$. Thus we have D_S^-.

8–37. Imagine that binding energies could be ignored. Find the masses of the u and d quarks from the masses of the proton and neutron.

SOLUTION. Quark composition of proton = uud and of neutron = udd. Thus, if we neglect binding energies, we may write

$$m_\mathrm{p} = 2m_\mathrm{u} + m_\mathrm{d}$$

and

$$m_\mathrm{n} = m_\mathrm{u} + 2m_\mathrm{d}.$$

Solving simultaneously, we find

$$m_\mathrm{u} = \tfrac{1}{3}(2m_\mathrm{p} - m_\mathrm{n}) = \tfrac{1}{3}\left(2(938\ \mathrm{MeV}/c^2) - 939.6\ \mathrm{MeV}/c^2\right) = 312\,\mathrm{MeV}/c^2.$$

and from either of the first two equations, $m_\mathrm{d} = 314\ \mathrm{MeV}/c^2$.

8–38. The text states that the reaction $\pi^- + \mathrm{p} \rightarrow \mathrm{K}^0 + \Lambda^0$ occurs with high probability, whereas the reaction $\pi^- + \mathrm{p} \rightarrow \mathrm{K}^0 + \mathrm{n}$ never occurs. Analyze these reactions at the quark level. Show that the first reaction conserves the total number of each type of quark, and the second reaction does not.

SOLUTION. In the first reaction, $\pi^- + \mathrm{p} \rightarrow \mathrm{K}^0 + \Lambda^0$, the quarks in the particles are: $\mathrm{\tilde{u}d} + \mathrm{uud} \rightarrow \mathrm{d\tilde{s}} + \mathrm{uds}$. There is a net of 1 up quark both before and after the reaction, a net of 2 down quarks both before and after, and a net of zero strange quarks both before and after. Thus, the reaction conserves the net number of each type of quark.

In the second reaction, $\pi^- + \mathrm{p} \rightarrow \mathrm{K}^0 + \mathrm{n}$, the quarks in the particles are: $\mathrm{\tilde{u}d} + \mathrm{uud} \rightarrow \mathrm{d\tilde{s}} + \mathrm{udd}$. In this case, there is a net of 1 up and 2 down quarks before the reaction but a net of 1 up, 3 down, and 1 anti-strange quark after the reaction. Thus, the reaction does not conserve the net number of each type of quark.

8–39. A Σ^0 particle traveling through matter strikes a proton; then a Σ^+ and gamma ray emerge, as well as a third particle. Use the quark model of each to determine the identity of the third particle.

SOLUTION. The process

$$\Sigma^0 + \mathrm{p} \to \Sigma^+ + \gamma + \mathrm{X}$$

is equivalent to

$$\mathrm{dds} + \mathrm{uud} \to \mathrm{uds} + 0 + ?$$

The left side has a net 3d, 2u and 1s. The right-hand side has 1d, 1u, and 1s leaving 2d and 1u missing. The unknown particle is a neutron, udd. Baryon and strangeness numbers are conserved.

8–40. What is the quark combination needed to produce a D^0 meson ($Q = B = S = 0$, $C = +1$)?

SOLUTION. For the D^0 we have charge, baryon number, strangeness, bottomness, and topness = 0. To get charm = +1 we need the c quark. To get baryon number = 0, we need an antiquark. Thus we have $\mathrm{D}^0 = \mathrm{c\tilde{u}}$.

8–41. The D_S^+ meson has $S = C = +1$, $B = 0$. What quark combination would produce it?

SOLUTION. For the D_S^+ we have baryon number, bottomness, and topness = 0. To get charm = +1 we need the c quark. To get baryon number = 0, we need an antiquark. To get strangeness = +1 we need the $\tilde{\mathrm{s}}$ quark. Thus we have $\mathrm{D}_S^+ = \mathrm{c\tilde{s}}$.

8–42. Fig. 8.5 presents a quark diagram of the process $\mathrm{p} + \pi^- \to \mathrm{n}$. Here all lines represent particles moving to the right, with right-pointing arrows for quarks and left-pointing for antiquarks. Moreover, quark lines never end in such a graph. Use this technique to draw possible graphs for the reactions

$$\begin{array}{lrcl} \text{(a)} & \pi^- + \mathrm{p} & \to & \pi^0 + \mathrm{n}, \\ \text{(b)} & \tilde{\mathrm{p}} + \mathrm{p} & \to & 2\,\pi^0. \end{array}$$

SOLUTION. (a) See Fig. 8.6.

(b) From Table B.2 we see that the π^0 could be either $\mathrm{u\tilde{u}}$ or $\mathrm{d\tilde{d}}$. Actually it is a combination of the two, $(\mathrm{u\tilde{u}} - \mathrm{d\tilde{d}})/\sqrt{2}$, but we will simplify the diagram by assuming that one π^0 is $\mathrm{u\tilde{u}}$ and the other is $\mathrm{d\tilde{d}}$ (Fig. 8.7). Note that high-energy $\mathrm{p\tilde{p}}$ collisions usually produce many particles.

8–43. Draw a possible quark diagram (see Fig. 8.5) for reaction

$$\mathrm{K}^- + \mathrm{p} \to \mathrm{K}^- + \mathrm{p}.$$

SOLUTION. See Fig. 8.8.

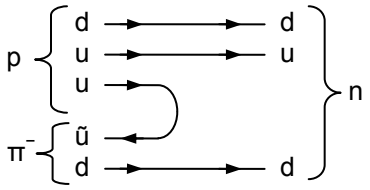


Figure 8.5. Quark representation of the strong interaction $p + \pi^- \to n$.

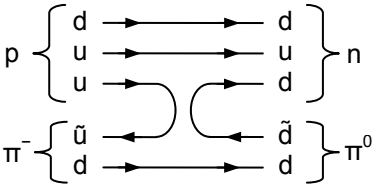


Figure 8.6. Problem 8–42a.

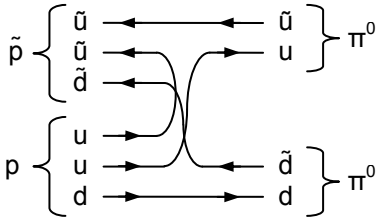


Figure 8.7. Problem 8–42b.

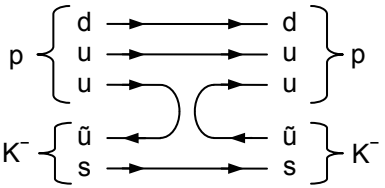


Figure 8.8. Problem 8–43.

8.5 General problems

8–44. What is the total energy of a proton whose kinetic energy is 25 GeV? What is its wavelength?

SOLUTION. The total energy of the proton is

$$E = K + mc^2 = 25 \text{ GeV} + 0.938 \text{ GeV} = 26 \text{ GeV}.$$

We find the momentum from

$$(pc)^2 = E^2 - (mc^2)^2 = (26 \text{ GeV})^2 - (0.938 \text{ GeV})^2,$$

which gives $pc = 26$ GeV.

We find the wavelength from

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar c}{pc} = \frac{1.24 \times 10^{-12} \text{ MeV} \cdot \text{m}}{(26 \text{ GeV})(10^3 \text{ MeV/GeV})} = 4.8 \times 10^{-17} \text{ m}.$$

8–45. Assume there are 5×10^{13} protons at 900 GeV stored in the 1.0-km-radius ring of the Tevatron. (a) How much current (amperes) is carried by this beam? (b) How fast would a 1 500 kg car have to move to carry the same kinetic energy as this beam?

SOLUTION. (a) At the high energy of 900 GeV the speed of the protons $\approx c$. Thus the time for one proton to make a revolution is

$$T = \frac{2\pi r}{c} = \frac{2\pi(1.0 \times 10^3 \text{ m})}{3.00 \times 10^8 \text{ m/s}} = 2.09 \times 10^{-5} \text{ s}.$$

The current carried by the beam of N protons is

$$I = \frac{Ne}{T} = \frac{(5 \times 10^{13})(1.60 \times 10^{-19} \text{ C})}{2.09 \times 10^{-5} \text{ s}} = 0.38 \text{ A}.$$

(b) We find the speed of the car from

$$\begin{aligned} K &= \tfrac{1}{2}mv^2; \\ (5 \times 10^{13})(900 \text{ GeV})(1.60 \times 10^{-10} \text{ J/GeV}) &= \tfrac{1}{2}(1\,500 \text{ kg})v^2, \end{aligned}$$

which gives $v = 98$ m/s $= 350$ km/h.

8–46. The 4.25-km-radius LEP tunnel will be reused to house the magnets for the Large Hadron Collider (LHC). If the design calls for proton beams of energy 7.0 TeV, what magnetic field will be required?

SOLUTION. Because

$$7.0 \text{ TeV} \gg m_{\text{p}}c^2,$$

we have

$$K = E = pc.$$

We use the result from Problem 8–13:

$$\begin{aligned} E(\text{eV}) &= Brc; \\ (7.0 \text{ TeV})(10^{12} \text{ eV/TeV}) &= B(4.25 \times 10^3 \text{ m})(3.00 \times 10^8 \text{ m/s}), \end{aligned}$$

which gives $B = 5.5$ T.

8–47. (a) How much energy is released when an electron and a positron annihilate each other? (b) How much energy is released when a proton and an antiproton annihilate each other?

SOLUTION. (a) We assume the momentum of the electron-positron pair is zero. The released energy is

$$E = 2m_{\mathrm{e}}c^2 = 2(0.511\ \mathrm{MeV}) = 1.022\ \mathrm{MeV}.$$

(b) We assume the momentum of the proton-antiproton pair is zero. The released energy is

$$E = 2m_{\mathrm{p}}c^2 = 2(938.3\ \mathrm{MeV}) = 1\,876.6\ \mathrm{MeV}.$$

8–48. Which of the following reactions are possible, and by what interaction could they occur? For those forbidden, explain why.

$$\begin{array}{lrcl}
\text{(a)} & \pi^- + \mathrm{p} & \to & \mathrm{K}^+ + \Sigma^- \\
\text{(b)} & \pi^+ + \mathrm{p} & \to & \mathrm{K}^+ + \Sigma^+ \\
\text{(c)} & \pi^- + \mathrm{p} & \to & \Lambda^0 + \mathrm{K}^0 + \pi^0 \\
\text{(d)} & \pi^+ + \mathrm{p} & \to & \Sigma^0 + \pi^0 \\
\text{(e)} & \pi^- + \mathrm{p} & \to & \pi + \mathrm{e}^- + \tilde{\nu}_{\mathrm{e}} \\
\text{(f)} & \pi^- + \mathrm{p} & \to & \mathrm{K}^0 + \mathrm{p} + \pi^0 \\
\text{(g)} & \mathrm{K}^- + \mathrm{p} & \to & \Lambda^0 + \pi^0 \\
\text{(h)} & \mathrm{K}^+ + \mathrm{n} & \to & \Sigma^+ + \pi^0 + \gamma \\
\text{(i)} & \mathrm{K}^+ & \to & \pi^0 + \pi^0 + \pi^+ \\
\text{(j)} & \pi^+ & \to & \mathrm{e}^+ + \nu_{\mathrm{e}}
\end{array}$$

SOLUTION. (a) For the reaction

$$\pi^- + \mathrm{p} \to \mathrm{K}^+ + \Sigma^-,$$

the conservation laws are

$$\begin{array}{rrcl}
\text{Charge :} & -1+1 & = & +1-1; \\
\text{Spin :} & 0+\frac{1}{2} & = & 0+\frac{1}{2}; \\
\text{Baryon number :} & 0+1 & = & 0+1; \\
\text{Lepton number :} & 0+0 & = & 0+0; \\
\text{Strangeness :} & 0+0 & = & +1-1.
\end{array}$$

Thus the reaction is possible, through the strong interaction.

(b) For the reaction

$$\pi^+ + \mathrm{p} \to \mathrm{K}^+ + \Sigma^+,$$

the conservation laws are

$$\begin{aligned}
\text{Charge}: \quad & +1+1 &=& \quad +1+1; \\
\text{Spin}: \quad & 0+\tfrac{1}{2} &=& \quad 0+\tfrac{1}{2}; \\
\text{Baryon number}: \quad & 0+1 &=& \quad 0+1; \\
\text{Lepton number}: \quad & 0+0 &=& \quad 0+0; \\
\text{Strangeness}: \quad & 0+0 &=& \quad +1-1.
\end{aligned}$$

Thus the reaction is possible, through the strong interaction.

(c) For the reaction

$$\pi^- + \mathrm{p} \to \Lambda^0 + \mathrm{K}^0 + \pi^0,$$

the conservation laws are

$$\begin{aligned}
\text{Charge}: \quad & -1+1 &=& \quad 0+0+0; \\
\text{Spin}: \quad & 0+\tfrac{1}{2} &=& \quad \tfrac{1}{2}+0+0; \\
\text{Baryon number}: \quad & 0+1 &=& \quad +1+0+0; \\
\text{Lepton number}: \quad & 0+0 &=& \quad 0+0+0; \\
\text{Strangeness}: \quad & 0+0 &=& \quad -1+1+0.
\end{aligned}$$

Thus the reaction is possible, through the strong interaction.

(d) For the reaction

$$\pi^+ + \mathrm{p} \to \Sigma^0 + \pi^0,$$

the conservation laws are

$$\text{Charge}: \quad +1+1 \quad \neq \quad 0+0;$$

Thus the reaction is forbidden, charge is not conserved.

(e) For the reaction

$$\pi^- + \mathrm{p} \to \pi + \mathrm{e}^- + \tilde{\nu}_\mathrm{e},$$

the conservation laws are

$$\begin{aligned}
\text{Charge}: \quad & -1+1 &=& \quad +1-1+0; \\
\text{Spin}: \quad & 0+\tfrac{1}{2} &=& \quad \tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}; \\
\text{Baryon number}: \quad & 0+1 &=& \quad +1+0+0; \\
\text{Lepton number}: \quad & 0+0 &=& \quad 0+1-1; \\
\text{Strangeness}: \quad & 0+0 &=& \quad 0+0+0.
\end{aligned}$$

Thus the reaction is possible, through the weak interaction.

(f) For the reaction

$$\pi^- + \mathrm{p} \to \mathrm{K}^0 + \mathrm{p} + \pi^0,$$

the conservation laws are

$$\text{Charge}: \quad -1+1 \quad \neq \quad 0+1+0;$$

Thus the reaction is forbidden, charge is not conserved.

(g) For the reaction

$$\mathrm{K}^- + \mathrm{p} \to \Lambda^0 + \pi^0,$$

the conservation laws are

$$\begin{aligned}
\text{Charge}: \quad & -1+1 &=& \quad 0+0; \\
\text{Spin}: \quad & 0+\tfrac{1}{2} &=& \quad \tfrac{1}{2}+0; \\
\text{Baryon number}: \quad & 0+1 &=& \quad +1+0; \\
\text{Lepton number}: \quad & 0+0 &=& \quad 0+0; \\
\text{Strangeness}: \quad & -1+0 &=& \quad -1+0.
\end{aligned}$$

Thus the reaction is possible, through the strong interaction.

(h) For the reaction

$$\mathrm{K}^+ + \mathrm{n} \to \Sigma^+ + \pi^0 + \gamma,$$

the conservation laws are

$$\begin{aligned}
\text{Charge}: \quad & +1+0 &=& \quad +1+0+0; \\
\text{Spin}: \quad & 0+\tfrac{1}{2} &=& \quad -\tfrac{1}{2}+0+1; \\
\text{Baryon number}: \quad & 0+1 &=& \quad +1+0+0; \\
\text{Lepton number}: \quad & 0+0 &=& \quad 0+0+0; \\
\text{Strangeness}: \quad & +1+0 &\neq& \quad -1+0+0.
\end{aligned}$$

Thus the reaction is forbidden, strangeness is not conserved.

(i) For the reaction

$$\mathrm{K}^+ \to \pi^0 + \pi^0 + \pi^+,$$

the conservation laws are

$$\begin{aligned} \text{Charge}: \quad +1 &= 0+0+1; \\ \text{Spin}: \quad 0 &= 0+0+0; \\ \text{Baryon number}: \quad 0 &= 0+0+0; \\ \text{Lepton number}: \quad 0 &= 0+0+0; \\ \text{Strangeness}: \quad +1 &= 0+0+0. \end{aligned}$$

Thus the reaction is possible, through the weak interaction.

(j) For the reaction

$$\pi^+ \to \mathrm{e}^+ + \nu_\mathrm{e},$$

the conservation laws are

$$\begin{aligned} \text{Charge}: \quad +1 &= +1+0; \\ \text{Spin}: \quad 0 &= \tfrac{1}{2}-\tfrac{1}{2}; \\ \text{Baryon number}: \quad 0 &= 0+0; \\ \text{Lepton number}: \quad 0 &= -1+1; \\ \text{Strangeness}: \quad 0 &= 0+0. \end{aligned}$$

Thus the reaction is possible, through the weak interaction.

8–49. For the decay

$$\Lambda^0 \to \mathrm{p} + \pi^-,$$

calculate (a) the Q-value (energy released), and (b) the kinetic energy of the p and π^-, assuming the Λ^0 decays from rest. (Use relativistic formulas.)

Solution. (a) For the decay

$$\Lambda^0 \to \mathrm{p} + \pi^-,$$

the Q-value is

$$\begin{aligned} Q &= \left(m(\Lambda^0) - m(\mathrm{p}) - m(\pi^-)\right) c^2 \\ &= \left((1\,115.7\ \mathrm{MeV}/c^2) - (938.3\ \mathrm{MeV}/c^2) - (139.6\ \mathrm{MeV}/c^2)\right) c^2 \\ &= 37.8\ \mathrm{MeV}. \end{aligned}$$

(b) The total kinetic energy after the decay of the stationary Λ^0 is the Q-value:

$$K_\mathrm{p} + K_\pi = Q = 37.8\ \mathrm{MeV}.$$

For energy conservation we have

$$m_\Lambda c^2 = E_\mathrm{p} + E_\pi,$$

or

$$E_\mathrm{p} = m_\Lambda c^2 - E_\pi.$$

For momentum conservation we have

$$0 = p_\mathrm{p} - p_\pi,$$

or

$$(p_\mathrm{p}c)^2 = (p_\pi c)^2 = E_\mathrm{p}^2 - (m_\mathrm{p}c^2)^2 = E_\pi^2 - (m_\pi c^2)^2.$$

When we combine this with the result from energy conservation, we get

$$\begin{aligned} E_\mathrm{p} &= \frac{(m_\Lambda c^2)^2 + (m_\mathrm{p}c^2)^2 - (m_\pi c^2)^2}{2m_\Lambda c^2} \\ &= \frac{(1\,115.7\ \mathrm{MeV})^2 + (938.3\ \mathrm{MeV})^2 - (139.6\ \mathrm{MeV})^2}{2(1\,115.7\ \mathrm{MeV})} \\ &= 943.7\ \mathrm{MeV}. \end{aligned}$$

For the kinetic energies we have

$$\begin{aligned} K_\mathrm{p} &= E_\mathrm{p} - m_\mathrm{p}c^2 = 943.7\ \mathrm{MeV} - 938.3\ \mathrm{MeV} = 5.4\ \mathrm{MeV}; \\ K_\pi &= Q - K_\mathrm{p} = 37.8\ \mathrm{MeV} - 5.4\ \mathrm{MeV} = 32.4\ \mathrm{MeV}. \end{aligned}$$

8–50. Symmetry breaking occurs in the electroweak theory at distances of about 10^{-18} m. Show that this corresponds to an energy that is on the order of the mass of the $\mathrm{W}^\pm$.

Solution. We estimate the energy from

$$\begin{aligned} \Delta x &\approx \frac{\hbar}{2\,\Delta p} = \frac{\hbar c}{2\,\Delta p\, c} = \frac{\hbar c}{2\,\Delta E}; \\ 10^{-18}\ \mathrm{m} &= \frac{(0.658 \times 10^{-15}\ \mathrm{eV \cdot s})(3.00 \times 10^8\ \mathrm{m/s})}{2\,\Delta E}, \end{aligned}$$

which gives

$$\Delta E = 0.99 \times 10^{11}\ \mathrm{eV} \approx 10^2\ \mathrm{GeV}.$$

This is on the order of the 80 GeV rest energy of the W particles.

8–51. The mass of a π^0 can be measured by observing the reaction

$$\pi^- + \mathrm{p} \rightarrow \pi^0 + \mathrm{n}$$

at very low incident π^- kinetic energy (assume it is zero). The neutron is observed to be emitted with a kinetic energy of 0.60 MeV. Use conservation of energy and momentum to determine the π^0 mass.

SOLUTION. For energy conservation we have

$$\begin{aligned} m_{\pi^-}c^2 + m_\mathrm{p}c^2 &= E_{\pi^0} + E_\mathrm{n} \\ &= E_{\pi^0} + K_\mathrm{n} + m_\mathrm{n}c^2; \\ 139.6\ \mathrm{MeV} + 938.3\ \mathrm{MeV} &= E_{\pi^0} + 0.60\ \mathrm{MeV} + 939.6\ \mathrm{MeV}, \end{aligned}$$

which gives $E_{\pi^0} = 137.7$ MeV.

For momentum conservation we have

$$\begin{aligned} 0 &= p_{\pi^0} - p_\mathrm{n}, \quad \text{or} \\ (p_\mathrm{n}c)^2 &= (p_{\pi^0}c)^2 \\ E_\mathrm{n}^2 - (m_\mathrm{n}c^2)^2 &= E_{\pi^0}^2 - (m_{\pi^0}c^2)^2; \\ (0.60\ \mathrm{MeV} + 939.6\ \mathrm{MeV})^2 & \\ - (939.6\ \mathrm{MeV})^2 &= (137.7\ \mathrm{MeV})^2 - (m_{\pi^0}c^2)^2, \end{aligned}$$

which gives $m_{\pi^0} = 133.5\ \mathrm{MeV}/c^2$.

8–52. Calculate the Q-value for each of the reactions, Eq. 8.4, for producing a pion.

SOLUTION. For the reaction

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{p} + \pi^0,$$

the Q-value is

$$Q = (2m_\mathrm{p} - 2m_\mathrm{p} - m_{\pi^0})\,c^2 = -m_{\pi^0}c^2 = -135.0\ \mathrm{MeV}.$$

For the reaction

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{n} + \pi^+,$$

the Q-value is

$$\begin{aligned} Q &= (2m_{\mathrm{p}} - m_{\mathrm{p}} - m_{\mathrm{n}} - m_{\pi^+})c^2 \\ &= 938.3\ \mathrm{MeV} - 939.6\ \mathrm{MeV} - 139.6\ \mathrm{MeV} = -140.9\ \mathrm{MeV}. \end{aligned}$$

8–53. Calculate the Q-value for the reaction

$$\pi^- + \mathrm{p} \to \Lambda^0 + \mathrm{K}^0,$$

when negative pions strike stationary protons. Estimate the minimum pion kinetic energy needed to produce this reaction.

Solution. For the reaction

$$\pi^- + \mathrm{p} \to \Lambda^0 + \mathrm{K}^0,$$

the Q-value is

$$\begin{aligned} Q &= \left(m(\pi^-) + m(\mathrm{p}) - m(\Lambda^0) - m(\mathrm{K}^0)\right) c^2 \\ &= \left((139.6\ \mathrm{MeV}/c^2) + (938.3\ \mathrm{MeV}/c^2) - (1\,115.7\ \mathrm{MeV}/c^2)\right. \\ &\quad \left. - (497.7\ \mathrm{MeV}/c^2)\right) c^2 \\ &= -535.5\ \mathrm{MeV}. \end{aligned}$$

The minimum initial kinetic energy produces the two particles moving together with the same speed, which we can consider to be a single particle with mass

$$M = m_\Lambda + m_{\mathrm{K}}.$$

For energy conservation we have

$$E_\pi + m_{\mathrm{p}}c^2 = E_M.$$

For momentum conservation we have

$$p_\pi + 0 = p_M,$$

or

$$(p_\pi c)^2 = (p_M c)^2 = E_\pi^2 - (m_\pi c^2)^2 = E_M^2 - (Mc^2)^2.$$

When we combine this with the result from energy conservation, we get

$$\begin{aligned} E_\pi &= \frac{(Mc^2)^2 - (m_\pi c^2)^2 - (m_\mathrm{p}c^2)^2}{2m_\mathrm{p}c^2} \\ &= \frac{(m_\Lambda c^2 + m_\mathrm{K}c^2)^2 - (m_\pi c^2)^2 - (m_\mathrm{p}c^2)^2}{2m_\mathrm{p}c^2} \\ &= \frac{(1\,115.7\ \mathrm{MeV} + 497.7\ \mathrm{MeV})^2 - (139.6\ \mathrm{MeV})^2}{2(938.3\ \mathrm{MeV})} \\ &\quad \frac{-(938.3\ \mathrm{MeV})^2}{2(938.3\ \mathrm{MeV})} \\ &= 907.6\ \mathrm{MeV}. \end{aligned}$$

For the threshold kinetic energy we have

$$K_\pi = E_\pi - m_\pi c^2 = 907.6\ \mathrm{MeV} - 139.6\ \mathrm{MeV} = 768.0\ \mathrm{MeV}.$$

8–54. How many fundamental fermions are there in a water molecule?

SOLUTION. We consider the quarks and leptons as the fundamental fermions. A water molecule consists of two hydrogen atoms and one oxygen atom, or $2 + 8 = 10$ protons, $2 + 8 = 10$ electrons, and 8 neutrons. Each of the protons and neutrons is made up of three quarks, so the number of fundamental fermions is

$$N = (10 + 8)(3) + 10 = 64.$$

8–55. Determine the maximum kinetic energy of (a) the positron, and (b) the π^-, in the decay

$$\mathrm{K}^0 \to \pi^- + \mathrm{e}^+ + \nu_\mathrm{e}.$$

SOLUTION. (a) For the decay

$$\mathrm{K}^0 \to \pi^- + \mathrm{e}^+ + \nu_\mathrm{e},$$

the positron will have maximum kinetic energy when the pion has no kinetic energy. Note that the neutrino, with essentially zero mass, cannot

have zero kinetic energy. The total kinetic energy after the decay of the stationary K^0 is the Q-value:

$$\begin{aligned} K_\mathrm{e} + K_\nu = Q &= \left(m(\mathrm{K}^0) - m(\pi^-) - m(\mathrm{e}^+)\right) c^2 \\ &= \left((497.7\ \mathrm{MeV}/c^2) - (139.6\ \mathrm{MeV}/c^2)\right. \\ &\quad \left. - (0.511\ \mathrm{MeV}/c^2)\right) c^2 \\ &= 357.6\ \mathrm{MeV}. \end{aligned}$$

For energy conservation we have

$$m_\mathrm{K} c^2 = m_\pi c^2 + E_\mathrm{e} + E_\nu,$$

or

$$E_\nu = (m_\mathrm{K} c^2 - m_\pi c^2) - E_\mathrm{e}.$$

The positron and neutrino must move in opposite directions. For momentum conservation we have

$$0 = p_\mathrm{e} - p_\nu,$$

or

$$(p_\mathrm{e} c)^2 = E_\mathrm{e}^2 - (m_\mathrm{e} c^2)^2 = (p_\nu c)^2 = E_\nu^2.$$

When we combine this with the result from energy conservation, we get

$$\begin{aligned} E_\mathrm{e} &= \frac{(m_\mathrm{K} c^2 - m_\pi c^2)^2 + (m_\mathrm{e} c^2)^2}{2(m_\mathrm{K} c^2 - m_\pi c^2)} \\ &= \frac{(497.7\ \mathrm{MeV} - 139.6\ \mathrm{MeV})^2 + (0.511\ \mathrm{MeV})^2}{2(497.7\ \mathrm{MeV} - 139.6\ \mathrm{MeV})} = 179.1\ \mathrm{MeV}. \end{aligned}$$

For the kinetic energies we have

$$\begin{aligned} K_\nu = E_\nu &= (497.7\ \mathrm{MeV} - 139.6\ \mathrm{MeV}) - 179.1\ \mathrm{MeV} \\ &= 179.0\ \mathrm{MeV}; \\ K_\mathrm{e} = Q - K_\nu &= 357.6\ \mathrm{MeV} - 179.0\ \mathrm{MeV} = 178.6\ \mathrm{MeV}. \end{aligned}$$

Alternatively,

$$K_\mathrm{e} = E_\mathrm{e} - m_\mathrm{e} c^2 = 179.1\ \mathrm{MeV} - 0.511\ \mathrm{MeV} = 178.6\ \mathrm{MeV}.$$

(b) For the decay

$$\mathrm{K}^0 \to \pi^- + \mathrm{e}^+ + \nu_\mathrm{e},$$

the pion will have maximum kinetic energy when the positron has no kinetic energy. Note that the neutrino, with essentially zero mass, cannot have zero kinetic energy. The total kinetic energy after the decay of the stationary K^0 is the Q-value:

$$\begin{aligned} K_\pi + K_\nu = Q &= \left(m(\mathrm{K}^0) - m(\pi^-) - m(\mathrm{e}^+)\right) c^2 \\ &= \left((497.7\ \mathrm{MeV}/c^2) - (139.6\ \mathrm{MeV}/c^2)\right. \\ &\quad \left. - (0.511\ \mathrm{MeV}/c^2)\right) c^2 \\ &= 357.6\ \mathrm{MeV}. \end{aligned}$$

For energy conservation we have

$$m_\mathrm{K} c^2 = m_\mathrm{e} c^2 + E_\pi + E_\nu,$$

or

$$E_\nu = (m_\mathrm{K} c^2 - m_\mathrm{e} c^2) - E_\pi.$$

The pion and neutrino must move in opposite directions. For momentum conservation we have

$$0 = p_\pi - p_\nu,$$

or

$$(p_\mathrm{p} c)^2 = E_\pi^2 - (m_\mathrm{p} c^2)^2 = (p_\nu c)^2 = E_\nu^2.$$

When we combine this with the result from energy conservation, we get

$$\begin{aligned} E_\mathrm{p} &= \frac{(m_\mathrm{K} c^2 - m_\mathrm{e} c^2)^2 + (m_\pi c^2)^2}{2(m_\mathrm{K} c^2 - m_\mathrm{e} c^2)} \\ &= \frac{(497.7\ \mathrm{MeV} - 0.511\ \mathrm{MeV})^2 + (139.6\ \mathrm{MeV})^2}{2(497.7\ \mathrm{MeV} - 0.511\ \mathrm{MeV})} = 268.2\ \mathrm{MeV}. \end{aligned}$$

For the kinetic energies we have

$$\begin{aligned} K_\nu &= E_\nu = (497.7\ \mathrm{MeV} - 0.511\ \mathrm{MeV}) - 268.2\ \mathrm{MeV} = 229.0\ \mathrm{MeV}; \\ K_\pi &= Q - K_\nu = 357.6\ \mathrm{MeV} - 229.0\ \mathrm{MeV} = 128.6\ \mathrm{MeV}. \end{aligned}$$

Alternatively,

$$K_\pi = E_\pi - m_\pi c^2 = 268.2 \text{ MeV} - 139.6 \text{ MeV} = 128.6 \text{ MeV}.$$

8–56. (a) Show that the so-called unification distance of 10^{-30} m in grand unified theory is equivalent to an energy of about 10^{14} GeV. Use either the uncertainty principle or de Broglie's wavelength formula, and explain how they apply. (b) Calculate what temperature this corresponds to.

SOLUTION. (a) If the momentum is taken to be the minimum allowed by the uncertainty principle, we have

$$E = pc = c\,\Delta p.$$

Thus we estimate the energy from

$$\begin{aligned} \Delta x &\approx \frac{\hbar}{2\,\Delta p} = \frac{\hbar c}{2c\,\Delta p} = \frac{\hbar c}{2E}; \\ 10^{-30} \text{ m} &= \frac{(0.658 \times 10^{-15} \text{ eV} \cdot \text{s})(3.00 \times 10^{8} \text{ m/s})}{2E}, \end{aligned}$$

which gives

$$E \approx 10^{23} \text{ eV} = 10^{14} \text{ GeV}.$$

(b) We find the temperature from

$$\begin{aligned} E &= k_\text{B} T; \\ (10^{14} \text{ GeV})(1.60 \times 10^{-10} \text{ J/GeV}) &= (1.38 \times 10^{-23} \text{J/K})\, T. \end{aligned}$$

which gives $T \approx 10^{27}$ K.

8–57. For the reaction

$$\text{p} + \text{p} \rightarrow 3\text{p} + \tilde{\text{p}},$$

where one of the initial protons is at rest, use relativistic formulas to show that the threshold energy is $6m_\text{p}c^2$, equal to three times the Q-value of the reaction, where m_p is the proton mass.

SOLUTION. For the reaction

$$\text{p} + \text{p} \rightarrow 3\text{p} + \tilde{\text{p}}$$

the Q-value is

$$Q = (2m_\mathrm{p} - 4m_\mathrm{p})\, c^2 = -2m_\mathrm{p}c^2.$$

The minimum initial kinetic energy produces the four particles moving together with the same speed, which we can consider to be a single particle with mass $M = 4m_\mathrm{p}$.

For energy conservation we have

$$E_\mathrm{p} + m_\mathrm{p}c^2 = E_M.$$

For momentum conservation we have

$$p_\mathrm{p} + 0 = p_M,$$

or

$$(p_\mathrm{p}c)^2 = (p_M c)^2 = E_\mathrm{p}^2 - (m_\mathrm{p}c^2)^2 = E_M^2 - (Mc^2)^2.$$

When we combine this with the result from energy conservation, we get

$$E_\mathrm{p} = \frac{(Mc^2)^2 - 2(m_\mathrm{p}c^2)^2}{2m_\mathrm{p}c^2} = \frac{(4m_\mathrm{p}c^2)^2 - 2(m_\mathrm{p}c^2)^2}{2m_\mathrm{p}c^2} = 7m_\mathrm{p}c^2.$$

For the threshold kinetic energy we have

$$K_\mathrm{p} = E_\mathrm{p} - m_\mathrm{p}c^2 = 7m_\mathrm{p}c^2 - m_\mathrm{p}c^2 = 6m_\mathrm{p}c^2.$$

This is three times the magnitude of the Q-value.

8–58. The lifetimes listed in Table B.1 are in terms of *proper time*, measured in a reference frame where the particle is at rest. If a τ-lepton is created with a kinetic energy of 450 MeV, how long would its track be as measured in the lab, on average, ignoring any collisions?

Solution. We find the speed of the τ-lepton from its kinetic energy:

$$\begin{aligned} K &= (\gamma - 1)\, mc^2; \\ 450\ \mathrm{MeV} &= (\gamma - 1)\,(1\,777\ \mathrm{MeV}), \end{aligned}$$

which gives

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = 1.25,$$

and $v = 0.603\,c$.

In the lab the lifetime of the τ-lepton will be dilated:

$$\Delta t = \gamma \Delta t_0.$$

Thus the length of the track will be

$$\begin{aligned} \Delta x &= v\,\Delta t = (0.603)(3.00 \times 10^8 \text{ m/s})(2.91 \times 10^{-13} \text{ s})(1.25) \\ &= 6.58 \times 10^{-5} \text{ m}. \end{aligned}$$

8–59. A particle at rest, with a rest-energy of $m_{\mathrm{P}}c^2$, decays into two fragments with rest energies of $m_{\mathrm{D1}}c^2$ and $m_{\mathrm{D2}}c^2$. Show that the kinetic energy of fragment D1 is

$$K_{\mathrm{D1}} = \frac{1}{2m_{\mathrm{P}}c^2}\left((m_{\mathrm{P}}c^2 - m_{\mathrm{D1}}c^2)^2 - (m_{\mathrm{D2}}c^2)^2\right).$$

Solution. For energy conservation of the particle's decay, we have

$$m_{\mathrm{P}}c^2 = E_{\mathrm{D1}} + E_{\mathrm{D2}} = K_{\mathrm{D1}} + m_{\mathrm{D1}}c^2 + K_{\mathrm{D2}} + m_{\mathrm{D2}}c^2.$$

The two decay fragments must move in opposite directions.

For momentum conservation we have

$$0 = p_{\mathrm{D1}} - p_{\mathrm{D2}},$$

or

$$(p_{\mathrm{D1}}c)^2 = (p_{\mathrm{D2}}c)^2 = E_{\mathrm{D1}}^2 - (m_{\mathrm{D1}}c^2)^2 = E_{\mathrm{D2}}^2 - (m_{\mathrm{D2}}c^2)^2.$$

When we use the result from energy conservation, we get

$$E_{\mathrm{D1}}^2 - (m_{\mathrm{D1}}c^2)^2 = (m_{\mathrm{P}}c^2 - E_{\mathrm{D1}})^2 - (m_{\mathrm{D2}}c^2)^2,$$

which gives

$$E_{\mathrm{D1}} = \frac{(m_{\mathrm{P}}c^2)^2 + (m_{\mathrm{D1}}c^2)^2 - (m_{\mathrm{D2}}c^2)^2}{2m_{\mathrm{P}}c^2}.$$

Thus the kinetic energy is

$$\begin{aligned} K_{\mathrm{D1}} &= E_{\mathrm{D1}} - m_{\mathrm{D1}}c^2 \\ &= \frac{(m_{\mathrm{P}}c^2)^2 + (m_{\mathrm{D1}}c^2)^2 - (m_{\mathrm{D2}}c^2)^2}{2m_{\mathrm{P}}c^2} - \frac{2m_{\mathrm{P}}c^2 m_{\mathrm{D1}}c^2}{2m_{\mathrm{P}}c^2} \\ &= \frac{(m_{\mathrm{P}}c^2 - m_{\mathrm{D1}}c^2)^2 - (m_{\mathrm{D2}}c^2)^2}{2m_{\mathrm{P}}c^2}. \end{aligned}$$

8–60. Use the quark model to describe the reaction

$$\tilde{\mathrm{p}} + \mathrm{n} \rightarrow \pi^- + \pi^0.$$

SOLUTION. From Table B.1 we see that the π^0 has charge $= 0$, $B = 0$, $L = 0$, and $S = 0$. From Table B.2 we see that the π^0 could be either $\mathrm{u}\tilde{\mathrm{u}}$ or $\mathrm{d}\tilde{\mathrm{d}}$. Actually it is a combination of the two, but we will simplify the analysis by assuming the π^0 is $\mathrm{d}\tilde{\mathrm{d}}$. Thus we can write the reaction as

$$\begin{aligned} \tilde{\mathrm{p}} + \mathrm{n} &\rightarrow \pi^- + \pi^0; \\ \tilde{\mathrm{u}}\tilde{\mathrm{u}}\tilde{\mathrm{d}} + \mathrm{udd} &\rightarrow \tilde{\mathrm{u}}\mathrm{d} + \mathrm{d}\tilde{\mathrm{d}}. \end{aligned}$$

Thus we see the annihilation of a $\mathrm{u}\tilde{\mathrm{u}}$ pair.

8–61. What fraction of the speed of light c is the speed of a 7.0 TeV proton?

SOLUTION. We find the speed of the proton from its kinetic energy:

$$\begin{aligned} K &= (\gamma - 1)\, mc^2; \\ 7.0 \times 10^6 \text{ MeV} &= (\gamma - 1)\,(938 \text{ MeV}), \end{aligned}$$

which gives

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = 7.46 \times 10^3.$$

This means the speed is very close to the speed of light. If we rearrange and use the approximation

$$\sqrt{1 - x} \approx 1 - \tfrac{1}{2}x,$$

we get

$$\beta = \sqrt{1 - \gamma^{-2}} \approx 1 - \frac{1}{2\gamma^2} = 1 - \frac{1}{2(7.46 \times 10^3)^2},$$

which gives

$$\beta = 1 - 9 \times 10^{-9}.$$

9 Astrophysics and Cosmology

9.1 Review

The night sky contains myriads of stars including those in the Milky Way, which is a "side view" of our Galaxy looking along the plane of the disc (Fig. 9.1). Our Galaxy includes about 10^{11} stars. Beyond our Galaxy are billions of other galaxies.

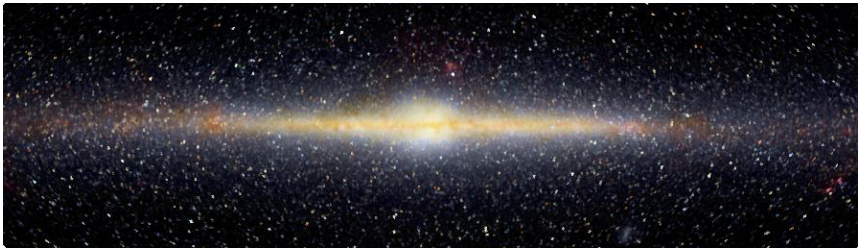


Figure 9.1. Infrared edge-on view photograph of our Galaxy, the Milky Way, taken by the COBE satellite, showing the central bulge of the Galaxy.

Astronomical distances are measured in light-years (1 ly $\approx 10^{13}$ km). The nearest star is about 4 ly away and the nearest other galaxy is 2 million ly away. Our galactic disc has a diameter of about 100 000 ly and the thickness of very roughly 2 000 ly. It has a bulging central nucleus and spiral arms. Our Sun, which seems to be just another star, is located more than halfway from the center to the edge, about 28 000 ly from the center. The Sun orbits the galactic center approximately once every 200 million years, so its speed is about 250 km/s relative to the center of the Galaxy. The total mass of all the stars in our Galaxy is about 3×10^{41} kg.

Example 9–1. Estimate the total mass of our Galaxy using the orbital data of the Sun (including our solar system) about the center of the Galaxy. Assume that most of the mass of the Galaxy approximated as a uniform sphere of mass (the central bulge, Fig. 9.1).

Solution. Our Sun and solar system orbit the center of the Galaxy, according to the best measurements as mentioned above, with a speed of about $v = 250$ km/s at a distance from the Galaxy center of about

$r = 28\,000$ ly. We use Newton's second law, $F = ma$, with a being the centripetal acceleration $a = v^2/r$, and F being the universal law of gravitation

$$\begin{aligned} F &= ma \\ G\frac{Mm}{r^2} &= m\frac{v^2}{r} \end{aligned}$$

where M is the mass of the Galaxy and m is the mass of our Sun and solar system. Solving this, we find

$$M = \frac{rv^2}{G} \approx \frac{(28\,000 \text{ ly})(10^{16} \text{ m/ly})(2.5 \times 10^5 \text{ m/s})^2}{6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2} \approx 3 \times 10^{41} \text{ kg}.$$

In terms of numbers of stars, if they are like our Sun ($m = 2.0 \times 10^{30}$ kg), there would be about $(3 \times 10^{41} \text{ kg})/(2 \times 10^{30} \text{ kg}) \approx 10^{11}$ or about 100 billion stars.

Galaxies tend to be grouped in galaxy clusters, with anywhere from a few to many thousands of galaxies in each cluster. Furthermore, clusters themselves seem to be organized into even larger aggregates: clusters of clusters of galaxies, or superclusters. The galaxies nearest us are about 2 million light-years away. The farthest detectable galaxies are thousands of times farther away, on the order of 10^{10} ly (Table 9.1).

Table 9.1. Heavenly distances

Object	Approximate distance from Earth (ly)
Moon	4×10^{-8}
Sun	1.6×10^{-5}
Farthest planet in our solar system (Pluto)	6×10^{-4}
Nearest star (Proxima Centauri)	4.3
Center of our Galaxy	3×10^4
Nearest galaxy (Andromeda)	2×10^6
Farthest galaxies	10^{10}

Example 9–2. Astronomers often think of their telescopes as time machines, looking back toward the origin of the universe. How far back do they look?

SOLUTION. The distance in light-years measures exactly how long in years the light has been traveling to reach us, so Table 9.1 tells us also how far back in time we are looking. For example, if we saw Proxima Centauri explode into a supernova today, then the event would have really occurred 4.3 years ago. The most distant objects, galaxies 10^{10} ly away, emitted the light we see now 10^{10} years ago, so what we see was how they were then, close to the beginning of the universe.

One basic technique for measuring the vast distances of objects in the universe employs simple geometry to measure the *parallax* of a star. By parallax we mean the apparent motion of a star, against the background of more distant stars, due to the Earth's motion about the Sun. As shown in Fig. 9.2, the sighting angle of a star relative to the plane of Earth's orbit (angle θ) is measured at different times of the year. Since we know the distance d from Earth to Sun, we can reconstruct the right triangles shown in the figure and can determine the distance D to the star.

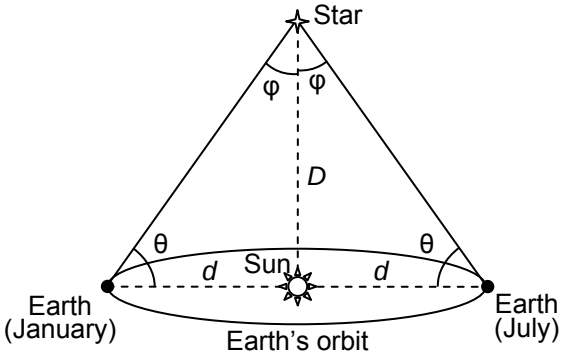


Figure 9.2. Distance to a star determined by parallax. (Not to scale.)

Example 9–3. Estimate the distance D to a star if the angle θ in Fig. 45-6 is measured to be $89.99994°$.

SOLUTION. The angle φ in Fig. 9.2 is $0.00006°$, or about 1.0×10^{-6} radians. Since $\tan\varphi = d/D$, then the distance D to the star is

$$D = \frac{d}{\tan\varphi} = \frac{d}{\varphi} = \frac{1.5 \times 10^{8}\ \text{km}}{1.0 \times 10^{-6}\ \text{rad}} = 1.5 \times 10^{14}\ \text{km},$$

or about 15 ly. (We can use $\tan\varphi \approx \varphi$ since φ is very small.)

Distances are often specified in parsecs (meaning *par*allax angle in *sec*onds of arc), where the *parsec* (pc) is defined as $1/\varphi$ and φ is given in seconds. $1\text{ pc} = 3.26\text{ ly} = (3.26\text{ ly})(9.46\times10^{15}\text{ m}) = 3.08\times10^{16}\text{ m}$.

Parallax can be used to determine the distance to stars as far as 100 light-years (≈ 30 parsecs). Beyond that distance, parallax angles are too small to measure. For greater distances, more subtle techniques must be employed.

A useful parameter for a star or galaxy is its *absolute luminosity*, L, by which we mean the total power radiated in watts. For Sun, $L = 3.9\times10^{26}$ W. Also important is the *apparent brightness*, b, defined as the power crossing unit area perpendicular to the path of the flight at the Earth. Given that energy is conserved, and ignoring any absorption in space, the total emitted power L at a distance d from the star will be spread over a sphere of surface area $4\pi d^2$. If we let d be the distance from the star to the Earth, then L must be equal to $4\pi d^2$ times b (power per unit area at Earth). That is,

$$b = \frac{L}{4\pi d^2}. \tag{9.1}$$

Example 9–4. Suppose a particular star has absolute luminosity equal to that of the Sun but is 10 pc away from Earth. By what factor will it appear dimmer than the Sun?

SOLUTION. The luminosity L is the same for both stars, so the apparent brightness depends only on their relative distances. We use the inverse square law as stated above in Eq. 9.1: the star appears dimmer by a factor

$$\frac{b_1}{b_2} = \frac{d_1^2}{d_2^2} = \frac{(1.5\times10^8\text{ km})^2}{(10\text{ pc})^2(3.26\text{ ly/pc})^2(10^{13}\text{ km/ly})^2} \approx 2\times10^{-13},$$

where d_1 and d_2 are the distances from Earth to the star and to the Sun, respectively.

For most stars, the absolute luminosity depends on the mass: the more massive the star, the greater its luminosity. Another important parameter of a star is its surface temperature, which can be determined from the spectrum of electromagnetic frequencies it emits, just as for a blackbody (Chapter 2). The relation is given by Wien's law (Eq. 2.1): the peak

wavelength in the spectrum of light emitted by a blackbody (and stars are fairly good approximations to blackbodies) is inversely proportional to its kelvin temperature. The surface temperatures of stars typically range about 3 500 K (reddish) to perhaps 50 000 K (bluish).

For most stars, the color is related to the absolute luminosity and therefore to the useful way to present this relationship is by the so-called Hertzsprung–Russell (H–R) diagram. On the H–R diagram, the horizontal axis shows the temperature T whereas the vertical axis is the luminosity L, and each star is represented by a point on the diagram, Fig. 9.3. Most stars fall along the diagonal band termed the *main sequence*. Starting at the lower right we find the coolest stars, reddish in color; they are the least luminous and therefore of low mass. Farther up toward the left we find hotter and more luminous stars that are yellowish-white, like our Sun. Still farther up we find still more massive and more luminous stars, bluish in color. Stars that fall on this diagonal band are called *main sequence stars*. There are also stars that fall outside the main sequence. Above and to the right we find extremely large stars, with high luminosities but with low (reddish) color temperature: these are called *red giants*. At the lower left, there are a few stars of low luminosity but with high temperature: these are the *white dwarfs*.

Example 9–5. Suppose that the distances to two nearby stars can be reasonably estimated and this data, together with their measured apparent brightnesses, suggests that the two stars have about the same absolute luminosity, L. The spectrum of one of the stars peaks at about 700 nm (so it is reddish). The spectrum of the other peaks at about 350 nm (bluish). Use Wien's law (Eq. 2.1) and the Stefan–Boltzmann equation to determine (a) the surface temperature of each star, and (b) how much larger one star is than the other.

Solution. (a) Wien's law states that $\lambda_P T = 2.90 \times 10^{-3}\ \mathrm{m \cdot K}$. So the temperature of the reddish star is

$$T_r = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{700 \times 10^{-9}\ \mathrm{m}} = 4\,140\ \mathrm{K}.$$

The temperature of the bluish star will be double this since its peak wave-

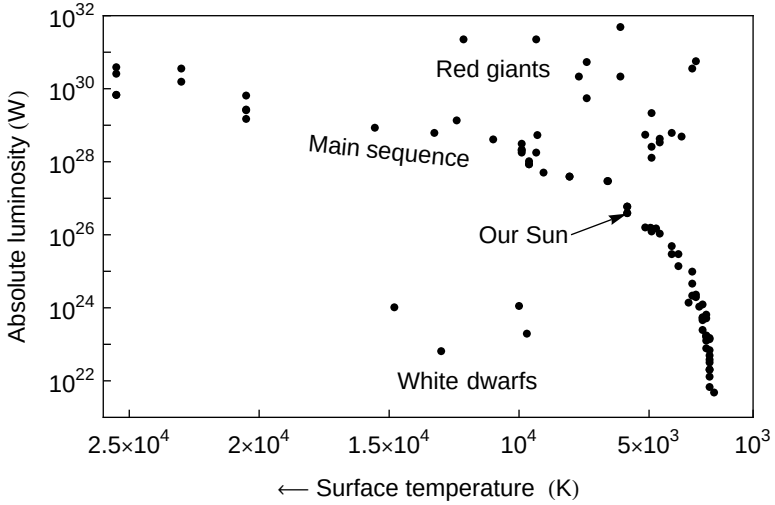


Figure 9.3. Hertzsprung–Russell (H–R) diagram.

length is half (350 nm vs. 700 nm); just to check:

$$T_{\mathrm{b}} = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{350 \times 10^{-9}\ \mathrm{m}} = 8\,280\ \mathrm{K}.$$

(b) The Stefan–Boltzmann equation states that the power dQ/dt radiated from a body is proportional to the fourth power of the kelvin temperature, T^4:

$$\frac{dQ}{dt} = e\sigma A T^4. \tag{9.2}$$

The factor $0 < e < 1$ is the *emissivity* of the material, A is the area of the emitting object, and $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$ is the *Stefan–Boltzmann constant*.

Now the temperature of the bluish star is double that of the reddish star, so the bluish one must radiate $2^4 = 16$ times as much energy per unit area. But we are given that they have the same luminosity (the same total power output), so the surface area of the blue star must be $\frac{1}{16}$ that of the

red one. Since the surface area of a sphere is $4\pi r^2$, we conclude that the radius of the reddish star is $\sqrt{16} = 4$ times larger than the radius of the bluish star (or $4^3 = 64$ times the volume).

Example 9–6. Suppose that detailed study of a certain star suggests that it most likely fits on the main sequence of an H–R diagram. Its measured apparent brightness is $b = 1.0 \times 10^{-12}$ W/m^2, and the peak wavelength of its spectrum is $\lambda_P \approx 600$ nm. Estimate its distance from us.

SOLUTION. The star's temperature, from Wien's law (Eq. 2.1), is

$$T \approx \frac{2.90 \times 10^{-3} \text{ m} \cdot \text{K}}{600 \times 10^{-9} \text{ m}} \approx 4\,800 \text{ K}.$$

A star on the main sequence of an H–R diagram at this temperature has absolute luminosity of about $L \approx 1 \times 10^{26}$ W, read off of Fig. 9.3. Then, from Eq. 9.1,

$$d = \sqrt{\frac{L}{4\pi b}} \approx \sqrt{\frac{1 \times 10^{26} \text{ W}}{4(3.14)(1.0 \times 10^{-12} \text{ W/m}^2)}} \approx 3 \times 10^{18} \text{ m}.$$

Its distance from us in light-years and parsecs is

$$d = \frac{3 \times 10^{18} \text{ m}}{10^{16} \text{ m/ly}} \approx 300 \text{ ly} \approx \frac{300 \text{ ly}}{3.26 \text{ ly/pc}} \approx 90 \text{ pc}.$$

Stars are believed to begin life as collapsing masses of hydrogen gas (protostars). As they contract, they heat up (potential energy is transformed to kinetic energy). When the temperature reaches 10 million degrees, nuclear fusion begins and forms heavier elements (nucleosynthesis), mainly helium at first. The energy released during these reactions balances the gravitational force, and the young star stabilizes as a main-sequence star. The tremendous luminosity of stars comes from the energy released during these thermonuclear reactions. After billions of years, as helium is collected in the core and hydrogen is used up, the core contracts and heats further. The envelope expands and cools, and the star becomes a red giant (larger diameter, redder color). So it will have moved to the right and upward on the H–R diagram, as shown in Fig. 9.4.

As a star's envelope expands, the core is shrinking and heating up. When the temperature reaches about 10^8 K, even helium nuclei, in spite

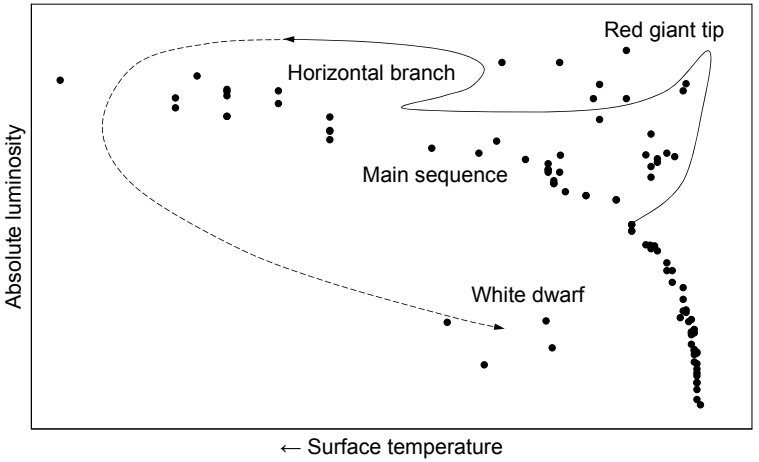


Figure 9.4. Evolution of a star like our Sun represented on an H–R diagram.

of their greater charge and hence greater electrical repulsion, can then undergo fusion. The reactions are

$$\begin{array}{lcll} {}^4_2\mathrm{He} + {}^4_2\mathrm{He} & \rightarrow & {}^8_4\mathrm{Be} + \gamma & (Q = -95 \text{ keV}) \\ {}^4_2\mathrm{He} + {}^8_4\mathrm{Be} & \rightarrow & {}^{12}_6\mathrm{C} + \gamma & (Q = 7.4 \text{ MeV}). \end{array} \tag{9.3}$$

The first reaction is (slightly) endothermic, and the ${}^6_4\mathrm{Be}$ formed is unstable and can decay quickly (10^{-16} s) back to two α particles. So the second reaction must occur very quickly after the first, and this is aided by the particularly large cross section for the second reaction. The net effect is

$$3\,{}^4_2\mathrm{He} \rightarrow {}^{12}_6\mathrm{C} \qquad (Q = 7.3 \text{ MeV}).$$

The burning of helium causes a rapid and major change in the star, and the star moves rapidly to the "horizontal branch" on the H–R diagram (see Fig. 9.4). Further reactions are possible, creating elements of higher and higher Z, up to $Z = 10$ or 12. If the mass of the star is greater than about 0.7 solar masses, then still higher Z elements can be formed. The star can get even hotter and in the range $T = 2.5$–5×10^9 K, nuclei as heavy as ${}^{56}_{26}\mathrm{Fe}$ and ${}^{56}_{28}\mathrm{Ni}$ can be made. But here the process of nucleosynthesis ends. As we

saw in Fig. 6.1, the average binding energy per nucleon begins to decrease for A greater than about 60. Further fusions would require energy, rather than release it.

The next stage of stellar evolution depends on the mass of the star. Stars of residual mass less than about 1. 4 solar masses cool further; and become white dwarfs, eventually fading and going out altogether. They typically follow the route shown in Fig. 9.4. Heavier stars contract further due to greater gravity: the density approaches nuclear density, the huge pressure forces electrons to combine with protons to form neutrons, and the star becomes essentially a huge nucleus of neutrons. This is a *neutron star*, and energy released from its final core contraction is believed to produce *supernovae* explosions. If the star's residual mass is greater than two or three solar masses, it may contract even further and form a *black hole*, which is so dense that no matter or light can escape from it.

A black hole is so dense that even light cannot escape from it. To become a black hole, a body of mass M must undergo *gravitational collapse*, contracting by gravitational self-attraction to within a radius called the *Schwarzschild radius*:

$$R = \frac{2GM}{c^2} \tag{9.4}$$

where G is the gravitational constant and c the speed of light. If a body collapses to within this radius, it is predicted by general relativity to rapidly ($\approx 10^{-5}$ s) collapse to a point at $r = 0$, forming an infinitely dense *singularity*. This prediction is uncertain, however, since in this realm we need to combine quantum mechanics with gravity, a unification of theories not yet fully achieved. The Schwarzschild radius also represents the *event horizon* of a black hole.

In the *general theory of relativity*, the *equivalence principle* states that an observer cannot distinguish acceleration from a gravitational field. Said another way, gravitational and inertial mass are the same. The theory predicts gravitational bending of light rays to a degree consistent with experiment. Gravity is treated as a curvature in space and time, the curvature being greater near massive bodies. The universe as a whole may be curved. If there is sufficient mass, the curvature of the universe is positive, and the universe is closed and finite; otherwise, it is open and infinite.

Distant galaxies display a redshift in their spectral lines, interpreted as a *Doppler shift*:

$$\lambda = \lambda_0 \sqrt{\frac{1 + v/c}{1 - v/c}}\,. \qquad \left(\begin{array}{l} \text{source and observer moving} \\ \text{away from each other} \end{array} \right) \tag{9.5}$$

where λ_0 is the emitted wavelength as seen in a reference frame at rest with respect to the source, and λ is the wavelength measured in a frame moving with velocity v away from the source along the line of sight. Thus, when a source emits light of a particular wavelength and the source is moving away from us, the wavelength appears longer to us: the color of the light (if it is visible) is shifted toward the red end of the visible spectrum. The amount of shift depends on the velocity of the source. For speeds not too close to the speed of light, it is easy to show (Problem 9–30) that the fractional change in wavelength is proportional to the speed of the source to or away from us:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\lambda - \lambda_0}{\lambda_0} \approx \frac{v}{c} \qquad (v \ll c). \tag{9.6}$$

The universe seems to be expanding, its galaxies racing away from each other at speeds (v) proportional to the distance (d) between them:

$$v = Hd, \tag{9.7}$$

which is known as *Hubble's law* ($H \approx 70$ km/s/Mpc is the *Hubble parameter*). This expansion of the universe suggests an explosive origin, the *Big Bang*, which probably occurred about 15 billion years ago.

Quasars ("quasistellar radio sources") are objects with a large redshift (suggesting great distance) and high luminosity (suggesting closeness or, more likely, extraordinary energy output).

The *cosmological principle* assumes that the universe, on a large scale, is homogeneous and isotropic.

Important evidence for the Big Bang model of the universe was the discovery of the *cosmic microwave background* radiation (CMB), which conforms to a blackbody radiation curve at a temperature of 2.7 K.

The *standard model* of the Big Bang provides a possible scenario as to how the universe developed as it expanded and cooled after the Big Bang

(Table 9.2). Starting at 10^{-43} seconds after the Big Bang, according to this model, there was a series of *phase transitions* during which previously unified forces of nature "condensed out" one by one. The *inflationary scenario* assumes that during one of these phase transitions, the universe underwent a brief but rapid exponential expansion. Until about 10^{-35} s, there was no distinction between quarks and leptons. Shortly thereafter, quarks were *confined* into hadrons (the *hadron era*). About 10^{-4} s after the Big Bang, the majority of hadrons disappeared, having combined with anti-hadrons, producing photons, leptons and energy, leaving mainly photons and leptons to freely move, thus introducing the *lepton era*. By the time the universe was about 10 s old, the electrons too had mostly disappeared, having combined with their antiparticles, and the universe became *radiation-dominated*. A couple of minutes later, nucleosynthesis began, but lasted only a few minutes. It then took several hundred thousand years before the universe was cool enough for electrons to combine with nuclei to form atoms. Also about this time, the background radiation had expanded and cooled so much that its total energy equaled the energy in matter. As the radiation cooled further, losing energy, the universe became *matter-dominated* (not in numbers, but in energy). Then stars and galaxies formed, producing a universe not much different than it is today—some 15 billion years later.

If the universe is open, it will continue to expand indefinitely. If it is closed, gravity is sufficiently strong to halt expansion and the universe will eventually begin to collapse back on itself, ending in a *Big Crunch*. Whether the universe is open or closed depends on whether its average mass density is above or below a critical density. The evidence today suggests that the universe is flat.

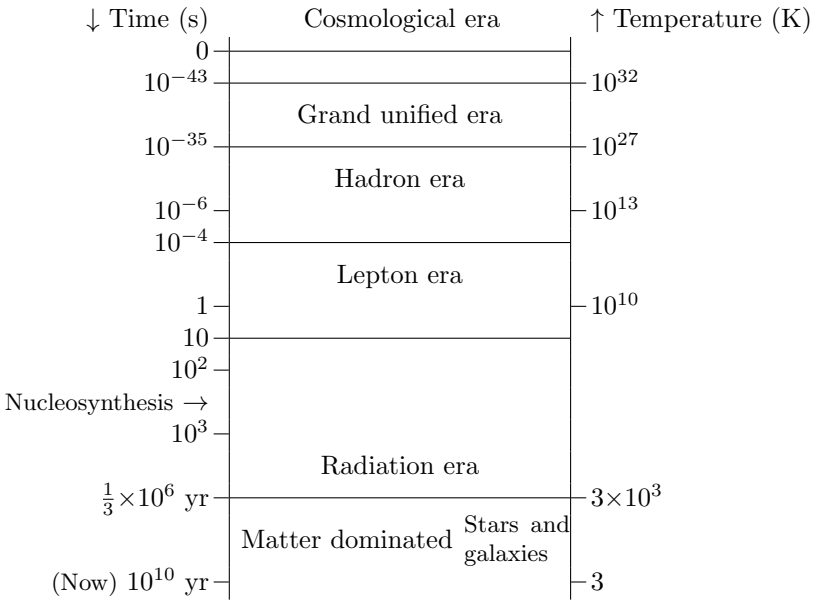
Example 9–7. Use energy conservation and the concept of escape velocity to estimate critical density of the universe.

SOLUTION. At the critical density, ρ_c, any given galaxy of mass m will just be able to "escape" away from our Galaxy. Escape can just occur if the total energy E of the galaxy satisfies

$$E = K + U = \frac{mv^2}{2} - G\frac{mM}{R} = 0.$$

Here R is the distance of that galaxy m from our Galaxy, and we assume on the enormous scale of the universe that each galaxy is essentially a

Table 9.2. Development of the universe after the Big Bang according to the standard model



point. The total mass M that pulls inward on m is the total mass of all galaxies within a sphere of radius R (recall Gauss theorem). If we assume the density of galaxies is roughly constant, then $M = \frac{4}{3}\pi\rho_c R^3$. Substituting this M into the above equation, and setting $v = HR$ (Hubble's law, Eq. 9.7), we obtain

$$\frac{GM}{R} = \frac{v^2}{2}$$

or

$$\frac{G\left(\frac{4}{3}\pi\rho_c R^3\right)}{R} = \frac{(HR)^2}{2}.$$

We solve for ρ_c:

$$\rho_c = \frac{3H^2}{8\pi G} \approx \frac{3\left((70\times10^3\ \text{m/s/Mpc})(1\ \text{Mpc}/3.08\times10^{22}\ \text{m})\right)^2}{8(3.14)(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)} \approx 10^{-26}\ \text{kg/m}^3.$$

Example 9–8. Suppose two neutrinos from SN1987a were detected on Earth (via the reaction $\tilde{\nu}_e + p \rightarrow n + e^+$) 10 seconds apart, with measured kinetic energies of about 20 MeV and 10 MeV. (a) Estimate the rest mass of the neutrino, m_ν, using this data, assuming both neutrinos were emitted at the same time. (b) Theoretical models of supernova explosions suggest that the neutrinos are emitted in a burst that lasts from a second or two up to perhaps 10 s. If we assume the neutrinos are not emitted simultaneously but rather at any time over a 10-s interval, what then can we say about the neutrino mass based on the data given above?

SOLUTION. (a) We expect the neutrino mass to be less than 100 eV (from laboratory measurements). Since our neutrinos have kinetic energy of 20 MeV and 10 MeV, we can make the approximation $m_\nu c^2 \ll E$, where E (the total energy) is essentially equal to the kinetic energy. From Eq. 1.12, we have

$$E = \gamma m_\nu c^2.$$

We solve this for v, the velocity of a neutrino with energy E:

$$v = c\left(1 - \frac{m_\nu^2 c^4}{E^2}\right)^{1/2} = c\left(1 - \frac{m_\nu^2 c^4}{2E^2} + \cdots\right),$$

where we have used the binomial expansion $(1+x)^{1/2} = 1 + \frac{1}{2}x + \ldots$, and we ignore higher-order terms since $m_\nu^2 c^4 \ll E^2$.

The time t for a neutrino to travel a distance d (= 170 000 ly) is

$$t = \frac{d}{v} = \frac{d}{c\,(1 - m_\nu^2 c^4/2E^2)} \approx \frac{d}{c}\left(1 + \frac{m_\nu^2 c^4}{2E^2}\right),$$

where again we used the binomial expansion $(1+x)^{-1} = 1 - x + \ldots$.

The difference in arrival times for our two neutrinos of energies $E_1 = 20$ MeV and $E_2 = 10$ MeV is

$$t_2 - t_1 = \frac{d}{c}\frac{m_\nu^2 c^4}{2}\left(\frac{1}{E_2^2} - \frac{1}{E_1^2}\right).$$

We solve this for $m_\nu c^2$ and set $t_2 - t_1 = 10$ s:

$$\begin{aligned} m_\nu c^2 &= \left(\frac{2c(t_2-t_1)}{d}\frac{E_1^2E_2^2}{E_1^2-E_2^2}\right)^{1/2} \\ &= \left(\frac{2(3\times10^8\text{ m/s})(10\text{ s})}{(1.7\times10^5\text{ ly})(10^{16}\text{ m/ly})}\frac{(400\text{ MeV})^2(100\text{ MeV})^2}{(400\text{ MeV})^2-(100\text{ MeV})^2}\right)^{1/2} \\ &= 22\times10^{-6}\text{ MeV} = 22\text{ eV}. \end{aligned}$$

We thus estimate the mass of the neutrino to be 22 eV/c^2, but there would of course be experimental uncertainties.

(b) If the two neutrinos of energy $E_1 = 20$ MeV and $E_2 = 10$ MeV were emitted at unknown times over a 10-s interval, then the 10-s difference in their arrival times could be due to a 10-s difference in their emission time. In this case our data are consistent with zero rest mass and it puts an approximate upper limit on the neutrino mass of 22 eV/c^2.

9.2 Stars and galaxies. Stellar evolution

9–1. Using the definitions of the parsec and the light-year, show that 1 pc = 3.26 ly.

Solution. Using the definition of the parsec, we find the equivalent distance in m:

$$\begin{aligned} D &= \frac{d}{\varphi}; \\ 1\text{ pc} &= \left(\frac{1.50\times10^{11}\text{ m}}{1.000''}\right)\left(\frac{3\,600''}{1^\circ}\right)\left(\frac{180^\circ}{\pi\text{ rad}}\right) \\ &= 3.094\times10^{16}\text{ m}. \end{aligned}$$

From the definition of the light-year, we have

$$1\text{ pc} = \frac{3.094\times10^{16}\text{ m}}{(3.00\times10^8\text{ m/s})(3.16\times10^7\text{ s/yr})} = 3.26\text{ ly}.$$

9–2. A star exhibits a parallax of 0.38 seconds of arc. How far away is it?

Solution. We find the distance from

$$D = \frac{1}{\varphi} = \left(\frac{1}{0.38''}\right)(3.26\text{ ly/pc}) = 8.6\text{ ly}.$$

9–3. The parallax angle of a star is $0.00019°$. How far away is the star?

SOLUTION. We find the distance from

$$D = \frac{1}{\varphi} = \left(\frac{1}{0.00019°}\right)\left(\frac{1°}{3\,600''}\right)(3.26 \text{ ly/pc}) = 4.8 \text{ ly}.$$

9–4. A star is 36 parsecs away. What is its parallax angle? State (a) in seconds of arc, and (b) in degrees.

SOLUTION. (a) We find the parallax angle from

$$D = \frac{1}{\varphi}; \quad 36 \text{ pc} = \frac{1}{\varphi},$$

which gives $\varphi = 0.028''$.

(b) We convert this to degrees:

$$\varphi = (0.028'')\left(\frac{1°}{3\,600''}\right) = (7.7 \times 10^{-6})°.$$

9–5. What is the parallax angle for a star that is 55 light-years away? How many parsecs is this?

SOLUTION. We find the parallax angle from

$$D = \frac{1}{\varphi}; \quad \frac{55 \text{ ly}}{3.26 \text{ ly/pc}} = \frac{1}{\varphi},$$

which gives $\varphi = 0.059''$.

The distance in parsecs is

$$D = \frac{55 \text{ ly}}{3.26 \text{ ly/pc}} = 17 \text{ pc}.$$

9–6. A star is 35 parsecs away. How long does it take for its light to reach us?

SOLUTION. We find the distance in light-years:

$$D = (35 \text{ pc})(3.26 \text{ ly/pc}) = 114 \text{ ly}.$$

Thus the light takes 114 yr to reach us.

9–7. If one star is twice as far away from us as a second star, will the parallax angle of the first star be greater or less than that of the second star? By what factor?

SOLUTION. The star farther away will subtend a smaller angle, so the parallax angle will be less.

From $D = 1/\varphi$, we see that

$$\frac{D_1}{D_2} = 2 = \frac{\varphi_2}{\varphi_1}\,,$$

or

$$\frac{\varphi_1}{\varphi_2} = \frac{1}{2}\,.$$

9–8. The rate energy reaches the Earth from the Sun (the "solar constant") is about 1.3×10^3 W/m^2. What is (a) the apparent brightness b of the Sun, and (b) the absolute luminosity L of the Sun?

SOLUTION. (a) The apparent brightness of the Sun is $b = 1.3 \times 10^3$ W/m^2.

(b) The absolute luminosity of the Sun is

$$L = bA = (1.3 \times 10^3 \text{ W/m}^2)4\pi(1.5 \times 10^{11} \text{ m})^2 = 3.7 \times 10^{26} \text{ W}.$$

9–9. What is the apparent brightness of the Sun as seen on Jupiter? (Jupiter is 5.2 times farther from the Sun than the Earth.)

SOLUTION. The apparent brightness is determined by the absolute luminosity and the distance:

$$L = bA = b4\pi D^2.$$

If we form the ratio for the apparent brightness at the Earth and at Jupiter, we get

$$\begin{aligned} \frac{b_\text{E}}{b_\text{J}} &= \left(\frac{D_\text{J}}{D_\text{E}}\right)^2; \\ \frac{(1.3 \times 10^3 \text{ W/m}^2)}{b_\text{J}} &= (5.2)^2, \end{aligned}$$

which gives $b_\text{J} = 48$ W/m^2.

9–10. Estimate the angular width that our Galaxy would subtend if observed from the nearest galaxy to us (Table 9.1). Compare to the angular width of the Moon from Earth.

SOLUTION. The diameter of our Galaxy is 100 000 ly so the angle subtended by our Galaxy is

$$\varphi_{\text{Galaxy}} = \frac{d_1}{D_1} = \frac{100\,000 \text{ ly}}{2 \times 10^6 \text{ ly}} = 0.05 \text{ rad} = 2.9^\circ.$$

The angle subtended by the Moon at the Earth is

$$\varphi_{\text{Moon}} = \frac{d_2}{D_2} = \frac{3.48 \times 10^6 \text{ m}}{3.84 \times 10^8 \text{ m}} = 9.06 \times 10^{-3} \text{ rad} = 0.52^\circ.$$

Thus we have

$$\varphi_{\text{Galaxy}} \approx 6\varphi_{\text{Moon}}.$$

9–11. When our Sun becomes a red giant, what will be its average density if it expands out to the orbit of Earth (1.5×10^{11} m from the Sun)?

SOLUTION. If we assume negligible mass change, as a red giant, the density of the Sun will be

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi r^3} = \frac{2 \times 10^{30} \text{ kg}}{\frac{4}{3}\pi(1.5 \times 10^{11} \text{ m})^3} = 1.4 \times 10^{-4} \text{ kg/m}^3.$$

9–12. When our Sun becomes a white dwarf, it is expected to be about the size of the Moon. What angular width will it subtend from Earth?

SOLUTION. The angle subtended at the Earth by the Sun as a white dwarf will be

$$\varphi = \frac{d}{D} = \frac{3.48 \times 10^6 \text{ m}}{1.5 \times 10^{11} \text{ m}} = 2.3 \times 10^{-5} \text{ rad} = 4.8''.$$

9–13. Calculate the density of a white dwarf whose mass is equal to the Sun's and whose radius is equal to the Earth's. How many times larger than Earth's density is this?

SOLUTION. The density of the white dwarf is

$$\rho_{\text{dwarf}} = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi r^3} = \frac{2.0 \times 10^{30} \text{ kg}}{\frac{4}{3}\pi(6.38 \times 10^6 \text{ m})^3} = 1.8 \times 10^9 \text{ kg/m}^3.$$

The ratio of densities will be the ratio of masses:

$$\frac{\rho_{\text{dwarf}}}{\rho_{\text{Earth}}} = \frac{M_{\text{dwarf}}}{M_{\text{Earth}}} = \frac{2.0 \times 10^{30}\ \text{kg}}{5.98 \times 10^{24}\ \text{kg}} = 3.3 \times 10^{5}.$$

9–14. A neutron star whose mass is 1.5 solar masses has a radius of about 11 km. Calculate its average density and compare to that for a white dwarf (Problem 9–13) and to that of nuclear matter.

SOLUTION. The density of the neutron star is

$$\rho_{\text{ns}} = \frac{M_{\text{ns}}}{V_{\text{ns}}} = \frac{M_{\text{ns}}}{\frac{4}{3}\pi r_{\text{ns}}^3} = \frac{(1.5)(2 \times 10^{30}\ \text{kg})}{\frac{4}{3}\pi(11 \times 10^{3}\ \text{m})^3} = 5.4 \times 10^{17}\ \text{kg/m}^3.$$

From the result for Problem 9–13 we see that

$$\rho_{\text{ns}} = (3 \times 10^{8})\rho_{\text{dwarf}}.$$

For the density of nuclear matter we calculate the density of a proton:

$$\rho_{\text{p}} = \frac{m_{\text{p}}}{\frac{4}{3}\pi r_{\text{p}}^3} = \frac{1.67 \times 10^{-27}\ \text{kg}}{\frac{4}{3}\pi(1.2 \times 10^{-15}\ \text{m})^3} = 2.3 \times 10^{17}\ \text{kg/m}^3.$$

Thus we see that $\rho_{\text{ns}} \approx \rho_{\text{nuclear matter}}$.

9–15. Calculate the Q-values for the He burning reactions of Eq. 9.3. (The mass of ${}^{8}_{4}\text{Be}$ is 8.005305 u.)

SOLUTION. For the reaction

$${}^{4}_{2}\text{He} + {}^{4}_{2}\text{He} \rightarrow {}^{8}_{4}\text{Be} + \gamma,$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M({}^{4}\text{He}) + M({}^{4}\text{He}) - M({}^{8}\text{Be})\right)c^2 \\ &= (2(4.002603\ \text{u}) - (8.005305\ \text{u}))\,c^2(931.5\ \text{MeV/u}c^2) \\ &= -0.0922\ \text{MeV} = -92.2\ \text{keV}. \end{aligned}$$

For the reaction

$${}^{4}_{2}\text{He} + {}^{8}_{4}\text{Be} \rightarrow {}^{12}_{6}\text{C} + \gamma,$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^4\mathrm{He}) + M(^8\mathrm{Be}) - M(^{12}\mathrm{C})\right) c^2 \\ &= ((4.002603\ \mathrm{u}) + (8.005305\ \mathrm{u}) - (12.000000\ \mathrm{u})) \\ &\quad \times c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= 7.366\ \mathrm{MeV}. \end{aligned}$$

Note that the total Q-value is 7.27 MeV.

9–16. In the later stages of stellar evolution, a star (if massive enough) will begin fusing carbon nuclei to form, for example, magnesium:

$$^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \rightarrow {}^{24}_{12}\mathrm{Mg} + \gamma.$$

(a) How much energy is released in this reaction (see Appendix C). (b) How much kinetic energy must each carbon nucleus have (assume equal) in a head-on collision they are just to touch (use Eq. 6.1) so that the strong force can come into play? (c) What temperature does this kinetic energy correspond to?

Solution. (a) For the reaction

$$^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \rightarrow {}^{24}_{12}\mathrm{Mg} + \gamma,$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{12}\mathrm{C}) + M(^{12}\mathrm{C}) - M(^{24}\mathrm{Mg})\right) c^2 \\ &= (2(12.000000\ \mathrm{u}) - (23.985042\ \mathrm{u}))\, c^2(931.5\ \mathrm{MeV/u}c^2) \\ &= 13.93\ \mathrm{MeV}. \end{aligned}$$

Thus 13.93 MeV is released.

(b) We find the radius of the carbon nucleus from

$$r = (1.2\ \mathrm{fm})\sqrt[3]{A} = (1.2\ \mathrm{fm})\sqrt[3]{12} = 2.75\ \mathrm{fm}.$$

If the two nuclei are just touching, the Coulomb potential energy must be the initial kinetic energies of the two nuclei:

$$2K = U = k_\mathrm{e}\frac{Z_\mathrm{C}Z_\mathrm{C}e^2}{2r} = \frac{(6)(6)(1.44\ \mathrm{MeV}\cdot\mathrm{fm})}{2(2.75\ \mathrm{fm})} = 9.4\ \mathrm{MeV}.$$

Thus each carbon nucleus has a kinetic energy of 4. 7 MeV.

(c) We find the temperature from

$$\begin{aligned} K &= \tfrac{3}{2}k_{\mathrm{B}}T; \\ (4.7\text{ MeV})(1.60\times10^{-13}\text{ J/MeV}) &= \tfrac{3}{2}(1.38\times10^{-23}\text{ J/K})T, \end{aligned}$$

which gives $T = 4\times10^{10}$ K.

9–17. Repeat Problem 9–16 for the reaction

$$^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} \rightarrow {}^{28}_{14}\mathrm{Si} + {}^{4}_{2}\mathrm{He}.$$

Solution. (a) For the reaction

$$^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} \rightarrow {}^{28}_{14}\mathrm{Si} + {}^{4}_{2}\mathrm{He},$$

we determine the Q-value:

$$\begin{aligned} Q &= \left(M(^{16}\mathrm{O}) + M(^{16}\mathrm{O}) - M(^{28}\mathrm{Si}) - M(^{4}\mathrm{He})\right)c^2 \\ &= (2(15.994915\text{ u}) - (27.976927\text{ u}) - (4.002603\text{ u})) \\ &\quad \times c^2(931.5\text{ MeV/u}c^2) \\ &= 9.594\text{ MeV}. \end{aligned}$$

Thus 9. 594 MeV is released.

(b) We find the radius of the oxygen nucleus from

$$r = (1.2\text{ fm})\sqrt[3]{A} = (1.2\text{ fm})\sqrt[3]{16} = 3.02\text{ fm}.$$

If the two nuclei are just touching, the Coulomb potential energy must be the initial kinetic energies of the two nuclei:

$$2K = U = k_{\mathrm{e}}\frac{Z_{\mathrm{O}}Z_{\mathrm{O}}e^2}{2r} = \frac{(8)(8)(1.44\text{ MeV}\cdot\text{fm})}{2(3.02\text{ fm})} = 15.25\text{ MeV}.$$

Thus each oxygen nucleus has a kinetic energy of 7. 6 MeV.

(c) We find the temperature from

$$\begin{aligned} K &= \tfrac{3}{2}k_{\mathrm{B}}T; \\ (7.6\text{ MeV})(1.60\times10^{-13}\text{ J/MeV}) &= \tfrac{3}{2}(1.38\times10^{-23}\text{ J/K})T, \end{aligned}$$

which gives $T = 6 \times 10^{10}$ K.

9–18. Suppose two stars of the same apparent brightness b are also believed to be the same size. The spectrum of one star peaks at 800 nm whereas that of the other peaks at 400 nm. Use Wien's law (Eq. 2.1) and the Stefan–Boltzmann equation (Eq. 9.2) to estimate their relative distances from us. [*Hint*: See Examples 9–5 and 9–6.]

SOLUTION. From Wien's displacement law we can compare the temperatures of the two stars:

$$\lambda_1 T_1 = \lambda_2 T_2,$$

or

$$\frac{T_2}{T_1} = \frac{\lambda_1}{\lambda_2} = \frac{(800\text{ nm})}{(400\text{ nm})} = 2.$$

If r is the radius of a star, the radiating area is $A = 4\pi r^2$. From the Stefan–Boltzmann law we have

$$L = b4\pi D^2 \propto AT^4 = 4\pi r^2 T^4.$$

If we form the ratio for the two stars, we get

$$\left(\frac{D_2}{D_1}\right)^2 = \left(\frac{T_2}{T_1}\right)^4 = (2)^4,$$

which gives $D_2 = 4D_1$.

9–19. Stars located in a certain cluster are assumed to be about the same distance from us. Two such stars have spectra that peak at $\lambda_1 = 500$ nm and $\lambda_2 = 700$ nm, and the ratio of their apparent brightness is $b_1/b_2 = 0.091$. Estimate their relative sizes (give ratio of their diameters). [*Hint*: Use the Stefan–Boltzmann equation, Eq. 9.2.]

SOLUTION. From Wien's displacement law we can compare the temperatures of the two stars:

$$\lambda_1 T_1 = \lambda_2 T_2,$$

or

$$\frac{T_2}{T_1} = \frac{\lambda_1}{\lambda_2} = \frac{(500\text{ nm})}{(700\text{ nm})} = \frac{5}{7}.$$

If r is the radius of a star, the radiating area is $A = 4\pi r^2$. From the Stefan–Boltzmann law we have

$$L = b4\pi D^2 \propto AT^4 = 4\pi r^2 T^4.$$

If we form the ratio for the two stars, we get

$$\begin{aligned} \frac{b_2}{b_1} &= \left(\frac{r_2}{r_1}\right)^2 \left(\frac{T_2}{T_1}\right)^4; \\ \frac{1}{0.091} &= \left(\frac{r_2}{r_1}\right)^2 \left(\frac{5}{7}\right)^4, \end{aligned}$$

which gives

$$\frac{r_2}{r_1} = 6.5,$$

so

$$\frac{d_2}{d_1} = 6.5.$$

9.3 General relativity

9–20. Describe a triangle, drawn on the surface of a sphere, for which the sum of the angles is (a) 360°, and (b) 180°.

Solution. (a) For the vertices of the triangle we choose the North pole and two points on a latitude line on opposite sides of the Earth, as shown on Fig. 9.5a.

(b) We can get 180° only approximately. For the vertices of the triangle we choose the North pole and two points on a latitude line very close together, so the angle at the North pole is negligible, as shown on Fig. 9.5b.

9–21. Show that the Schwarzschild radius for a star with mass equal to that (a) of our Sun is 2.95 km, and (b) of Earth is 8.9 mm.

Solution. (a) The Schwarzschild radius for a star with mass equal to that of our Sun is (Eq. 9.4)

$$\begin{aligned} R &= \frac{2GM_S}{c^2} = \frac{2(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(1.99 \times 10^{30}\ \mathrm{kg})}{(3.00 \times 10^8\ \mathrm{m/s})^2} \\ &= 2.95 \times 10^3\ \mathrm{m} = 2.95\ \mathrm{km}. \end{aligned}$$

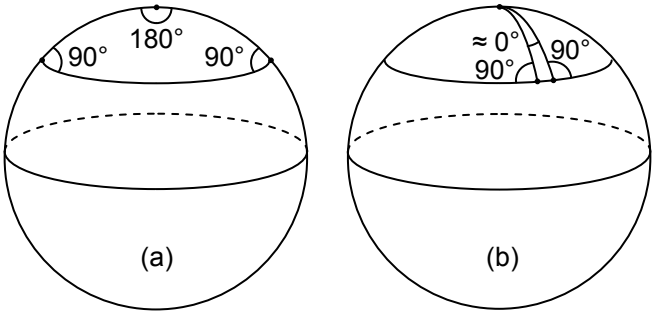


Figure 9.5. Problem 9–20.

(b) The Schwarzschild radius for a star with mass equal to that of the Earth is

$$\begin{aligned} R &= \frac{2GM_{\mathrm{E}}}{c^2} = \frac{2(6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2})(5.97\times10^{24}\ \mathrm{kg})}{(3.00\times10^{8}\ \mathrm{m/s})^2} \\ &= 8.9\times10^{-3}\ \mathrm{m} = 8.9\ \mathrm{mm}. \end{aligned}$$

9–22. What is the Schwarzschild radius for a typical galaxy?

Solution. If we use the data for our galaxy, we have

$$R = \frac{2GM}{c^2} = \frac{2(6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2})(3\times10^{41}\ \mathrm{kg})}{(3.00\times10^{8}\ \mathrm{m/s})^2} = 4\times10^{14}\ \mathrm{m}.$$

9–23. What is the maximum sum-of-the-angles for a triangle on a sphere?

Solution. If we consider a triangle with its three vertices on a great circle, such as one through the North and South poles as shown in the diagram (Fig. 9.6), we see that the sum of the angles is 540°.

9–24. Calculate the escape velocity, using Newtonian mechanics, from a body that has collapsed to its Schwarzschild radius.

Solution. To escape from an object with mass M and radius r, the kinetic energy of the body must be greater than the gravitational potential

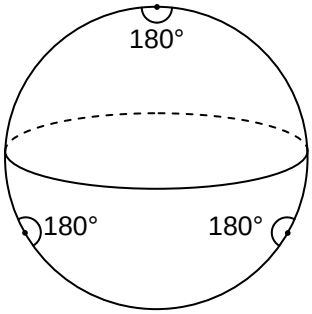


Figure 9.6. Problem 9–23.

energy at the surface:

$$\begin{aligned} \frac{mv^2}{2} &\geqslant \frac{GMm}{r}, \quad \text{or} \\ v_{\text{esc}}^2 &= \frac{2GM}{r}. \end{aligned}$$

At the Schwarzschild radius we have

$$v_{\text{esc}}^2 = \frac{2GM}{(2GM/c^2)} = c^2,$$

or $v_{\text{esc}} = c$.

9.4 The expanding universe

9–25. If a galaxy is traveling away from us at 1.0 percent of the speed of light, roughly how far away is it?

Solution. We find the distance from Hubble's law:

$$\begin{aligned} v &= Hd; \\ (0.010)(3.00 \times 10^8 \text{ m/s}) &= \frac{70 \times 10^3 \text{ m/s/Mpc}}{(10^6 \text{ pc/Mpc})(3.26 \text{ ly/pc})}\, d, \end{aligned}$$

which gives $d = 1.4 \times 10^8$ ly.

9–26. The redshift of a galaxy indicates a velocity of 3 500 km/s. How far away is it?

SOLUTION. We find the distance from Hubble's law:

$$\begin{aligned} v &= Hd; \\ 3.5 \times 10^6 \text{ m/s} &= \frac{70 \times 10^3 \text{ m/s/Mpc}}{(10^6 \text{ pc/Mpc})(3.26 \text{ ly/pc})}\, d, \end{aligned}$$

which gives $d = 1.6 \times 10^8$ ly.

9–27. Estimate the speed of a galaxy (relative to us) that is near the "edge" of the universe, say 12 billion light-years away.

SOLUTION. We estimate the speed of the galaxy from Hubble's law:

$$v = Hd = \frac{70 \times 10^3 \text{ m/s/Mpc}}{3.26 \text{ ly/pc}} (12 \times 10^3 \text{ Mly}) = 2.6 \times 10^8 \text{ m/s} = 0.86c.$$

9–28. Estimate the observed wavelength for the 656-nm line in the Balmer series of hydrogen in the spectrum of a galaxy whose distance from us is (a) 1.0×10^6 ly, (b) 1.0×10^8 ly, (c) 1.0×10^{10} ly.

SOLUTION. (a) We find the receding speed of the galaxy from

$$\begin{aligned} v &= Hd = \frac{70 \times 10^3 \text{ m/s/Mpc}}{3.26 \text{ ly/pc}} (1.0 \text{ Mly}) = 2.15 \times 10^4 \text{ m/s} \\ &= (7.2 \times 10^{-5})c. \end{aligned}$$

For the Doppler shift of the wavelength we have

$$\begin{aligned} \frac{\lambda}{\lambda_0} &= \sqrt{\frac{1+\beta}{1-\beta}}; \\ \frac{\lambda}{656 \text{ nm}} &= \sqrt{\frac{1+7.2 \times 10^{-5}}{1-7.2 \times 10^{-5}}}, \end{aligned}$$

which gives $\lambda = 656$ nm.

(b) We find the receding speed of the galaxy from

$$\begin{aligned} v &= Hd = \frac{70 \times 10^3 \text{ m/s/Mpc}}{3.26 \text{ ly/pc}} (1.0 \times 10^2 \text{ Mly}) = 2.15 \times 10^6 \text{ m/s} \\ &= (7.2 \times 10^{-3})c. \end{aligned}$$

For the Doppler shift of the wavelength we have

$$\begin{aligned} \frac{\lambda}{\lambda_0} &= \sqrt{\frac{1+\beta}{1-\beta}}\,; \\ \frac{\lambda}{656\text{ nm}} &= \sqrt{\frac{1+7.2\times10^{-3}}{1-7.2\times10^{-3}}}\,, \end{aligned}$$

which gives $\lambda = 661$ nm.

(c) We find the receding speed of the galaxy from

$$\begin{aligned} v &= Hd = \frac{70\times10^3\text{ m/s/Mpc}}{3.26\text{ ly/pc}}\,(1.0\times10^4\text{ Mly}) = 2.15\times10^8\text{ m/s} \\ &= 0.72\,c. \end{aligned}$$

For the Doppler shift of the wavelength we have

$$\begin{aligned} \frac{\lambda}{\lambda_0} &= \sqrt{\frac{1+\beta}{1-\beta}}\,; \\ \frac{\lambda}{656\text{ nm}} &= \sqrt{\frac{1+0.72}{1-0.72}}\,, \end{aligned}$$

which gives

$$\lambda = 1.63\times10^3\text{ nm} = 1.63\ \mu\text{m}.$$

9–29. Estimate the speed of a galaxy, and its distance from us, if the wavelength for the hydrogen line at 434 nm is measured on Earth as being 610 nm.

Solution. We find the speed from the Doppler shift:

$$\begin{aligned} \frac{\lambda}{\lambda_0} &= \sqrt{\frac{1+\beta}{1-\beta}}\,; \\ \frac{610\text{ nm}}{434\text{ nm}} &= \sqrt{\frac{1+\beta}{1-\beta}}\,, \end{aligned}$$

which gives $v = 0.328\,c$.

We find the distance from Hubble's law:

$$\begin{aligned} v &= Hd; \\ (0.328)(3.00 \times 10^8 \text{ m/s}) &= \frac{70 \times 10^3 \text{ m/s/Mpc}}{(10^6 \text{ pc/Mpc})(3.26 \text{ ly/pc})}\, d, \end{aligned}$$

which gives $d = 4.6 \times 10^9$ ly.

9–30. Starting from Eq. 9.5, show that the Doppler shift in wavelength is $\Delta\lambda/\lambda_0 \approx \beta$ (Eq. 9.6) for $v \ll c$. [*Hint*: Use the binomial expansion.]

SOLUTION. The Doppler shift is

$$\frac{\lambda}{\lambda_0} = \sqrt{\frac{1+\beta}{1-\beta}}.$$

From the binomial expansion for small x, we have

$$(1-x)^{-1} \approx 1 + x.$$

Thus when $\beta \ll 1$, we get

$$\frac{\lambda}{\lambda_0} = \sqrt{\frac{1+\beta}{1-\beta}} \approx \sqrt{(1+\beta)^2} = 1 + \beta,$$

or

$$\lambda \approx \lambda_0 + \beta\lambda_0.$$

Thus the fractional wavelength change is

$$\frac{\lambda - \lambda_0}{\lambda_0} = \frac{\Delta\lambda}{\lambda_0} \approx \beta.$$

9.5 The Big Bang. The standard cosmological model

9–31. Calculate the wavelength at the peak of the blackbody radiation distribution at 2.7 K using the Wien law.

SOLUTION. We find the peak wavelength from

$$\begin{aligned} \lambda T &= 2.90 \times 10^{-3} \text{ m} \cdot \text{K}; \\ \lambda(2.7 \text{ K}) &= 2.90 \times 10^{-3} \text{ m} \cdot \text{K}, \end{aligned}$$

which gives

$$\lambda = 1.1 \times 10^{-3} \text{ m} = 1.1 \text{ mm}.$$

9–32. The critical density for closure of the universe is $\rho_c \approx 10^{-26}$ kg/m^3. State ρ_c in terms of the average number of nucleons per cubic meter.

SOLUTION. We express the critical density in terms of the nucleon density from

$$\begin{aligned} \rho_c &= \frac{M}{V} = \frac{nm_n}{V}\,; \\ 10^{-26} \text{ kg/m}^3 &= \frac{n(1.67 \times 10^{-27} \text{ kg})}{V}\,, \end{aligned}$$

which gives

$$\frac{n}{V} = 6 \text{ nucleons/m}^3.$$

9–33. The size of the universe (the average distance between galaxies) at any one moment is believed to have been inversely proportional to the absolute temperature. Estimate the size of the universe, compared to today, at (a) $t = 10^6$ yr, (b) $t = 1$ s, (c) $t = 10^{-6}$ s, and (d) $t = 10^{-35}$ s.

SOLUTION. If $d \propto T^{-1}$, when we form the ratio for two different times, we get

$$\frac{d}{d_0} = \frac{T_0}{T}\,,$$

where $T_0 = 2.7$ K is the temperature today.

(a) From Table 9.2 we estimate the temperature to be 3 000 K, so we have

$$\frac{d}{d_0} = \frac{T_0}{T} = \frac{2.7 \text{ K}}{3\,000 \text{ K}} \approx 10^{-3}.$$

(b) From Table 9.2 we estimate the temperature to be 10^{10} K, so we have

$$\frac{d}{d_0} = \frac{T_0}{T} = \frac{2.7 \text{ K}}{10^{10} \text{ K}} \approx 10^{-10}.$$

(c) From Table 9.2 we estimate the temperature to be 10^{13} K, so we have

$$\frac{d}{d_0} = \frac{T_0}{T} = \frac{2.7 \text{ K}}{10^{13} \text{ K}} \approx 10^{-13}.$$

(d) From Table 9.2 we estimate the temperature to be 10^{27} K, so we have

$$\frac{d}{d_0} = \frac{T_0}{T} = \frac{2.7\ \text{K}}{10^{27}\ \text{K}} \approx 10^{-27}.$$

9–34. At approximately what time had the universe cooled below the threshold temperature for producing (a) kaons ($M \approx 500$ MeV/c^2), (b) Υ ($M \approx 9\,500$ MeV/c^2), and (c) muons ($M \approx 100$ MeV/c^2)?

SOLUTION. We find the equivalent temperature from

$$E = Mc^2 = \tfrac{3}{2}k_{\text{B}}T,$$

or

$$T = \frac{2}{3}\,\frac{Mc^2}{k_{\text{B}}}\,.$$

(a) For the kaon threshold we have

$$T = \frac{2}{3}\,\frac{Mc^2}{k_{\text{B}}} = \frac{2}{3}\,\frac{(500\ \text{MeV})(1.60\times10^{-13}\ \text{J/MeV})}{1.38\times10^{-23}\ \text{J/K}} = 4\times10^{12}\ \text{K}.$$

We estimate the time from Table 9.2: $t \approx 10^{-5}$ s.

(b) For the Υ threshold we have

$$T = \frac{2}{3}\,\frac{Mc^2}{k_{\text{B}}} = \frac{2}{3}\,\frac{(9\,500\ \text{MeV})(1.60\times10^{-13}\ \text{J/MeV})}{1.38\times10^{-23}\ \text{J/K}} = 7\times10^{13}\ \text{K}.$$

We estimate the time from Table 9.2: $t \approx 10^{-7}$ s.

(c) For the muon threshold we have

$$T = \frac{2}{3}\,\frac{Mc^2}{k_{\text{B}}} = \frac{2}{3}\,\frac{(100\ \text{MeV})(1.60\times10^{-13}\ \text{J/MeV})}{1.38\times10^{-23}\ \text{J/K}} = 8\times10^{11}\ \text{K}.$$

We estimate the time from Table 9.2: $t \approx 10^{-4}$ s.

9.6 General problems

9–35. Suppose that three main-sequence stars could undergo the three changes represented by the three arrows, A, B, and C, in the H–R diagram of Fig. 9.7. For each case, describe the changes in temperature, luminosity, and size.

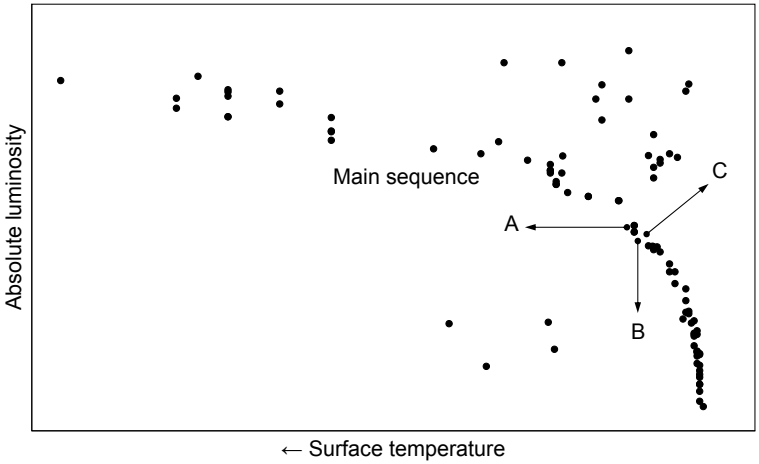


Figure 9.7. Problem 9–35.

SOLUTION. The absolute luminosity depends on the radius of the star and the temperature:

$$L \propto AT^4 = 4\pi r^2 T^4.$$

(a) For case A we have: temperature increases, luminosity is constant, and size decreases.

(b) For case B we have: temperature is constant, luminosity decreases, and size decreases.

(c) For case C we have: temperature decreases, luminosity increases, and size increases.

9–36. Assume that the nearest stars to us have an intrinsic luminosity about the same as the Sun's. Their apparent brightness, however, is about 10^{11} times fainter than the Sun. From this, estimate the distance to the nearest stars. (Newton did this calculation, although he made a numerical error of a factor of 100.)

SOLUTION. For the luminosity we have $L = b4\pi d^2$. If we form the

ratio with constant luminosity, we get

$$\left(\frac{d_{\text{star}}}{d_{\text{Sun}}}\right)^2 = \frac{b_{\text{Sun}}}{b_{\text{star}}};$$
$$\left(\frac{d_{\text{star}}}{(1.5\times10^{11}\text{ m})/(9.5\times10^{15}\text{ m/ly})}\right)^2 = 10^{11},$$

which gives $d_{\text{star}} = 5$ ly.

9–37. Use conservation of angular momentum to estimate the angular velocity of a neutron star which has collapsed to a diameter of 10 km, from a star whose radius was equal to that of our Sun (7×10^8 m), of mass 1.5 times that of the Sun, and which rotated (like our Sun) about once a month.

SOLUTION. For the conservation of angular momentum, with constant mass, we have

$$\begin{aligned} I\omega &= I_0\omega_0; \\ \tfrac{2}{5}Mr^2\omega &= \tfrac{2}{5}Mr_0^2\omega_0; \\ (5\text{ km})^2\omega &= (7\times10^5\text{ km})^2(1\text{ rev/mo}), \end{aligned}$$

which gives

$$\begin{aligned} \omega &= 2\times10^{10}\text{ rev/mo} = \frac{2\times10^{10}\text{ rev/mo}}{(30\text{ day/mo})(24\text{ h/day})(3\,600\text{ s/h})} \\ &= 8\times10^3\text{ rev/s}. \end{aligned}$$

9–38. By what factor does the rotational kinetic energy change when the star in Problem 9–37 collapses to a neutron star?

SOLUTION. We simplify the ratio of rotational kinetic energies by using the conservation of angular momentum:

$$\frac{K}{K_0} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}I_0\omega_0^2} = \frac{\omega}{\omega_0} = \frac{2\times10^{10}\text{ rev/mo}}{1\text{ rev/mo}} = 2\times10^{10}.$$

9–39. A certain pulsar, believed to be a neutron star of mass 1.5 times that of the Sun, with diameter 10 km, is observed to have a rotation speed of 1.0 rev/s. If it loses rotational kinetic energy at the rate of 1 part in 10^9

per day, which is all transformed into radiation, what is the power output of the star?

SOLUTION. The kinetic energy of the neutron star is

$$\begin{aligned} K &= \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}\left(\tfrac{2}{5} Mr^2\right)\omega^2 \\ &= \tfrac{1}{2}\left(\tfrac{2}{5}\right)(1.5)(2.0\times 10^{30}\text{ kg})(5.0\times 10^{3}\text{ m})^2 \\ &\quad \times \left((1.0\text{ rev/s})(2\pi\text{ rad/rev})\right)^2 \\ &= 5.92\times 10^{38}\text{ J}. \end{aligned}$$

The power output is

$$P = \frac{E}{t} = \frac{(10^{-9}\text{ day}^{-1})(5.92\times 10^{38}\text{ J})}{86\,400\text{ s/day}} = 7\times 10^{24}\text{ W}.$$

9–40. Estimate the rate of which hydrogen atoms would have to be created, according to the steady-state model, to maintain the present density of the universe of about 10^{-27} kg/m^3, assuming the universe is expanding with Hubble constant $H = 70$ km/s/Mpc.

SOLUTION. If R is the radius of the universe, the volume is

$$V = \tfrac{4}{3}\pi R^3.$$

We find the increase in radius from Hubble's law:

$$\Delta R = v\,\Delta t = HR\,\Delta t.$$

Thus the rate at which the universe is expanding is

$$\frac{\Delta V}{\Delta t} = 4\pi R^2 \frac{\Delta R}{\Delta t} = 4\pi R^3 H = 3VH.$$

The rate at which hydrogen atoms have to be created is given by

$$\frac{\Delta m}{\Delta t} = mH\frac{\Delta N}{\Delta t} = \rho\,\frac{\Delta V}{\Delta t} = \rho\,3VH.$$

If we find the rate per unit volume, we have

$$\begin{aligned}\frac{1}{V}\frac{\Delta N}{\Delta t} &= \frac{3\rho H}{m_{\rm H}} \\ &= \frac{3(10^{-27}\ {\rm kg/m^3})(70\ {\rm km/s/Mpc})}{(1.67\times10^{-27}\ {\rm kg/atom})(3.16\times10^{7}\ {\rm s/yr})} \\ &\quad\times\frac{(10^3\ {\rm m/km})^3}{(3.26\times10^{6}\ {\rm ly/Mpc})(9.46\times10^{12}\ {\rm km/ly})} \\ &= 0.13\ {\rm atoms/km^3/yr}.\end{aligned}$$

Thus hydrogen atoms would need to be created at the rate 1 atom/km^3 every 8 years.

9–41. Estimate what neutrino rest mass (in eV) would provide the critical density to close the universe. Assume the neutrino density is, like photons, about 10^9 times that of nucleons, and that nucleons make up only (a) 2 percent of the mass needed, or (b) 5 percent of the mass needed.

SOLUTION. We use M for the mass of the universe and n for the number of nucleons.

(a) If nucleons provide 2% of the mass, we have

$$\begin{aligned}0.02\,M &= nm_{\rm n}, \quad \text{and} \\ 0.98\,M &= (10^9 n)m_\nu.\end{aligned}$$

When we combine these, we get

$$\frac{n(939\times10^6\ {\rm eV})}{0.02} = \frac{(10^9 n)m_\nu}{0.98}\,,$$

which gives $m_\nu = 46$ eV.

(b) If nucleons provide 5% of the mass, we have

$$\begin{aligned}0.05M &= nm_{\rm n}, \quad \text{and} \\ 0.95M &= (10^9 n)m_\nu.\end{aligned}$$

When we combine these, we get

$$\frac{n(939\times10^6\ {\rm eV})}{0.05} = \frac{(10^9 n)m_\nu}{0.95}\,,$$

which gives $m_\nu = 18$ eV.

9–42. Two stars, whose spectra peak at 600 nm and 400 nm, respectively, both lie on the main sequence. Use Wien's law, the Stefan–Boltzmann equation, and the H–R diagram (Fig. 9.3) to estimate the ratio of their diameters. [*Hint*: See Examples 9–5 and 9–6.]

SOLUTION. We find the temperature of the stars from

$$\begin{aligned} \lambda T &= 2.90 \times 10^{-3}\ \text{m} \cdot \text{K}; \\ (600 \times 10^{-9}\ \text{m}) T_1 &= 2.90 \times 10^{-3}\ \text{m} \cdot \text{K}, \end{aligned}$$

which gives $T_1 = 4\,800$ K;

$$(400 \times 10^{-9}\ \text{m}) T_2 = 2.90 \times 10^{-3}\ \text{m} \cdot \text{K},$$

which gives $T_2 = 7\,300$ K.

From the H–R diagram, we estimate the luminosities:

$$L_1 = 10^{26}\ \text{W}, \quad \text{and} \quad L_2 = 2 \times 10^{27}\ \text{W}.$$

From the Stefan–Boltzmann law, the absolute luminosity depends on the radius of the star and the temperature:

$$L \propto AT^4 = 4\pi r^2 T^4.$$

When we form the ratio, we get

$$\begin{aligned} \frac{L_2}{L_1} &= \left(\frac{r_2}{r_1}\right)^2 \left(\frac{T_2}{T_1}\right)^4 = \left(\frac{r_2}{r_1}\right)^2 \left(\frac{\lambda_1}{\lambda_2}\right)^4; \\ \frac{2 \times 10^{27}\ \text{W}}{10^{26}\ \text{W}} &= \left(\frac{r_2}{r_1}\right)^2 \left(\frac{600\ \text{nm}}{400\ \text{nm}}\right)^4, \end{aligned}$$

which gives $r_2/r_1 = 2$.

9–43. The farthest we can measure with parallax is about 30 parsecs. What is our minimum angular resolution (in degrees), based on this information?

SOLUTION. We find half the subtended angle from

$$\varphi = \frac{1}{30}\,\text{pc} = \left(\frac{1}{30}\right)'' \left(\frac{1^\circ}{3\,600''}\right) \approx (1 \times 10^{-5})^\circ.$$

Thus the minimum subtended angle is $\approx (2 \times 10^{-5})^\circ$.

9–44. Through some coincidence, the Balmer lines from singly ionized helium in a distant star happen to overlap with the Balmer lines from hydrogen (Fig. 38-20) in the Sun. How fast is the star receding from us?

SOLUTION. The wavelengths for the same transition in hydrogen-like atoms are given by

$$\lambda^{-1} = (\text{constant}) Z^2.$$

Thus we have

$$\frac{\lambda'}{\lambda} = \frac{\lambda_{\text{H}}}{\lambda_{\text{He}}} = \left(\frac{Z_{\text{He}}}{Z_{\text{H}}}\right)^2 = \left(\frac{2}{1}\right)^2 = \sqrt{\frac{1+\beta}{1-\beta}},$$

which gives $v = 0.88c$.

9–45. What is the temperature that corresponds to the 1.8 TeV collisions at the Fermilab collider? To what era in cosmological history does this correspond?

SOLUTION. We find the temperature from

$$E = \tfrac{3}{2} k_{\text{B}} T,$$

or

$$T = \frac{2}{3}\,\frac{Mc^2}{k_{\text{B}}} = \frac{2}{3}\,\frac{(1.8\ \text{TeV})(1.60 \times 10^{-7}\ \text{J/TeV})}{1.38 \times 10^{-23}\ \text{J/K}} = 1.4 \times 10^{16}\ \text{K}.$$

From Table 9.2 we see that this is the hadron era.

9–46. Astronomers have recently measured the rotation of gas around what might be a supermassive black hole of about 2 billion solar masses at the center of a galaxy. If the radius from the galactic center to the gas clouds is 60 light-years, what Doppler shift $\Delta\lambda/\lambda_0$ do you estimate they saw?

SOLUTION. We find the speed of the gas clouds from the radial acceleration provided by the gravitational attraction:

$$\frac{GMm}{R^2} = \frac{mv^2}{R},$$

or

$$v^2 = \frac{GM}{R},$$

where

$$M = (2 \times 10^9)(2.0 \times 10^{30}\ \text{kg}) = 4.0 \times 10^{39}\ \text{kg},$$

and

$$R = (60\ \text{ly})(9.46 \times 10^{15}\ \text{m/ly}) = 5.7 \times 10^{17}\ \text{m}.$$

To account for the gases moving in opposite directions on the two sides of the cloud, we have

$$\begin{aligned} \frac{v}{c} &= \pm\frac{1}{c}\sqrt{\frac{GM}{R}} \\ &= \pm\frac{1}{3.00 \times 10^8\ \text{m/s}}\sqrt{\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(4.0 \times 10^{39}\ \text{kg})}{5.7 \times 10^{17}\ \text{m}}} \\ &= \pm 2.3 \times 10^{-3}. \end{aligned}$$

Because $v \ll c$, we use the result from Problem 9–30:

$$\frac{\Delta\lambda}{\lambda_0} \approx \frac{v}{c} = \pm 2.3 \times 10^{-3}.$$

This will be in addition to the shift from any overall receding of the galaxy.

9–47. Astronomers use an *apparent magnitude* (m) scale to describe apparent brightness. It uses a logarithmic scale, where a higher number corresponds to a less bright star. (For example, the Sun has magnitude -27, whereas most stars have positive magnitudes.) On this scale, a change in apparent magnitude by $+5$ corresponds to a decrease in apparent brightness by a factor of 100. If Venus has an apparent magnitude of -4.4, whereas Sirius has an apparent magnitude of -1.4, which is brighter? What is the ratio of the apparent brightness of these two objects?

SOLUTION. Because Venus has a more negative value, Venus is brighter.

We write the logarithmic scale as

$$m = k \lg b.$$

We find the value of k from

$$\begin{aligned} m_2 - m_1 &= k(\lg b_2 - \lg b_1) = k \lg\left(\frac{b_2}{b_1}\right); \\ +5 &= k \lg\left(\frac{1}{100}\right), \end{aligned}$$

which gives $k = -2.5$.

For Venus and Sirius, we have

$$\begin{aligned} m_{\mathrm{V}} - m_{\mathrm{S}} &= k \lg \left(\frac{b_{\mathrm{V}}}{b_{\mathrm{S}}} \right); \\ -4.4 - (-1.4) &= -2.5 \lg \left(\frac{b_{\mathrm{V}}}{b_{\mathrm{S}}} \right), \end{aligned}$$

which gives

$$\frac{b_{\mathrm{V}}}{b_{\mathrm{S}}} = 16.$$

9–48. How large would the Sun be if its density equaled the critical density of the universe, $\rho_c \approx 10^{-26}$ kg/m^3? Express your answer in light-years and compare with the Earth–Sun distance and the size of our Galaxy.

SOLUTION. For the critical density we have

$$\begin{aligned} \rho_{\mathrm{c}} &= \frac{M}{V} = \frac{M}{\frac{4}{3}\pi r^3}; \\ 10^{-26}\ \mathrm{kg/m}^3 &= \frac{2.0 \times 10^{30}\ \mathrm{kg}}{\frac{4}{3}\pi r^3}, \end{aligned}$$

which gives

$$r = 3.6 \times 10^{18}\ \mathrm{m} = 4 \times 10^2\ \mathrm{ly}.$$

When we compare to the Earth-Sun distance we get

$$\frac{r}{r_{\mathrm{ES}}} = \frac{3.6 \times 10^{18}\ \mathrm{m}}{1.5 \times 10^{11}\ \mathrm{m}} = 2.4 \times 10^7.$$

When we compare to our Galaxy we get

$$\frac{r}{r_{\mathrm{Galaxy}}} = \frac{4 \times 10^2\ \mathrm{ly}}{50\,000\ \mathrm{ly}} = 8 \times 10^{-3}.$$

9–49. Calculate the Schwarzschild radius using classical (Newtonian) gravitational theory, by calculating the minimum radius R for a sphere of mass M such that a photon (rest mass = 0) can escape from the surface. (General relativity gives $R = 2GM/c^2$.)

SOLUTION. We find the equivalent mass of the photon from

$$m = \frac{E}{c^2}\,.$$

To escape from a mass M and radius R, the energy of the photon must be greater than the gravitational potential energy at the surface:

$$mc^2 = \frac{GMm}{R}\,,$$

so the Schwarzschild radius is

$$R \geqslant \frac{GM}{c^2}\,.$$

9–50. Estimate the radius of a white dwarf whose mass is equal to that of the Sun by the following method, assuming there are N nucleons and $\frac{1}{2}N$ electrons (why $\frac{1}{2}$?): (a) Use Fermi–Dirac statistics (Chapter 5) to show that the total energy of all the electrons is

$$E_{\mathrm{e}} = \frac{3}{5}\left(\frac{N}{2}\right)\frac{\hbar^2}{2m_{\mathrm{e}}}\left(\frac{3\pi^2 N}{2V}\right)^{2/3}\,.$$

[*Hint*: See Eqs. 5.13 and 5.14; we assume electrons fill energy levels from 0 up to the Fermi energy.] (b) The nucleons contribute to the total energy mainly via the gravitational force (note that the Fermi energy for nucleons is negligible compared to that for electrons—why?). Use a gravitational form of Gauss's law to show that the total gravitational potential energy of a uniform sphere of radius R is

$$-\frac{3GM^2}{5R},$$

by considering the potential energy of a spherical shell of radius r due only to the mass inside it (why?) and integrate from $r = 0$ to $r = R$. (c) Write the total energy as a sum of these two terms, and set $dE/dR = 0$ to find the equilibrium radius, and evaluate it for a mass equal to the Sun's $(2.0 \times 10^{30}$ kg).

SOLUTION. If there are N nucleons, approximately half of them will be protons. Thus there will be $\frac{1}{2}N$ electrons.

From Eqs. 5.13 and 5.14, the average energy of an electron for a system where the energy levels are filled to the Fermi energy is $\frac{3}{5}E_{\mathrm{F}}$. Thus the total energy of the electrons is

$$E_{\mathrm{e}} = \left(\frac{3}{5}E_{\mathrm{F}}\right)\left(\frac{N}{2}\right) = \frac{3}{5}\left(\frac{N}{2}\right)\frac{\hbar^2}{2m_{\mathrm{e}}}\left(\frac{3\pi^2 N}{2V}\right)^{2/3}.$$

For a spherical volume, we get

$$E_{\mathrm{e}} = \frac{3}{5}\left(\frac{N}{2}\right)\frac{\hbar^2}{2m_{\mathrm{e}}}\left(\frac{9\pi N}{8R^3}\right)^{2/3}.$$

The Fermi energy for the nucleons will have a similar expression; however, because $m_{\mathrm{n}} \gg m_{\mathrm{e}}$, their Fermi energy is negligible.

For electric charges, Gauss's law shows that the electric field at the surface of a spherical charge distribution is the same as that of a point charge with the total charge of the sphere. Thus the potential at the surface is

$$V = k_{\mathrm{e}}\frac{q}{r} = k_{\mathrm{e}}\frac{\rho(\frac{4}{3}\pi r^3)}{r} = k_{\mathrm{e}}\rho\frac{4}{3}\pi r^2.$$

If we compare the Coulomb force

$$k_{\mathrm{e}}\frac{qQ}{r^2}$$

with the gravitational force

$$G\frac{mM}{r^2},$$

we see that the gravitational potential of a spherical mass will be

$$-G\rho\frac{4}{3}\pi r^2,$$

where we use the negative sign because like masses attract, while like charges repel.

When we bring in the next dm as a spherical shell,

$$dm = \rho(4\pi r^2)\,dr,$$

the increase in potential energy is

$$dE_{\rm g} = -G\rho\frac{4}{3}\pi r^2\,dm = -G\rho^2\frac{(4\pi)^2}{3}\,r^4\,dr.$$

To find the total gravitational energy we integrate from $r = 0$ to $r = R$:

$$\begin{aligned} E_{\rm g} &= -\int_0^R \frac{1}{3}\,G\rho^2(4\pi)^2 r^4\,dr = -\frac{1}{15}\,G\rho^2(4\pi)^2R^5 \\ &= -\frac{1}{15}\,G\left(\frac{3M}{4\pi R^3}\right)^2(4\pi)^2R^5 = -\frac{3}{5}\frac{GM^2}{R}\,. \end{aligned}$$

Thus the total energy is

$$E = E_{\rm e} + E_{\rm g} = \frac{3\hbar^2N\,(9\pi N)^{2/3}}{80m_{\rm e}R^2} - \frac{3GM^2}{5R}\,.$$

We find the equilibrium radius from $dE/dR = 0$:

$$\frac{dE}{dR} = -\frac{3\hbar^2N\,(9\pi N)^{2/3}}{40m_{\rm e}R^3} + \frac{3GM^2}{5R^2} = 0,$$

which gives

$$R = \frac{\hbar^2N\,(9\pi N)^{2/3}}{8Gm_{\rm e}M^2}\,.$$

Because the mass of the electrons is negligible, $N = M/m_{\rm n}$. Thus we have

$$R = \frac{\hbar^2\,(9\pi)^{2/3}}{8Gm_{\rm e}m_{\rm n}^{5/3}M^{1/3}}\,.$$

For a mass equal to that of the Sun, we get

$$\begin{aligned} R &= \frac{(1.055\times10^{-34}\ \mathrm{J\cdot s})^2(9\pi)^{2/3}}{8(6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2})(9.11\times10^{-31}\ \mathrm{kg})} \\ &\quad\times\frac{1}{(1.67\times10^{-27}\ \mathrm{kg})^{5/3}(2.0\times10^{30}\ \mathrm{kg})^{1/3}} \\ &= 7.2\times10^6\ \mathrm{m} = 7.2\times10^3\ \mathrm{km}. \end{aligned}$$

9–51. Determine the radius of a neutron star using the same argument as in Problem 9–50 but for N neutrons only. Show that the radius of a neutron star, of 1.5 solar masses, is about 11 km.

SOLUTION. If we have only $N = M/m_{\rm n}$ neutrons, the Fermi energy is

$$E_{\rm n} = \left(\frac{3}{5}E_{\rm F}\right)N = \frac{3}{5}N\frac{\hbar^2}{2m_{\rm n}}\left(\frac{3\pi^2 N}{V}\right)^{2/3} = \frac{3\hbar^2 N}{40m_{\rm n}R^2}\left(18\pi N\right)^{2/3}.$$

The gravitational energy will be the same, so the total energy is

$$E = E_{\rm n} + E_{\rm g} = \frac{3\hbar^2 N(18\pi N)^{2/3}}{40m_{\rm n}R^2} - \frac{3GM^2}{5R}.$$

We find the equilibrium radius from $dE/dR = 0$:

$$\frac{dE}{dR} = -\frac{3\hbar^2 N(18\pi N)^{2/3}}{20m_{\rm n}R^3} + \frac{3GM^2}{5R^2} = 0,$$

which gives

$$R = \frac{\hbar^2 N(18\pi N)^{2/3}}{4Gm_{\rm n}M^2}.$$

In terms of the mass, this is

$$R = \frac{\hbar^2(18\pi)^{2/3}}{4Gm_{\rm n}^{8/3}M^{1/3}}.$$

For a mass of 1.5 solar masses, we get

$$\begin{aligned} R &= \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2(18\pi)^{2/3}}{4(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.67\times10^{-27}\ \text{kg})^{8/3}} \\ &\quad\times\frac{1}{\left((1.5)(2.0\times10^{30}\ \text{kg})\right)^{1/3}} \\ &= 1.1\times10^4\ \text{m} = 11\ \text{km}. \end{aligned}$$

9–52. (a) Use special relativity and Newton's law of gravitation to show that a photon of mass $m = E/c^2$ just grazing the Sun will be deflected by an angle $\Delta\theta$ given by

$$\Delta\theta = \frac{2GM}{c^2R}$$

where G is the gravitational constant, R and M are the radius and mass of the Sun, and c is the speed of light. (b) Put in values and show $\Delta\theta = 0.87''$. (General relativity predicts an angle twice as large, $1.74''$.)

SOLUTION. (a) A photon of energy E will have a mass

$$m = \frac{E}{c^2}\,,$$

and a momentum

$$p = \frac{E}{c} = mc.$$

We assume the deflection is small. Using the coordinate system shown in Fig. 9.8, this means $y \approx R$ and $r^2 \approx x^2 + R^2$. We want the deflection in the y-direction. When the photon is at the position x, the vertical component of the gravitational force will produce an impulse $F_y\,dt$ in the time $dt = dx/c$.

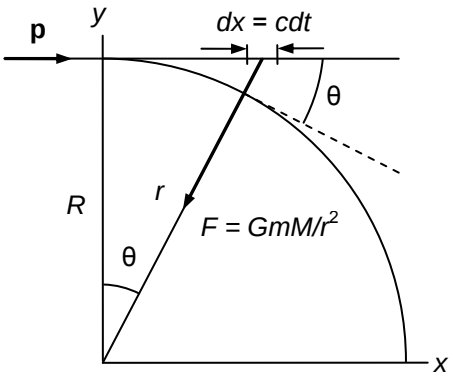


Figure 9.8. Problem 9–52.

Thus we have

$$dp_y = -\frac{GMm}{r^2}\,\cos\theta\,dt = -\frac{GMm}{cr^2}\,\frac{R}{r}\,dx = -\frac{GMmR}{c}\,\frac{dx}{(x^2+R^2)^{3/2}}\,.$$

To find the total deflection we integrate over the path from $x = -\infty$

to $x = +\infty$:

$$\begin{aligned}\Delta p &= -\int_{-\infty}^{+\infty} \frac{GMmR}{c} \frac{dx}{(x^2+R^2)^{3/2}} \\ &= -\frac{GMmR}{c} \left(\frac{x}{R^2\sqrt{x^2+R^2}} \right) \Bigg|_{-\infty}^{+\infty} \\ &= -\frac{2GMm}{cR} = -\frac{2GMp}{c^2R} \,.\end{aligned}$$

For the magnitude of the deflection angle we have

$$\Delta\theta = -\frac{\Delta p_y}{p} = \frac{2GM}{c^2R} \,.$$

(b) When we use the data for the Sun, we get

$$\begin{aligned}\Delta\theta &= \frac{2(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(2.0 \times 10^{30}\ \mathrm{kg})}{(3.00 \times 10^8\ \mathrm{m/s})^2(7.0 \times 10^8\ \mathrm{m})} \\ &\quad \times \left(\frac{180^\circ}{\pi} \right) \left(\frac{3\,600''}{1^\circ} \right) \\ &= 0.87''.\end{aligned}$$

A Physical Constants and Other Useful Data

Table A.1. Fundamental constants

Quantity	Symbol	Approximate value
Speed of light in vacuum	c	2.998×10^{8} m/s
Gravitational constant	G	6.673×10^{-11} N $\cdot$ m^2/kg^2
Planck's constant	h	6.626×10^{-34} J $\cdot$ s
		$= 4.136 \times 10^{-15}$ eV $\cdot$ s
Planck's constant reduced	$\hbar = \dfrac{h}{2\pi}$	1.055×10^{-34} J $\cdot$ s
		$= 0.658 \times 10^{-15}$ eV $\cdot$ s
Charge of electron	e	1.602×10^{-19} C
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ T $\cdot$ m/A
Permittivity of free space	$\varepsilon_0 = \dfrac{1}{\mu_0 c^2}$	8.854×10^{-12} C^2/N $\cdot$ m^2
Coulomb constant	$k_\mathrm{e} = \dfrac{1}{4\pi\varepsilon_0}$	8.988×10^{9} N $\cdot$ m^2/C^2
		$= 1.440 \times 10^{-9}$ eV $\cdot$ m/e^2
Boltzmann's constant	k_B	1.381×10^{-23} J/K
		$= 0.8617 \times 10^{-4}$ eV/K
Avogadro's number	N_A	6.022×10^{23} mol^{-1}
Gas constant	$R = k_\mathrm{B} N_\mathrm{A}$	8.315 J/mol $\cdot$ K
Electron rest mass	m_e	0.9109×10^{-30} kg
		$= 0.000549$ u
		$= 0.5110$ MeV/c^2
Proton rest mass	m_p	1.6726×10^{-27} kg
		$= 1.00728$ u
		$= 938.3$ MeV/c^2
Neutron rest mass	m_n	1.6749×10^{-27} kg
		$= 1.008665$ u
		$= 939.6$ MeV/c^2
Bohr magneton	$\mu_\mathrm{B} = \dfrac{e\hbar}{2m_\mathrm{e}}$	9.274×10^{-24} J/T

continued

Table A.1. Fundamental constants *(continued)*

Quantity	Symbol	Approximate value
		$= 5.788 \times 10^{-5}$ eV/T
Nuclear magneton	$\mu_{\mathrm{N}} = \dfrac{e\hbar}{2m_{\mathrm{p}}}$	5.051×10^{-27} J/T
		$= 3.152 \times 10^{-8}$ eV/T
Bohr radius	$r_{\mathrm{B}} = \dfrac{\hbar^2}{k_{\mathrm{e}} m_{\mathrm{e}} e^2}$	5.292×10^{-11} m
Compton wavelength	$\lambda_{\mathrm{C}} = \dfrac{2\pi\hbar}{m_{\mathrm{e}} c}$	2.426×10^{-12} m
Rydberg constant	$R_\infty = \dfrac{k_e^2 m_{\mathrm{e}} e^4}{4\pi\hbar^3 c}$	1.097×10^{7} m^{-1}
Stefan–Boltzmann constant	σ	5.6704×10^{-8} W/m$^2 \cdot$ K^4

Table A.2. Some unit conversions

Unit	Approximate value
1 atomic mass unit (u)	1.6605×10^{-27} kg $= 931.5$ MeV/c^2
1 electron volt (eV)	1.6022×10^{-19} J
1 day	8.64×10^{4} s
1 year (yr)	3.156×10^{7} s
1 light-year (ly)	9.46×10^{15} m
1 parsec (pc)	3.26 ly $= 3.09 \times 10^{16}$ m
1 atmosphere (atm)	1.013×10^{5} Pa

At the atomic scale, it is most convenient to have energies in electron-volts and wavelengths in nanometers. A useful expression for the energy of a photon in terms of its wavelength is

$$E = 2\pi\hbar f = \frac{2\pi\hbar c}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\lambda}.$$

A Coulomb factor that appears in the analysis of electron energies is

$$k_e e^2 = \frac{e^2}{4\pi\varepsilon_0} = 2.30 \times 10^{-28} \text{ J} \cdot \text{m} = 1.44 \text{ MeV} \cdot \text{fm}.$$

B Properties of Elementary Particles

Table B.1. Particles (stable under strong decay)

Particle /Antiparticle	Spin	Rest mass (MeV/c^2)	B	L_e	L_μ	L_τ	S	Lifetime (s)	Decay modes
					Gauge bosons				
γ/γ	1	0	0	0	0	0	0	stable	
$\mathrm{W}^+/\mathrm{W}^-$	1	80.41×10^3	0	0	0	0	0	3×10^{25}	$\mathrm{e}\nu_\mathrm{e}$, $\mu\nu_\mu$, $\tau\nu_\tau$, hadrons
$\mathrm{Z}^0/\mathrm{Z}^0$	1	91.19×10^3	0	0	0	0	0	3×10^{25}	$\mathrm{e}^+\mathrm{e}^-$, $\mu_+\mu^-$, $\tau^+\tau^-$, hadrons
					Leptons				
$\mathrm{e}^-/\mathrm{e}^+$	$\frac{1}{2}$	0.511	0	+1	0	0	0	stable	
$\nu_\mathrm{e}/\tilde{\nu}_\mathrm{e}$	$\frac{1}{2}$	0	0	+1	0	0	0	stable	
μ^-/μ^+	$\frac{1}{2}$	105.7	0	0	+1	0	0	2.20×10^{-6}	$\mathrm{e}^-\tilde{\nu}_\mathrm{e}\nu_\mu$
$\nu_\mu/\tilde{\nu}_\mu$	$\frac{1}{2}$	0	0	0	+1	0	0	stable	
τ^-/τ^+	$\frac{1}{2}$	1777	0	0	0	+1	0	2.91×10^{-13}	$\mu^-\tilde{\nu}_\mu\nu_\tau$, $\mathrm{e}^-\tilde{\nu}_\mathrm{e}\nu_\tau$, hadrons $+\,\nu_\tau$
$\nu_\tau/\tilde{\nu}_\tau$	$\frac{1}{2}$	0	0	0	0	+1	0	stable	
					Hadrons: mesons				
π^+/π^-	0	139.6	0	0	0	0	0	2.60×10^{-8}	$\mu^+\nu_\mu$
π^0/π^0	0	135.0	0	0	0	0	0	0.84×10^{-16}	2γ
$\mathrm{K}^+/\mathrm{K}^-$	0	493.7	0	0	0	0	+1	1.24×10^{-8}	$\mu^+\nu_\mu$, $\pi^+\pi^0$
$\mathrm{K}^0_\mathrm{S}/\tilde{\mathrm{K}}^0_\mathrm{S}$	0	497.7	0	0	0	0	+1	0.89×10^{-10}	$\pi^+\pi^-$, $2\pi^0$
$\mathrm{K}^0_\mathrm{L}/\tilde{\mathrm{K}}^0_\mathrm{L}$	0	497.7	0	0	0	0	+1	5.17×10^{-8}	$\pi^\pm\mathrm{e}^\mp\tilde{\nu}_\mathrm{e}$, $\pi^\pm\mu^\mp\tilde{\nu}_\mu$, 3π

continued

Table B.1. Particles (stable under strong decay) *(continued)*

Particle /Antiparticle	Spin	Rest mass (MeV/c^2)	B	L_e	L_μ	L_τ	S	Lifetime (s)	Decay modes
η^0/η^0	0	547.3	0	0	0	0	0	5×10^{-19}	2γ, $3\pi^0$, $\pi^+\pi^-\pi^0$
Hadrons: baryons									
p/$\tilde{\mathrm{p}}$	$\frac{1}{2}$	938.3	+1	0	0	0	0	stable	
n/$\tilde{\mathrm{n}}$	$\frac{1}{2}$	939.6	+1	0	0	0	0	887	$\mathrm{pe}^-\tilde{\nu}_\mathrm{e}$
$\Lambda^0/\tilde{\Lambda}^0$	$\frac{1}{2}$	1 115.7	+1	0	0	0	−1	2.63×10^{-10}	$\mathrm{p}\pi^-$, $\mathrm{n}\pi^0$
$\Sigma^+/\tilde{\Sigma}^-$	$\frac{1}{2}$	1 189.4	+1	0	0	0	−1	0.80×10^{-10}	$\mathrm{p}\pi^0$, $\mathrm{n}\pi^+$
$\Sigma^0/\tilde{\Sigma}^0$	$\frac{1}{2}$	1 192.6	+1	0	0	0	−1	7.4×10^{-20}	$\Lambda^0\gamma$
$\Sigma^-/\tilde{\Sigma}^+$	$\frac{1}{2}$	1 197.4	+1	0	0	0	−1	1.48×10^{-10}	$\mathrm{n}\pi^-$
$\Xi^0/\tilde{\Xi}^0$	$\frac{1}{2}$	1 314.9	+1	0	0	0	−2	2.90×10^{-10}	$\Lambda^0\pi^0$
Ξ^-/Ξ^+	$\frac{1}{2}$	1 321.3	+1	0	0	0	−2	1.64×10^{-10}	$\Lambda^0\pi^-$
Ω^-/Ω^+	$\frac{1}{2}$	1 672.5	+1	0	0	0	−3	0.82×10^{-10}	$\Xi^0\pi^-$, $\Lambda^0\mathrm{K}^-$, $\Xi^-\pi^0$

Table B.2. Quarks and antiquarks

Name	Symbol	Spin	Charge	Baryon number	Strangeness	Charm	Bottomness	Topness
Quarks/Antiquarks								
Up	u/$\tilde{\mathrm{u}}$	$\frac{1}{2}$	$+\frac{2}{3}e/-\frac{2}{3}e$	$\frac{1}{3}/-\frac{1}{3}$	0	0	0	0
Down	d/$\tilde{\mathrm{d}}$	$\frac{1}{2}$	$-\frac{1}{3}e/+\frac{1}{3}e$	$\frac{1}{3}/-\frac{1}{3}$	0	0	0	0
Strange	s/$\tilde{\mathrm{s}}$	$\frac{1}{2}$	$-\frac{1}{3}e/+\frac{1}{3}e$	$\frac{1}{3}/-\frac{1}{3}$	−1/+1	0	0	0

continued

Table B.2. Quarks and antiquarks *(continued)*

Name	Symbol	Spin	Charge	Baryon number	Strangeness	Charm	Bottomness	Topness
Charmed	c/$\tilde{\text{c}}$	$\frac{1}{2}$	$+\frac{2}{3}e/-\frac{2}{3}e$	$\frac{1}{3}/-\frac{1}{3}$	0	$+1/-1$	0	0
Bottom	b/$\tilde{\text{b}}$	$\frac{1}{2}$	$-\frac{1}{3}e/+\frac{1}{3}e$	$\frac{1}{3}/-\frac{1}{3}$	0	0	$-1/+1$	0
Top	t/$\tilde{\text{t}}$	$\frac{1}{2}$	$+\frac{2}{3}e/-\frac{2}{3}e$	$\frac{1}{3}/-\frac{1}{3}$	0	0	0	$+1/-1$

Table B.3. Partial list of hadrons associated with charm and bottomness ($L_{\text{e}} = L_\mu = L_\tau = 0$)

Particle /Antiparticle	Spin	Rest mass (MeV/c^2)	Baryon number	Strangeness	Charm	Bottomness	Lifetime (s)	Decay modes
				Mesons				
D^+/D^-	0	1 869.4	0	0	+1	0	10.6×10^{-13}	K + others, e + others
$\text{D}^0/\tilde{\text{D}}^0$	0	1 864.6	0	0	+1	0	4.2×10^{-13}	K + others, μ or e + others
$\text{D}_S^+/\text{D}_S^-$	0	1 969	0	+1	+1	0	4.7×10^{-13}	K + others
J/ψ /self	1	3 096.9	0	0	0	0	0.8×10^{-20}	hadrons, e^+e^-, $\mu^+\mu^-$
Υ/self	1	9 460.4	0	0	0	0	1.3×10^{-20}	hadrons, $\mu^+\mu^-$, e^+e^-, $\tau^+\tau^-$

continued

Table B.3. Hadrons associated with charm and bottomness *(continued)*

Particle /Antiparticle	Spin	Rest mass (MeV/c^2)	Baryon number	Strangeness	Charm	Bottomness	Lifetime (s)	Decay modes
$\mathrm{B}^-/\mathrm{B}^+$	0	5 279	0	0	0	−1	1.5×10^{-12}	D^0 + others
$\mathrm{B}^0/\tilde{\mathrm{B}}^0$	0	5 279	0	0	0	−1	1.5×10^{-12}	D^0 + others
				Baryons				
Λ_c^+/Λ_c^-	$\frac{1}{2}$	2 285	+1	0	+1	0	2.0×10^{-13}	hadrons (e.g., Λ + others)
$\Sigma_c^{++}/\Sigma_c^{--}$	$\frac{1}{2}$	2 453	+1	0	+1	0	?	$\Lambda_c^+\pi^+$
Σ_c^+/Σ_c^-	$\frac{1}{2}$	2 454	+1	0	+1	0	?	$\Lambda_c^+\pi^0$
$\Sigma_c^0/\tilde{\Sigma}_c^0$	$\frac{1}{2}$	2 452	+1	0	+1	0	?	$\Lambda_c^+\pi^-$
$\Lambda_b^0/\tilde{\Lambda}_b^0$	$\frac{1}{2}$	5 640	+1	0	0	−1	1.1×10^{-12}	$\mathrm{J}/\psi\Lambda^0$, $\mathrm{pD}^0\pi^-$, $\Lambda_c^+\pi^+\pi^-\pi^-$

Table B.4. The elementary particles as seen today

	1-st generation	2-nd generation	3-rd generation
Quarks	u, d	s, c	b, t
Leptons	e, ν_e	μ, ν_μ	τ, ν_τ
Gauge bosons	γ	$\mathrm{W}^\pm$, Z^0	gluons

C Chemical Elements and Isotopes

Table C.1. Selected isotopes. The atomic masses are those for the neutral atom, including the Z electrons. (Names of elements with atomic number 112 to 118 are provisional.)

Z	Element	Symbol	A	Atomic mass (u)	half-life, $T_{1/2}$ (if radioactive)
0	(Neutron)	n	1	1.008665	10.4 min
1	Hydrogen	H	1	1.007825	
	Deuterium	d or D	2	2.014102	
	Tritium	t or T	3	3.016049	12.33 yr
2	Helium	He	3	3.016029	
			4	4.002603	
3	Lithium	Li	6	6.015122	
			7	7.016004	
4	Beryllium	Be	7	7.016929	53.12 days
			9	9.012182	
5	Boron	B	10	10.012937	
			11	11.009305	
6	Carbon	C	11	11.011434	20.39 min
			12	12.000000	
			13	13.003355	
			14	14.003242	5 730 yr
7	Nitrogen	N	13	13.005739	9.965 min
			14	14.003074	
			15	15.000108	
8	Oxygen	O	15	15.003065	122.24 s
			16	15.994915	
			18	17.999160	

continued

Table C.1. Selected isotopes *(continued)*

Z	Element	Symbol	A	Atomic mass (u)	half-life, $T_{1/2}$ (if radioactive)
9	Fluorine	F	19	18.998403	
10	Neon	Ne	20	19.992440	
			22	21.991386	
11	Sodium	Na	22	21.994437	2.6019 yr
			23	22.989770	
			24	23.990963	14.9590 h
12	Magnesium	Mg	24	23.985042	
13	Aluminum	Al	27	26.981538	
14	Silicon	Si	28	27.976927	
			31	30.975363	157.3 min
15	Phosphorus	P	31	30.973762	
			32	31.973907	14.262 days
16	Sulfur	S	32	31.972071	
			35	34.969032	87.32 days
17	Chlorine	Cl	35	34.968853	
			37	36.965903	
18	Argon	Ar	40	39.962383	
19	Potassium	K	39	38.963707	
			40	39.963999	1.28×10^{9} yr
20	Calcium	Ca	40	39.962591	
21	Scandium	Sc	45	44.955910	
22	Titanium	Ti	48	47.947947	
23	Vanadium	V	51	50.943964	
24	Chromium	Cr	52	51.940512	
25	Manganese	Mn	55	54.938049	
26	Iron	Fe	56	55.934942	
27	Cobalt	Co	59	58.933200	
			60	59.933822	5.2714 yr

continued

Table C.1. Selected isotopes *(continued)*

Z	Element	Symbol	A	Atomic mass (u)	half-life, $T_{1/2}$ (if radioactive)
28	Nickel	Ni	58	57.935348	
			60	59.930791	
29	Copper	Cu	63	62.929601	
			65	64.927794	
30	Zinc	Zn	64	63.929147	
			66	65.926037	
31	Gallium	Ga	69	68.925581	
32	Germanium	Ge	72	71.922076	
			74	73.921178	
33	Arsenic	As	75	74.921596	
34	Selenium	Se	80	79.916522	
35	Bromine	Br	79	78.918338	
36	Krypton	Kr	84	83.911507	
37	Rubidium	Rb	85	84.911789	
38	Strontium	Sr	86	85.909262	
			88	87.905614	
			90	89.907737	28.79 yr
39	Yttrium	Y	89	88.905848	
40	Zirconium	Zr	90	89.904704	
41	Niobium	Nb	93	92.906377	
42	Molybdenum	Mo	98	97.905408	
43	Technetium	Tc	97	96.9064	4.2×10^6 yr
			98	97.9072	6.6×10^6 yr
			99	98.9063	2.1×10^5 yr
44	Ruthenium	Ru	102	101.904349	
45	Rhodium	Rh	103	102.905504	
46	Palladium	Pd	106	105.903483	
47	Silver	Ag	107	106.905093	

continued

Table C.1. Selected isotopes *(continued)*

Z	Element	Symbol	A	Atomic mass (u)	half-life, $T_{1/2}$ (if radioactive)
			109	108.904756	
48	Cadmium	Cd	113	112.904402	7.7×10^{15} yr
			114	113.903358	
49	Indium	In	115	114.903878	4.41×10^{14} yr
50	Tin	Sn	120	119.902197	
51	Antimony	Sb	121	120.903818	
52	Tellurium	Te	130	129.906223	7.9×10^{20} yr
53	Iodine	I	127	126.904468	
			131	130.906124	8.0207 days
54	Xenon	Xe	132	131.904154	
			136	135.907220	
55	Cesium	Cs	133	132.905446	
56	Barium	Ba	137	136.905821	
			138	137.905241	
57	Lanthanum	La	139	138.906348	
58	Cerium	Ce	140	139.905434	
59	Praseodymium	Pr	141	140.907647	
60	Neodymium	Nd	142	141.907718	
61	Promethium	Pm	145	144.912744	17.7 yr
			147	146.9151	2.623 yr
62	Samarium	Sm	152	151.919728	
63	Europium	Eu	153	152.921226	
64	Gadolinium	Gd	158	157.924101	
65	Terbium	Tb	159	158.925343	
66	Dysprosium	Dy	164	163.929171	
67	Holmium	Ho	165	164.930319	
68	Erbium	Er	166	165.930290	
69	Thulium	Tm	169	168.934211	

continued

Table C.1. Selected isotopes *(continued)*

Z	Element	Symbol	A	Atomic mass (u)	half-life, $T_{1/2}$ (if radioactive)
70	Ytterbium	Yb	174	173.938858	
71	Lutecium	Lu	175	174.940767	
72	Hafnium	Hf	180	179.946549	
73	Tantalum	Ta	181	180.947996	
74	Tungsten	W	184	183.950933	
75	Rhenium	Re	187	186.955751	4.35×10^{10} yr
76	Osmium	Os	191	190.960927	15.4 days
			192	191.961479	
77	Iridium	Ir	191	190.960591	
			193	192.962923	
78	Platinum	Pt	195	194.964774	
79	Gold	Au	197	196.966551	
80	Mercury	Hg	199	198.968262	
			202	201.970625	
81	Thallium	Tl	205	204.974412	
82	Lead	Pb	206	205.974449	
			207	206.975880	
			208	207.976635	
			210	209.984173	22.3 yr
			211	210.988731	36.1 min
			212	211.991887	10.64 h
			214	213.999798	26.8 min
83	Bismuth	Bi	209	208.980383	
			211	210.987258	2.14 min
84	Polonium	Po	209	208.9824	102 yr
			210	209.982857	138.376 day
			214	213.995185	164.3 μs
85	Astatine	At	210	209.9871	8.1 h

continued

Table C.1. Selected isotopes *(continued)*

Z	Element	Symbol	A	Atomic mass (u)	half-life, $T_{1/2}$ (if radioactive)
			211	210.9875	7.21 h
86	Radon	Rn	211	210.9906	14.6 h
			220	220.0114	55.6 s
			222	222.0176	3.823 day
87	Francium	Fr	223	223.019731	21.8 min
88	Radium	Ra	223	223.0185	11.43 day
			224	224.0202	3.66 day
			226	226.0254	1 599 yr
			228	228.0311	5.76 yr
89	Actinium	Ac	227	227.0278	21.773 yr
90	Thorium	Th	230	230.0331	7.54×10^{6} yr
			232	232.038050	1.405×10^{10} yr
91	Protactinium	Pa	231	231.035878	3.276×10^{4} yr
92	Uranium	U	232	232.037146	68.9 yr
			233	233.039628	1.592×10^{5} yr
			235	235.043923	7.038×10^{8} yr
			236	236.045561	2.342×10^{7} yr
			238	238.050782	4.468×10^{9} yr
			239	239.054287	23.45 min
93	Neptunium	Np	237	237.048166	2.144×10^{6} yr
			239	239.052931	2.3565 day
94	Plutonium	Pu	238	238.0496	87.7 yr
			239	239.0522	2.410×10^{4} yr
			240	240.0538	6.56×10^{3} yr
			241	241.0569	14.4 yr
			242	242.0587	3.75×10^{5} yr
			244	244.0642	8.00×10^{7} yr
95	Americium	Am	241	241.0568	432.7 yr

continued

Table C.1. Selected isotopes *(continued)*

Z	Element	Symbol	A	Atomic mass (u)	half-life, $T_{1/2}$ (if radioactive)
			243	243.061373	7 370 yr
96	Curium	Cm	243	243.0614	29.1 yr
			244	244.0628	18.1 yr
			245	245.0655	8.48×10^3 yr
			246	246.0672	4.76×10^3 yr
			247	247.0704	1.56×10^7 yr
			248	248.0723	3.48×10^5 yr
97	Berkelium	Bk	247	247.070298	1 380 yr
98	Californium	Cf	249	249.0749	351 yr
			250	250.0764	13.1 yr
			251	251.0796	9.0×10^2 yr
			252	252.0816	2.65 yr
99	Einsteinium	Es	252	252.082972	472 day
100	Fermium	Fm	257	257.095099	101 day
101	Mendelevium	Md	258	258.098425	51.5 day
			260	260.1037	27.8 day
102	Nobelium	No	259	259.10102	58 min
103	Lawrencium	Lr	262	262.10969	3.6 hr
104	Rutherfordium	Rf	267	267.1215	1.3 min
105	Dubnium	Db	268	268.1255	0.7 day
106	Seaborgium	Sg	271	271.1335	21 s
107	Bohrium	Bh	272	272.1380	10 s
108	Hassium	Hs	277	277.150	16.5 min
109	Meitnerium	Mt	276	276.1512	0.7 s
110	Darmstadtium	Ds	281	281.162	9.6 s
111	Roentgenium	Rg	280	280.1645	3.6 s
112	Ununbium	Uub	285	285.174	34 s
113	Ununtrium	Uut	284	284.178	0.5 s

continued

Table C.1. Selected isotopes *(continued)*

Z	Element	Symbol	A	Atomic mass (u)	half-life, $T_{1/2}$ (if radioactive)
114	Ununquadium	Uuq	289	289.189	2.7 s
115	Ununpentium	Uup	288	288.192	87×10^{-3} s
116	Ununhexium	Uuh	293		0.05 s
118	Ununoctium	Uuo	294		1.8×10^{-3} s

Table C.2. Electron configurations of the neutral gaseous atoms in the ground state. (Given by subshells in concise form, by subshells written out, and by number of electrons per shell.)

			${}_1$H hydrogen: $1s^1$		
$1s^1$					
1					
			${}_2$He helium: $1s^2$		
$1s^2$					
2					
			${}_3$Li lithium: $1s^2\,2s^1$		
$1s^2$	$2s^1$				
2	1				
			${}_4$Be beryllium: $1s^2\,2s^2$		
$1s^2$	$2s^2$				
2	2				
			${}_5$B boron: $1s^2\,2s^2\,2p^1$		
$1s^2$	$2s^2\ 2p^1$				
2	3				
			${}_6$C carbon: $1s^2\,2s^2\,2p^2$		
$1s^2$	$2s^2\ 2p^2$				
2	4				
			${}_7$N nitrogen: $1s^2\,2s^2\,2p^3$		
$1s^2$	$2s^2\ 2p^3$				
2	5				

continued

Table C.2. Electron configurations *(continued)*

				$_8$O oxygen: $1s^2\,2s^2\,2p^4$			
$1s^2$	$2s^2\ 2p^4$						
2	6						
				$_9$F fluorine: $1s^2\,2s^2\,2p^5$			
$1s^2$	$2s^2\ 2p^5$						
2	7						
				$_{10}$Ne neon: $1s^2\,2s^2\,2p^6$			
$1s^2$	$2s^2\ 2p^6$						
2	8						
				$_{11}$Na sodium: $[\mathrm{Ne}]\,3s^1$			
$1s^2$	$2s^2\ 2p^6$	$3s^1$					
2	8	1					
				$_{12}$Mg magnesium: $[\mathrm{Ne}]\,3s^2$			
$1s^2$	$2s^2\ 2p^6$	$3s^2$					
2	8	2					
				$_{13}$Al aluminium: $[\mathrm{Ne}]\,3s^2\,3p^1$			
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^1$					
2	8	3					
				$_{14}$Si silicon: $[\mathrm{Ne}]\,3s^2\,3p^2$			
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^2$					
2	8	4					
				$_{15}$P phosphorus: $[\mathrm{Ne}]\,3s^2\,3p^3$			
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^3$					
2	8	5					
				$_{16}$S sulfur: $[\mathrm{Ne}]\,3s^2\,3p^4$			
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^4$					
2	8	6					
				$_{17}$Cl chlorine: $[\mathrm{Ne}]\,3s^2\,3p^5$			
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^5$					
2	8	7					
				$_{18}$Ar argon: $[\mathrm{Ne}]\,3s^2\,3p^6$			
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6$					
2	8	8					

continued

Table C.2. Electron configurations *(continued)*

$_{19}$K potassium: $[\mathrm{Ar}]\,4s^1$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6$	$4s^1$		
2	8	8	1		
$_{20}$Ca calcium: $[\mathrm{Ar}]\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6$	$4s^2$		
2	8	8	2		
$_{21}$Sc scandium: $[\mathrm{Ar}]\,3d^1\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^1$	$4s^2$		
2	8	9	2		
$_{22}$Ti titanium: $[\mathrm{Ar}]\,3d^2\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^2$	$4s^2$		
2	8	10	2		
$_{23}$V vanadium: $[\mathrm{Ar}]\,3d^3\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^3$	$4s^2$		
2	8	11	2		
$_{24}$Cr chromium: $[\mathrm{Ar}]\,3d^5\,4s^1$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^5$	$4s^1$		
2	8	13	1		
$_{25}$Mn manganese: $[\mathrm{Ar}]\,3d^5\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^5$	$4s^2$		
2	8	13	2		
$_{26}$Fe iron: $[\mathrm{Ar}]\,3d^6\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^6$	$4s^2$		
2	8	14	2		
$_{27}$Co cobalt: $[\mathrm{Ar}]\,3d^7\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^7$	$4s^2$		
2	8	15	2		
$_{28}$Ni nickel: $[\mathrm{Ar}]\,3d^8\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^8$	$4s^2$		
2	8	16	2		
$_{29}$Cu copper: $[\mathrm{Ar}]\,3d^{10}\,4s^1$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^1$		
2	8	18	1		

continued

Table C.2. Electron configurations *(continued)*

$_{30}$Zn zinc: [Ar] $3d^{10}\,4s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2$		
2	8	18	2		
$_{31}$Ga gallium: [Ar] $3d^{10}\,4s^2\,4p^1$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^1$		
2	8	18	3		
$_{32}$Ge germanium: [Ar] $3d^{10}\,4s^2\,4p^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^2$		
2	8	18	4		
$_{33}$As arsenic: [Ar] $3d^{10}\,4s^2\,4p^3$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^3$		
2	8	18	5		
$_{34}$Se selenium: [Ar] $3d^{10}\,4s^2\,4p^4$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^4$		
2	8	18	6		
$_{35}$Br bromine: [Ar] $3d^{10}\,4s^2\,4p^5$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^5$		
2	8	18	7		
$_{36}$Kr krypton: [Ar] $3d^{10}\,4s^2\,4p^6$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6$		
2	8	18	8		
$_{37}$Rb rubidium: [Kr] $5s^1$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6$	$5s^1$	
2	8	18	8	1	
$_{38}$Sr strontium: [Kr] $5s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6$	$5s^2$	
2	8	18	8	2	
$_{39}$Y yttrium: [Kr] $4d^1\,5s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^1$	$5s^2$	
2	8	18	9	2	
$_{40}$Zr zirconium: [Kr] $4d^2\,5s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^2$	$5s^2$	
2	8	18	10	2	

continued

Table C.2. Electron configurations *(continued)*

$_{41}$Nb niobium: $[\mathrm{Kr}]\,4d^4\,5s^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^4$	$5s^1$		
2	8	18	12	1		
$_{42}$Mo molybdenum: $[\mathrm{Kr}]\,4d^5\,5s^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^5$	$5s^1$		
2	8	18	13	1		
$_{43}$Tc technetium: $[\mathrm{Kr}]\,4d^5\,5s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^5$	$5s^2$		
2	8	18	13	2		
$_{44}$Ru ruthenium: $[\mathrm{Kr}]\,4d^7\,5s^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^7$	$5s^1$		
2	8	18	15	1		
$_{45}$Rh rhodium: $[\mathrm{Kr}]\,4d^8\,5s^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^8$	$5s^1$		
2	8	18	16	1		
$_{46}$Pd palladium: $[\mathrm{Kr}]\,4d^{10}$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$			
2	8	18	18			
$_{47}$Ag silver: $[\mathrm{Kr}]\,4d^{10}\,5s^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^1$		
2	8	18	18	1		
$_{48}$Cd cadmium: $[\mathrm{Kr}]\,4d^{10}\,5s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2$		
2	8	18	18	2		
$_{49}$In indium: $[\mathrm{Kr}]\,4d^{10}\,5s^2\,5p^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^1$		
2	8	18	18	3		
$_{50}$Sn tin: $[\mathrm{Kr}]\,4d^{10}\,5s^2\,5p^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^2$		
2	8	18	18	4		
$_{51}$Sb antimony: $[\mathrm{Kr}]\,4d^{10}\,5s^2\,5p^3$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^3$		
2	8	18	18	5		

continued

Table C.2. Electron configurations *(continued)*

$_{52}$Te tellurium: [Kr] $4d^{10}\,5s^2\,5p^4$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^4$		
2	8	18	18	6		
$_{53}$I iodine: [Kr] $4d^{10}\,5s^2\,5p^5$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^5$		
2	8	18	18	7		
$_{54}$Xe xenon: [Kr] $4d^{10}\,5s^2\,5p^6$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^6$		
2	8	18	18	8		
$_{55}$Cs caesium: [Xe] $6s^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^6$	$6s^1$	
2	8	18	18	8	1	
$_{56}$Ba barium: [Xe] $6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	18	8	2	
$_{57}$La lanthanum: [Xe] $5d^1\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}$	$5s^2\ 5p^6\ 5d^1$	$6s^2$	
2	8	18	18	9	2	
$_{58}$Ce cerium: [Xe] $4f^1\,5d^1\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^1$	$5s^2\ 5p^6\ 5d^1$	$6s^2$	
2	8	18	19	9	2	
$_{59}$Pr praseodymium: [Xe] $4f^3\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^3$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	21	8	2	
$_{60}$Nd neodymium: [Xe] $4f^4\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^4$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	22	8	2	
$_{61}$Pm promethium: [Xe] $4f^5\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^5$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	23	8	2	
$_{62}$Sm samarium: [Xe] $4f^6\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^6$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	24	8	2	

continued

Table C.2. Electron configurations *(continued)*

$_{63}$Eu europium: $[\mathrm{Xe}]\,4f^7\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^7$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	25	8	2	
$_{64}$Gd gadolinium: $[\mathrm{Xe}]\,4f^7\,5d^1\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^7$	$5s^2\ 5p^6\ 5d^1$	$6s^2$	
2	8	18	25	9	2	
$_{65}$Tb terbium: $[\mathrm{Xe}]\,4f^9\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^9$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	27	8	2	
$_{66}$Dy dysprosium: $[\mathrm{Xe}]\,4f^{10}\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{10}$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	28	8	2	
$_{67}$Ho holmium: $[\mathrm{Xe}]\,4f^{11}\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{11}$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	29	8	2	
$_{68}$Er erbium: $[\mathrm{Xe}]\,4f^{12}\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{12}$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	30	8	2	
$_{69}$Tm thulium: $[\mathrm{Xe}]\,4f^{13}\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{13}$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	31	8	2	
$_{70}$Yb ytterbium: $[\mathrm{Xe}]\,4f^{14}\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6$	$6s^2$	
2	8	18	32	8	2	
$_{71}$Lu lutetium: $[\mathrm{Xe}]\,4f^{14}\,5d^1\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^1$	$6s^2$	
2	8	18	32	9	2	
$_{72}$Hf hafnium: $[\mathrm{Xe}]\,4f^{14}\,5d^2\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^2$	$6s^2$	
2	8	18	32	10	2	
$_{73}$Ta tantalum: $[\mathrm{Xe}]\,4f^{14}\,5d^3\,6s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^3$	$6s^2$	
2	8	18	32	11	2	

continued

Table C.2. Electron configurations *(continued)*

$_{74}$W tungsten: [Xe] $4f^{14}\,5d^4\,6s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^4$	$6s^2$
2	8	18	32	12	2
$_{75}$Re rhenium: [Xe] $4f^{14}\,5d^5\,6s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^5$	$6s^2$
2	8	18	32	13	2
$_{76}$Os osmium: [Xe] $4f^{14}\,5d^6\,6s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^6$	$6s^2$
2	8	18	32	14	2
$_{77}$Ir iridium: [Xe] $4f^{14}\,5d^7\,6s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^7$	$6s^2$
2	8	18	32	15	2
$_{78}$Pt platinum: [Xe] $4f^{14}\,5d^9\,6s^1$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^9$	$6s^1$
2	8	18	32	17	1
$_{79}$Au gold: [Xe] $4f^{14}\,5d^{10}\,6s^1$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^1$
2	8	18	32	18	1
$_{80}$Hg mercury: [Xe] $4f^{14}\,5d^{10}\,6s^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2$
2	8	18	32	18	2
$_{81}$Tl thallium: [Xe] $4f^{14}\,5d^{10}\,6s^2\,6p^1$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^1$
2	8	18	32	18	3
$_{82}$Pb lead: [Xe] $4f^{14}\,5d^{10}\,6s^2\,6p^2$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^2$
2	8	18	32	18	4
$_{83}$Bi bismuth: [Xe] $4f^{14}\,5d^{10}\,6s^2\,6p^3$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^3$
2	8	18	32	18	5
$_{84}$Po polonium: [Xe] $4f^{14}\,5d^{10}\,6s^2\,6p^4$					
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^4$
2	8	18	32	18	6

continued

Table C.2. Electron configurations *(continued)*

$_{85}$At astatine: [Xe] $4f^{14}\,5d^{10}\,6s^2\,6p^5$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^5$	
2	8	18	32	18	7	
$_{86}$Rn radon: [Xe] $4f^{14}\,5d^{10}\,6s^2\,6p^6$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^6$	
2	8	18	32	18	8	
$_{87}$Fr francium: [Rn] $7s^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^6$	$7s^1$
2	8	18	32	18	8	1
$_{88}$Ra radium: [Rn] $7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	18	8	2
$_{89}$Ac actinium: [Rn] $6d^1\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^6\ 6d^1$	$7s^2$
2	8	18	32	18	9	2
$_{90}$Th thorium: [Rn] $6d^2\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}$	$6s^2\ 6p^6\ 6d^2$	$7s^2$
2	8	18	32	18	10	2
$_{91}$Pa protactinium: [Rn] $5f^2\,6d^1\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^2$	$6s^2\ 6p^6\ 6d^1$	$7s^2$
2	8	18	32	20	9	2
$_{92}$U uranium: [Rn] $5f^3\,6d^1\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^3$	$6s^2\ 6p^6\ 6d^1$	$7s^2$
2	8	18	32	21	9	2
$_{93}$Np neptunium: [Rn] $5f^4\,6d^1\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^4$	$6s^2\ 6p^6\ 6d^1$	$7s^2$
2	8	18	32	22	9	2
$_{94}$Pu plutonium: [Rn] $5f^6\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^6$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	24	8	2
$_{95}$Am americium: [Rn] $5f^7\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^7$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	25	8	2

continued

Table C.2. Electron configurations *(continued)*

$_{96}$Cm curium: [Rn] $5f^7\,6d^1\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^7$	$6s^2\ 6p^6\ 6d^1$	$7s^2$
2	8	18	32	25	9	2
$_{97}$Bk berkelium: [Rn] $5f^9\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^9$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	27	8	2
$_{98}$Cf californium: [Rn] $5f^{10}\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{10}$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	28	8	2
$_{99}$Es einsteinium: [Rn] $5f^{11}\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{11}$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	29	8	2
$_{100}$Fm fermium: [Rn] $5f^{12}\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{12}$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	30	8	2
$_{101}$Md mendelevium: [Rn] $5f^{13}\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{13}$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	31	8	2
$_{102}$No nobelium: [Rn] $5f^{14}\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6$	$7s^2$
2	8	18	32	32	8	2
$_{103}$Lr lawrencium: [Rn] $5f^{14}\,7s^2\,7p^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6$	$7s^2\ 7p^1$
2	8	18	32	32	8	3
$_{104}$Rf rutherfordium: [Rn] $5f^{14}\,6d^2\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6\ 6d^2$	$7s^2$
2	8	18	32	32	10	2
$_{105}$Db dubnium: [Rn] $5f^{14}\,6d^3\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6\ 6d^3$	$7s^2$
2	8	18	32	32	11	2
$_{106}$Sg seaborgium: [Rn] $5f^{14}\,6d^4\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6\ 6d^4$	$7s^2$
2	8	18	32	32	12	2

continued

Table C.2. Electron configurations *(continued)*

$_{107}$Bh bohrium: [Rn] $5f^{14}\,6d^5\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6\ 6d^5$	$7s^2$
2	8	18	32	32	13	2
$_{108}$Hs hassium: [Rn] $5f^{14}\,6d^6\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6\ 6d^6$	$7s^2$
2	8	18	32	32	14	2
$_{109}$Mt meitnerium: [Rn] $5f^{14}\,6d^7\,7s^2$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6\ 6d^7$	$7s^2$
2	8	18	32	32	15	2
$_{110}$Ds darmstadtium: [Rn] $5f^{14}\,6d^9\,7s^1$						
$1s^2$	$2s^2\ 2p^6$	$3s^2\ 3p^6\ 3d^{10}$	$4s^2\ 4p^6\ 4d^{10}\ 4f^{14}$	$5s^2\ 5p^6\ 5d^{10}\ 5f^{14}$	$6s^2\ 6p^6\ 6d^9$	$7s^1$
2	8	18	32	32	17	1

1 H 1.00794																1 H 1.00794	2 He 4.002602
3 Li 6.941	4 Be 9.012182											5 B 10.811	6 C 12.0107	7 N 14.00674	8 O 15.9994	9 F 18.9984032	10 Ne 20.1797
11 Na 22.989770	12 Mg 24.3050											13 Al 26.981538	14 Si 28.0855	15 P 30.973761	16 S 32.066	17 Cl 35.4527	18 Ar 39.948
19 K 39.0983	20 Ca 40.078	21 Sc 44.955910	22 Ti 47.867	23 V 50.9415	24 Cr 51.9961	25 Mn 54.938049	26 Fe 55.845	27 Co 58.933200	28 Ni 58.6934	29 Cu 63.546	30 Zn 65.39	31 Ga 69.723	32 Ge 72.61	33 As 74.92160	34 Se 78.96	35 Br 79.904	36 Kr 83.80
37 Rb 85.4678	38 Sr 87.62	39 Y 88.90585	40 Zr 91.224	41 Nb 92.90638	42 Mo 95.94	43 Tc (98)	44 Ru 101.07	45 Rh 102.90550	46 Pd 106.42	47 Ag 107.8682	48 Cd 112.411	49 In 114.818	50 Sn 118.710	51 Sb 121.760	52 Te 127.60	53 I 126.90447	54 Xe 131.29
55 Cs 132.90545	56 Ba 137.327	57 La 138.9055	72 Hf 178.49	73 Ta 180.9479	74 W 183.84	75 Re 186.207	76 Os 190.23	77 Ir 192.217	78 Pt 195.078	79 Au 196.96655	80 Hg 200.59	81 Tl 204.3833	82 Pb 207.2	83 Bi 208.98038	84 Po (209)	85 At (210)	86 Rn (222)
87 Fr (223)	88 Ra (226)	89 Ac (227)	104 Rf (261)	105 Db (262)	106 Sg (263)	107 Bh (262)	108 Hs (265)	109 Mt (266)	110 (269)	111 (272)	112 (277)		114 (289) (287)		116 (289)		118 (293)

58 Ce 140.116	59 Pr 140.90765	60 Nd 144.24	61 Pm (145)	62 Sm 150.36	63 Eu 151.964	64 Gd 157.25	65 Tb 158.92534	66 Dy 162.50	67 Ho 164.93032	68 Er 167.26	69 Tm 168.93421	70 Yb 173.04	71 Lu 174.967
90 Th 232.0381	91 Pa 231.03588	92 U 238.0289	93 Np (237)	94 Pu (244)	95 Am (243)	96 Cm (247)	97 Bk (247)	98 Cf (251)	99 Es (252)	100 Fm (257)	101 Md (258)	102 No (259)	103 Lr (262)

Figure C.1. Periodic table of the elements

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About the Author



Almaz Mustafin is Professor of Physics at K. I. Satpaev Kazakh National Technical University. He is hereditary educator: his grandfather was a school teacher and both his parents were university professors. He studied in the School of General and Applied Physics of the Moscow Institute of Physics and Technology. Upon graduating from MIPT he spent two years as a Research Assistant at the I. E. Tamm Theoretical Department of the P. N. Lebedev Physical Institute, Russian Academy of Sciences. He earned his Candidate of Sciences degree from the M. V. Lomonosov Moscow State University where he pursued his post-graduate research under the advice of Prof. D. S. Chernavskii. In 1984 he joined the research staff of the Kazakh National Academy of Sciences, Almaty, where he worked until 2001—initially at the Institute of Theoretical and Applied Mathematics and later on at the Space Research Institute. During that period he also spent a year at Kyoto University, Japan, as a recipient of the fellowship by Japan Society for the Promotion of Science. He received his habilitation doctorate from the ITAM in 2001. He joined the faculty of KazNTU in 2007 after six years teaching and doing research in the Department of Physics and Institute for Research in Engineering and Applied Physics at the University of Maryland, College Park. He also held teaching appointments at the George Washington University, DC, in 2001–2002 and in 2007. Prof. Mustafin's research area is nonlinear dynamics with applications in molecular biophysics, evolutionary cybernetics, and laser physics. In addition to his work on this book, he is author of two monographs in mathematical modeling and co-author of a lab manual on the physics of sound for undergraduates (Wiley, 2007). He is a member of the American Physical Society. Prof. Mustafin is an enthusiastic swimmer and hiker. When not in the classroom or bending over a computer, he and his wife Aliya (who holds her doctorate in Economics from Kyoto University, Japan, and also is Professor at KazNTU) explore opportunities for new projects and enjoy reading and traveling.

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